Diagrammatic Matrix Algebra

Lets first recall how matrix multiplication works. Matrices are arrays of elements of an arithmetic or an algebra. Here we will begin by assuming that the matrix elements occur in ordinary numbers (integers, rationals, reals or complex numbers) or their algebra. Two 2 x 2 arrays are multiplied by the following formula.

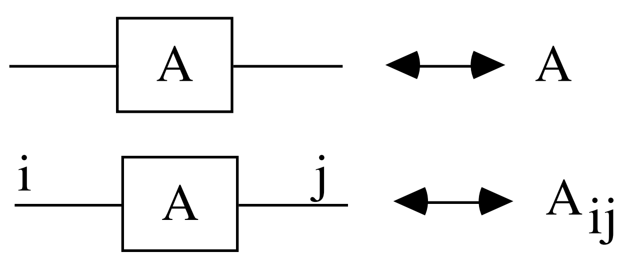
$$\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix} = \begin{pmatrix} a_{00}b_{00}+a_{01}b_{10} & a_{00}b_{01}+a_{01}b_{11} \\ a_{10}b_{00}+a_{11}b_{10} & a_{10}b_{01}+a_{11}b_{11} \end{pmatrix}$$

We denote a matrix **$A = (A_{ij})$** by a global letter (**A** in this case), and by an indication of the form of the elements of the array, **A_{ij}**. The subscripts range over the set **{0,1}** in the case of a 2 x 2 matrix, as shown above. The rule for multiplying two matrices is

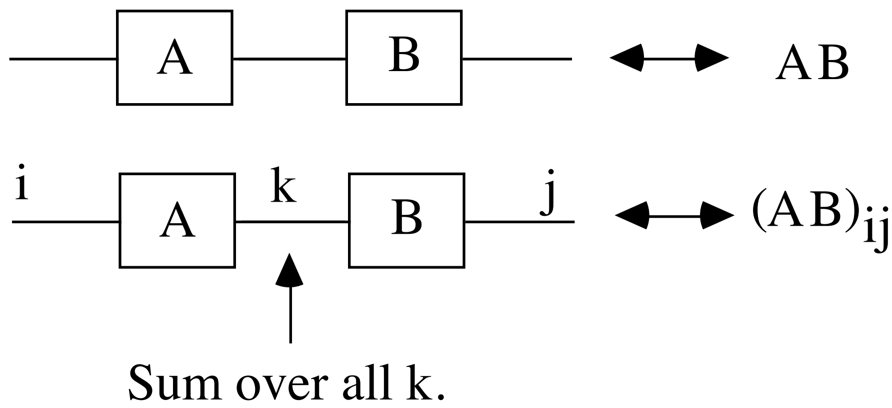
$$(AB)_{ij} = \Sigma_k A_{ik}B_{kj}.$$

where the summation is over the index set for the matrix size that we are using. Compare this formula with the arrangement of indices and sums in the explicit matrix product given above.

We now give a diagrammatic interpretation for matrix algebra. Each individual matrix is represented by a box with (input and output) lines that correspond to the matrix indices.



Matrix multiplication is represented by attaching the output line from one box to the input line of the other box.



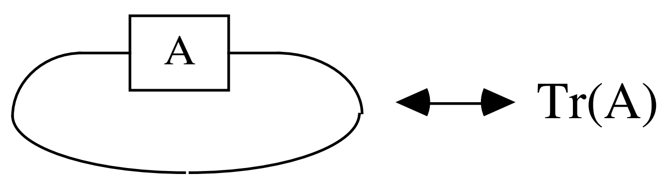
$$(AB)_{ij} = \Sigma_k A_{ik} B_{kj}$$

Lines tying one box to another correspond to internal indices in the matrix product, and so one sums over all possible choices of index for such internal lines.

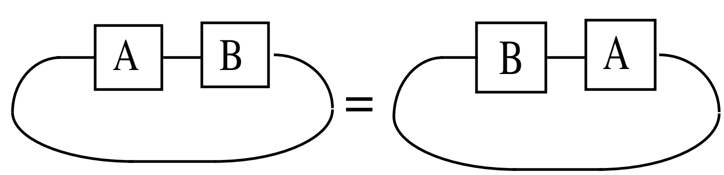
With these diagrammatic conventions in place, one can often make very efficient insight into properties of matrix composition. For example, the trace of a matrix A is given by the formula

$$\mathbf{Tr(A) = \Sigma_i A_{ii}.}$$

Here is the diagram.

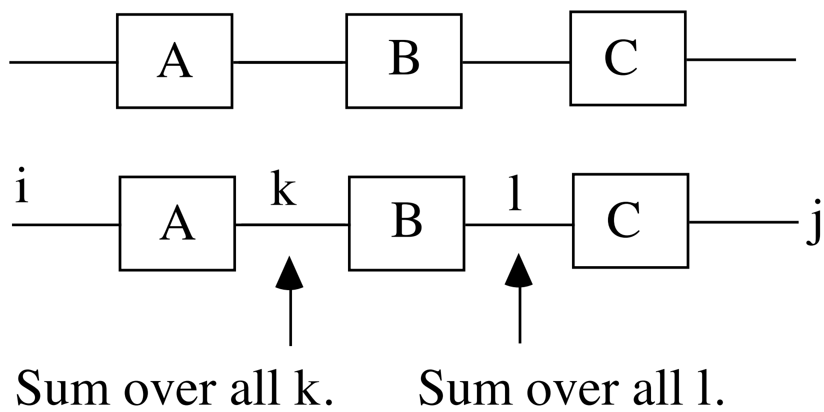


With this diagrammatic for the trace of A, we easily prove that **Tr(AB) = Tr(BA)** by putting two boxes in a circular connection pattern.



Tr(AB) = Tr(BA)

Another application of these diagrams is to the proof that matrix multiplication is associative. For we see that the diagram for the product of three matrices A, B and C is given by



$$(ABC)_{ij} = \sum_{k,l} A_{ik} B_{kl} C_{lj}$$

Matrix Multiplication is associative because the products of matrix entries are associative.

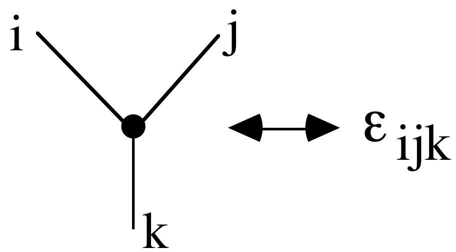
The Epsilon Matrix

One of my favorite matrices is the "epsilon tensor" ε_{ijk}. This matrix has three indices, each of which can take the values 1, 2 or 3. The values of the epsilon are as follows

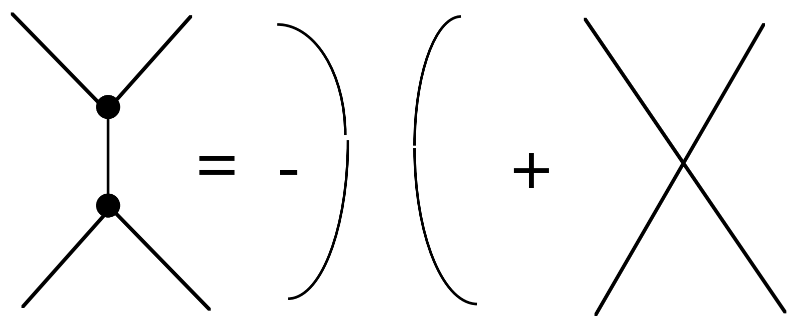
$$\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = +1$$

$$\varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1.$$

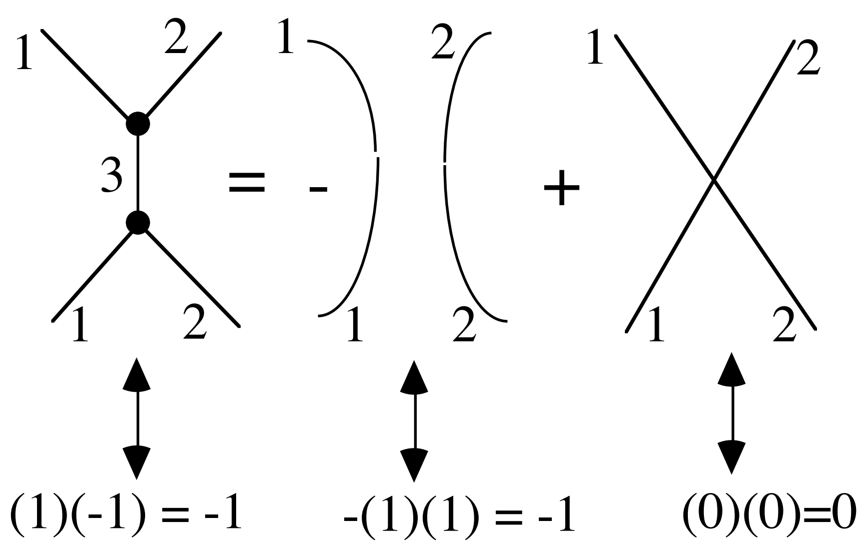
Otherwise (if there is any repetition of indices) the epsilon is zero. Note that epsilon is invariant under cyclic permutation of the indices. We diagram epsilon by using a trivalent vertex.



There is a magic identity about the epsilon, which translates into diagrammatic language as



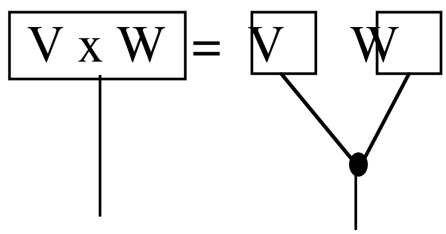
A single line represents the identity matrix. That is, when the two endpoints of the line have the same index value, then the value of the matrix element is one, otherwise it is zero. You can see the truth of this diagrammatic identity by assigning some values to the lines. For example:



Now the *cross product* of two three dimensional vectors is defined by the epsilon:

$$(\mathbf{V} \times \mathbf{W})_k = \sum_k \varepsilon_{ijk} V_i W_j.$$

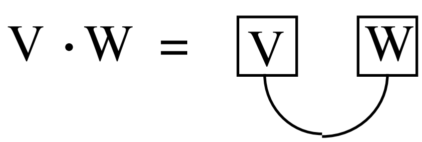
Here one sums over the repeated index **k**. Note that a vector, having only one index is represented by a box with one line. In diagrams the vector cross product is given as follows.



Similarly, the *dot product* of two vectors is given by the formula

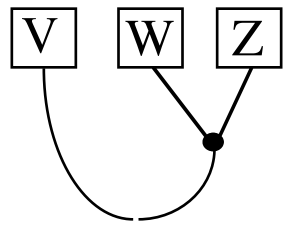
$$\mathbf{V} \bullet \mathbf{W} = \sum_k V_k W_k.$$

In diagrams, we have:

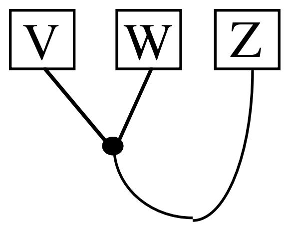


Now we are prepared to see some identities about the vector cross product and the dot product.

$V \cdot (W \times Z) =$



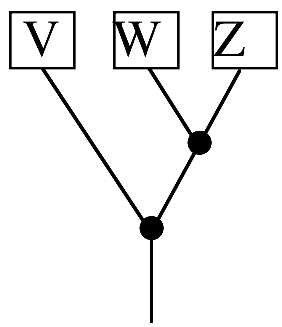
$(V \times W) \cdot Z =$



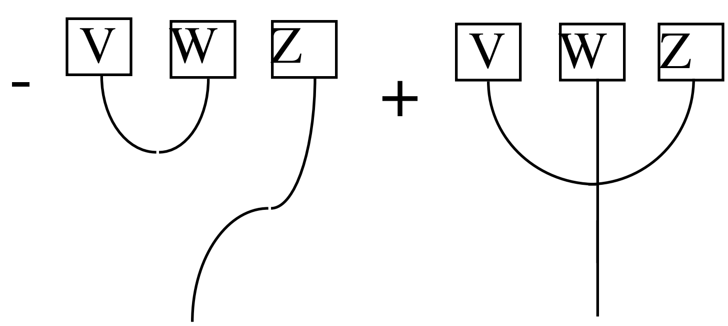
$$V \cdot (W \times Z) = (V \times W) \cdot Z$$

The diagrams deform to one another in the plane. The epsilon is invariant under cyclic permutation of its indices.
Here is one that uses the basic epsilon identity.

$V \times (W \times Z) =$



=



$$= - (V \cdot W)Z + (V \cdot Z)W$$

Vector algebra becomes transparent through the use of diagrammatic matrices.