

MARGULIS NUMBERS AND NUMBER FIELDS

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ABSTRACT. Let M be a closed, orientable hyperbolic 3-manifold such that (1) $H_1(M; \mathbb{Z}_s) = 0$ for $s = 2, 3, 7$. Suppose that (2) M is non-Haken, or more generally that it has integral traces. If the trace field of M is quadratic then 0.395 is a Margulis number for M . If the trace field is cubic then 0.3 is a Margulis number for M .

If K is any number field, then for all but finitely many closed, orientable hyperbolic 3-manifolds M which satisfy (1) and (2) and have trace field K , the number 0.183 is a Margulis number for M . Furthermore, if K is any number field, there is a real number ϵ with $0 < \epsilon \leq 0.3$, having the following property. Let M be any closed hyperbolic 3-manifold which satisfies (1) and (2) and has trace field K . Then about every primitive closed geodesic in M having length $l < \epsilon$ there is an embedded tube having radius $R(l)$, where $R(l)$ is an explicitly defined function such that $\sinh^2 R(l)$ is asymptotic to $(.01869\dots)/l$.

1. INTRODUCTION

Let M be a closed hyperbolic n -manifold. We may write $M = \mathbb{H}^n/\Gamma$ where $\Gamma \leq \text{Isom}_+(\mathbb{H}^n)$ is discrete, cocompact and torsion-free. The group Γ is uniquely determined by M up to conjugacy in $\text{Isom}_+(\mathbb{H}^n)$. We define a *Margulis number* for M , or for Γ , to be a positive real number μ with the following property:

1.0.1. *If P is a point of \mathbb{H}^n , and if x_1 and x_2 are elements of Γ such that $d(P, x_i \cdot P) < \mu$ for $i = 1, 2$, then x_1 and x_2 commute.*

Here d denotes the hyperbolic distance on \mathbb{H}^n .

The Margulis Lemma [7, Chapter D] implies that for every $n \geq 2$ there is a positive constant which is a Margulis number for every closed hyperbolic n -manifold. The largest such number, $\mu(n)$, is called the *Margulis constant* for closed hyperbolic n -manifolds.

Margulis numbers play a central role in the geometry of hyperbolic manifolds. If μ is a Margulis number for M then the points of M where the injectivity radius is less than $\mu/2$ form a disjoint union of “tubes” about closed geodesics whose geometric structure can be precisely described. Topologically they are open $(n - 1)$ -ball bundles over S^1 . This observation and the Margulis Lemma can be used to show, for example, that for every $V > 0$ there is a finite collection of compact orientable 3-manifolds M_1, \dots, M_N , whose boundary components are tori, such that every closed, orientable hyperbolic 3-manifold of volume at most V can be obtained by a Dehn filling of one of the M_i .

The value of $\mu(3)$ is not known; the best known lower bound, due to Meyerhoff [19], is $0.104\dots$. Marc Culler has informed me that the Margulis number for the Weeks manifold appears to be $0.774\dots$, which is therefore an upper bound for $\mu(3)$.

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Under suitable topological restrictions on a hyperbolic 3-manifold M , one can give lower bounds for the Margulis number that are much larger than 0.104. For example, it follows from the results of [11], combined with those of [9] or [2], that if M is closed and orientable and $\pi_1(M)$ has no 2-generator finite-index subgroup, then $\log 3$ is a Margulis number for M .

The theme of the present paper is that new estimates for Margulis numbers can come from *number-theoretical* restrictions on a hyperbolic 3-manifold. Before explaining this further I must give a few definitions.

Throughout this paper I will fix a surjective (continuous) homomorphism $\Pi : \mathrm{SL}_2(\mathbb{C}) \rightarrow \mathrm{Isom}_+(\mathbb{H}^3)$ with kernel $\{\pm I\}$. If M is a closed hyperbolic 3-manifold, and if we write $M = \mathbb{H}^3/\Gamma$ as above, then $\tilde{\Gamma} \doteq \Pi^{-1}(\Gamma)$ is also uniquely determined by M up to conjugacy in $\mathrm{Isom}_+(\mathbb{H}^3)$; in particular the set of all traces of elements of $\tilde{\Gamma}$ is uniquely determined. According to [18, Theorem 3.1.2], these traces generate a finite extension of \mathbb{Q} , called the *trace field* of M , or of Γ .

I will say that M , or Γ , has *integral traces* if the traces of elements of Γ are all algebraic integers. For example, it follows from the proof of [23, Proposition 5] that if the closed orientable hyperbolic 3-manifold M is not a Haken manifold, then M has integral traces. Note also that if M has integral traces then any finite-sheeted covering \tilde{M} of M also has integral traces (although \tilde{M} may well be a Haken manifold if M is non-Haken).

The Margulis number of a hyperbolic 3-manifold M is a quantity associated with its geometric structure. In the previous literature, while number-theoretic invariants such as the trace field have interacted with topological properties of M such as its commensurability class [18], there seems to have few interactions if any between such number-theoretic invariants and quantitative geometric properties of M . In this paper I will show that restrictions on the trace field give stronger information than is otherwise available about the Margulis number and related quantities.

I will illustrate this by stating the following four theorems, which are applications of the methods of this paper. The first two concern the case where the trace field is of low degree.

Theorem 1.1. *Let M be a closed orientable hyperbolic 3-manifold having integral traces, such that $H_1(M; \mathbb{Z}_s) = 0$ for $s = 2, 3, 7$. Suppose that the trace field of Γ is a quadratic number field. Then 0.3925 is a Margulis number for Γ .*

This will be proved in Section 9.

Theorem 1.2. *Let M be a closed orientable hyperbolic 3-manifold having integral traces, such that $H_1(M; \mathbb{Z}_s) = 0$ for $s = 2, 3, 7$. Suppose that the trace field of M is a cubic field. Then 0.3 is a Margulis number for M .*

This will be proved in Section 10.

For examples of closed, orientable hyperbolic 3-manifolds having quadratic and cubic trace fields, see [18, Appendix 13.5].

The next two theorems concern trace fields of arbitrary degree.

Theorem 1.3. *If $K \subset \mathbb{C}$ is an arbitrary number field, there is a real number $\epsilon = \epsilon_K$ with $0 < \epsilon \leq 0.3$, having the following property. Let M be any closed hyperbolic 3-manifold which has trace field K , has integral traces and satisfies $H_1(M; \mathbb{Z}_s) = 0$ for $s = 2, 3, 7$. Let C be a primitive closed*

geodesic in M having length $l < \epsilon$. Then there is an embedded tube about C having radius R , where R is defined by

$$(1.3.1) \quad \sinh^2 R = \frac{\cosh((\log 3)/3) - \cosh \sqrt{\frac{4\pi l}{\sqrt{3}}}}{\cosh \sqrt{\frac{4\pi l}{\sqrt{3}}} - 1}.$$

(In particular C is embedded.)

Theorem 1.3 is closely related to Margulis numbers, although it does not explicitly involve them. Indeed, Proposition 11.4, which we state and prove in Section 11, implies that if M is a hyperbolic 3-manifold having $(\log 3)/3$ as a Margulis number, then any closed geodesic of length $l < (\log 3)/3$ in M is the core of an embedded tube of radius R , where R is defined by (1.3.1). Note that as $l \rightarrow 0$ the right hand side of (1.3.1) is asymptotic to A/l , where $A = \sqrt{3}(\cosh((\log 3)/3) - 1)/(2\pi) = 0.01869\dots$. In contrast, if one uses Proposition 11.4 and the Margulis number 0.104 to obtain a lower bound R' for the embedded tube radius about a closed geodesic of sufficiently small length l , then $\sinh^2 R'$ is asymptotic to A'/l , where $A' = \sqrt{3}(\cosh(0.104) - 1)/2\pi = 0.00149\dots$

Alan Reid has pointed out to me that the methods of [1] may be used to show that certain number fields K have the property that for every $\epsilon > 0$ there is a closed, orientable manifold having trace field K and containing a closed geodesic of length $< \epsilon$. In these cases the conclusion of Theorem 1.3 is non-vacuous.

For the Margulis number itself, we have the following result.

Theorem 1.4. *Let K be a number field. Then up to isometry there are at most finitely many closed, orientable hyperbolic 3-manifolds with the following properties:*

- (1) M has integral traces, and K is its trace field;
- (2) $H_1(M; \mathbb{Z}_p) = 0$ for $p = 2, 3$ and 7 ; and
- (3) 0.183 is not a Margulis number for M .

Theorems 1.1 and 1.2 are applications of a general result which we state below as Theorem 1.6. Theorems 1.3 and 1.4 are not formally consequences of Theorem 1.6, but they rely on most of the ingredients in the proof of the latter theorem. The statement of Theorem 1.6 involves the following technical definition.

Definition 1.5. Let \mathcal{O} be the ring of integers in a number field. An element τ of \mathcal{O} will be called *nifty* if either

- (i) τ and $\tau^2 - 2$ are both non-units in \mathcal{O} , or
- (ii) $\tau - 1$ and $\tau + 1$ are both non-units in \mathcal{O} .

The element τ will be called *swell* if (i) holds.

Theorem 1.6 will be stated in terms of subgroups of $\mathrm{SL}_2(\mathbb{C})$ rather than $\mathrm{Isom}_+(\mathbb{H}^3)$; the transition between the statement and the applications to subgroups of $\mathrm{Isom}_+(\mathbb{H}^3)$ will involve observations to be made in Subsection 2.4 below.

Theorem 1.6. *Let $\Gamma \leq \mathrm{SL}_2(\mathbb{C})$ be a non-elementary torsion-free discrete group having integral traces, and suppose that $H_1(\Gamma; \mathbb{Z}_p) = 0$ for $p = 2, 3$ and 7 . Let \mathcal{O} denote the ring of integers in the trace field of Γ . Let x and y be non-commuting elements of Γ . Suppose that each of the elements*

x and y is a power of an element of Γ whose trace is a nifty element of \mathcal{O} . Then for any point $P \in \mathbb{H}^3$ we have

$$\max(d(P, x \cdot P), d(P, y \cdot P)) > 0.3.$$

Furthermore, if each of the elements x and y is a power of an element of Γ whose trace is a swell element of \mathcal{O} , then for any $P \in \mathbb{H}^3$ we have

$$\max(d(P, \Pi(x) \cdot P), d(P, \Pi(y) \cdot P)) > 0.3925.$$

In Sections 9 and 10, Theorems 1.1 and 1.2 will be deduced from Theorem 1.6. Using observations made in 2.4 below, one can interpret Theorems 1.1 and 1.2 as saying that if K is either an imaginary quadratic field, or a cubic extension of \mathbb{Q} which is not a subfield of \mathbf{R} , and if Γ is a discrete torsion-free subgroup of $\mathrm{SL}_2(\mathbb{C})$ whose trace field is K , then for any non-commuting elements x and y of Γ and for any $P \in \mathbb{H}^3$ we have

$$(1.6.1) \quad \max(d(P, x \cdot P), d(P, y \cdot P)) > \mu,$$

where we set $\mu = 0.395$ in the quadratic case and $\mu = 0.3$ in the cubic case. If K is quadratic and the traces of x and y are both swell, or if K is cubic and the traces of x and y are both nifty, this follows from Theorem 1.6. All the work in Sections 9 and 10 consists of classifying the non-swell elements in an imaginary quadratic field and the non-nifty elements in a cubic extension of \mathbb{Q} which is not a subfield of \mathbf{R} , and showing that when the trace of x or y is among these elements, the inequality 1.6.1 follows from either a direct geometric estimate or a variant of the proof of Theorem 1.6.

In Sections 11, Theorems 1.3 and 1.4 will be deduced from the methods used to prove Theorem 1.6. The key ingredient in doing this is Theorem 11.2, which asserts that the ring of integers of an arbitrary number contains at most finitely many non-nifty elements. This will be in turn be deduced from a fundamental finiteness theorem for solutions to the S -unit equation in a number field, due to Siegel and Mahler.

The proof of Theorem 1.6 occupies Sections 3–8. Here, in order to indicate the basic idea of the proof, 1.6, I will give a sketch under a large number of simplifying assumptions. I will confine the discussion to the case in which trace x is swell, and in place of the hypothesis that $H_1(\Gamma; \mathbb{Z}_p) = 0$ for $p = 2, 3$ and 7 , I will make the stronger assumption that $H_1(\Gamma; \mathbb{Z}) = 0$. It is easy to reduce the proof to the case where $\Gamma \leq \mathrm{SL}_2(\mathcal{O})$, where \mathcal{O} is the ring of integers in a number field containing the trace field. The assumption that τ is swell means that τ and $\tau^2 - 2$ are non-units in \mathcal{O} , so that there are prime ideals \mathfrak{p}_1 and \mathfrak{p}_2 in \mathcal{O} such that $\tau \in \mathfrak{p}_1$ and $\tau^2 - 2 \in \mathfrak{p}_2$. For $i = 1, 2$ let k_i denote the finite field $\mathcal{O}/\mathfrak{p}_i$. As an additional simplifying assumption, I will limit the discussion to the case where k_1 and k_2 are both of odd characteristic.

The quotient homomorphism $\mathcal{O} \rightarrow k_i$ defines a homomorphism $\mathrm{SL}_2(\mathcal{O}) \rightarrow \mathrm{PSL}_2(k_i)$, whose restriction to Γ I will denote here by h_i . Set $G_i = h_i(d\Gamma)$ and $\xi_i = h_i(x)$, and let t_i denote the trace of a representative of ξ_i in $\mathrm{SL}_2(k_i)$. Our choice of the p_i guarantees that $t_1 = 0$ and that $t_2^2 = 2$. Using the assumption of odd characteristic one can deduce that the orders of ξ_1 and ξ_2 in G_1 and G_2 are 2 and 4 respectively.

In particular the G_i are non-trivial. The assumption that $H_1(\Gamma; \mathbb{Z}) = 0$ then implies that the G_i are non-solvable. The non-solvable subgroups of $\mathrm{SL}_2(k)$, where k is a finite field, were classified by Dickson (see [17, II.8.27], and in particular they are simple. The simplicity of the G_i , together with the information that ξ_1 and ξ_2 have different orders, allows one to deduce that the diagonal

homomorphism $h : \gamma \mapsto (h_1(\gamma), h_2(\gamma))$ is a surjection from Γ to $G \doteq G_1 \times G_2$. Note that $\xi \doteq (\xi_1, \xi_2) = h(x) \in G$ has order 4.

Dickson's classification given in [13] guarantees that the 2-Sylow subgroup T_i of each G_i is non-cyclic. This implies in turn that if T is a 2-Sylow subgroup of G then $H_1(T; \mathbb{Z}_2)$ has \mathbb{Z}_2 -rank at least 4. Since ξ has order 4, we may choose T to contain ξ . Furthermore, if we set $\eta = h(y)$, we have $\eta\xi^4\eta^{-1} = 1 \in T$. Hence the finite-index subgroup $\tilde{T} \doteq h^{-1}(T)$ of Γ contains x and yx^4y^{-1} .

The surjectivity of h implies that $\dim_{\mathbb{Z}_2} \tilde{T} \geq \dim_{\mathbb{Z}_2} T \geq 4$. As \tilde{T} is the fundamental group of an orientable hyperbolic 3-manifold, one can then deduce, using [25, Proposition 1.1], that the two-generator subgroup $\langle x, yx^4y^{-1} \rangle$ has infinite index in \tilde{T} ; and then, using [4, Theorem 7.1] and the assumption that x and y do not commute, that $\langle x, yx^4y^{-1} \rangle$ is a free group of rank 2.

The freeness of $\langle x, yx^4y^{-1} \rangle$ can be used to prove the inequality $\max(d(P, x \cdot P), d(P, y \cdot P)) > 0.3925$, by means of [3, Theorem 4.1], which is in turn a consequence of the main theorem of [4] together with the tameness theorem of Agol [2] and Calegari-Gabai [9]. As a final simplification of the present sketch, I will limit myself to establishing the weaker lower bound of 0.375 for $\max(d(P, x \cdot P), d(P, y \cdot P))$.

Theorem 2.2 below, which is a special case of [3, Theorem 4.1], asserts that if X and Y are elements of $\text{Isom}_+(\mathbb{C})$ that generate a rank-2 free discrete group, we have

$$1/(1 + \exp(d(P, X \cdot P))) + 1/(1 + \exp(d(P, Y \cdot P))) \leq 1/2$$

for any $P \in \mathbb{H}^3$. If we assume that $\max(d(P, x \cdot P), d(P, y \cdot P)) \leq 0.375$, and take $X = \Pi(x)$ and $Y = \Pi(yx^4y^{-1})$, the triangle inequality gives $d(P, Y \cdot P) \leq 6 \times 0.375$ and hence

$$\begin{aligned} \frac{1}{1 + \exp(d(P, x \cdot P))} + \frac{1}{1 + \exp(d(P, yx^4y^{-1} \cdot P))} &\leq \frac{1}{1 + \exp(0.375)} + \frac{1}{1 + \exp(6 \times 0.375)} \\ &= 0.502\dots, \end{aligned}$$

a contradiction. This completes the sketch.

Sections 3 and 4, following the preliminary Section 2, are largely devoted to general algebraic background for the technical arguments in Sections 5 and 6, which are refinements of the algebraic steps described in the sketch above. The main results of these sections are Lemmas 5.8 and 5.9, which concern finite-index subgroups of general subgroups of $\text{SL}_2(\mathcal{O})$ where \mathcal{O} is the ring of integers in a number field, and Propositions 6.3 and 6.4, which adapt the results of Section 5 to the 3-manifold situation. Most of Sections 7 and 8 are devoted to the spherical geometry and hyperbolic trigonometry needed to refine the application of the triangle inequality in the sketch above.

I am grateful to Marc Culler, Jason DeBlois, Ben McReynolds, Alan Reid and Steven Smith for helping to make this a better paper. Smith helped me with the finite group theory needed in Sections 3 and 5. It was Reid who told me about Nagell's paper [20], which halved the amount of calculation needed for the results in Section 10, and about the result of Siegel and Mahler which made the results of Section 11 possible.

2. CONVENTIONS AND PRELIMINARY OBSERVATIONS

2.1. I will say that two elements of a group Γ are *independent* if they generate a rank-2 free subgroup of Γ .

As in the introduction, hyperbolic distance in \mathbb{H}^3 will be denoted by d . On the other hand, I will denote by d_s the spherical distance function on S^2 , in which the distance between antipodal points is equal to π .

The following result, which was mentioned in the Introduction, is crucial for all the main results of this paper.

Theorem 2.2. *Let P be a point of \mathbb{H}^3 , and let X and Y be independent elements of $\text{Isom}_+(\mathbb{C})$ that generate a discrete group. Then we have*

$$(2.2.1) \quad 1/(1 + \exp(d(P, X \cdot P))) + 1/(1 + \exp(d(P, Y \cdot P))) \leq 1/2.$$

Proof. This is the case $k = 2$ of [3, Theorem 4.1]. □

2.3. As was already mentioned in the introduction, I will fix throughout the paper a surjective (continuous) homomorphism $\Pi : \text{SL}_2(\mathbb{C}) \rightarrow \text{Isom}_+(\mathbb{H}^3)$ with kernel $\{\pm \text{id}\}$. The natural action of $\text{Isom}_+(\mathbb{H}^3)$ on \mathbb{H}^3 pulls back via Π to an action of $\text{SL}_2(\mathbb{C})$ on \mathbb{H}^3 . Both these actions will be denoted by $(g, z) \mapsto g \cdot z$. In particular we have $\Pi(A) \cdot z = A \cdot z$ for any $A \in \text{SL}_2(\mathbb{C})$ and any $z \in \mathbb{H}^3$.

2.4. If $\Gamma \leq \text{SL}_2(\mathbb{C})$ is cocompact and torsion-free, then Π maps Γ isomorphically onto a cocompact (and torsion-free) subgroup of $\text{Isom}_+(\mathbb{H}^3)$. Conversely, it follows from [10, Proposition 3.1.1] that any cocompact torsion-free subgroup of $\text{Isom}_+(\mathbb{H}^3)$ is the isomorphic image under Π of a cocompact (and torsion-free) subgroup of $\text{SL}_2(\mathbb{C})$.

2.5. I will denote the complex length of a loxodromic isometry $g \in \text{Isom}_+(\mathbb{H}^3)$ by $\text{Clength } g$. Recall that $\text{Clength } g$ is a complex number defined modulo $2\pi i$ and having positive real part. If z is a point of \mathbb{H}^3 , and if we set $D = \text{dist}(z, g \cdot z)$, then the distance from z to the axis of g is equal to $\omega(\text{Clength } g, D)$, where ω is the function defined by

$$(2.5.1) \quad \omega(l + i\theta, D) = \text{arcsinh} \left(\left(\frac{\cosh D - \cosh l}{\cosh l - \cos \theta} \right)^{1/2} \right)$$

whenever $D \geq l > 0$ and $\theta \in \mathbf{R}$. The formula (2.5.1) shows that for any θ and any $l > 0$, the function $\omega(l, \theta, \cdot)$ is a continuous, monotonically increasing function on (l, ∞) .

If $A \in \text{SL}_2(\mathbb{C})$ and $\text{trace } A \in [-2, 2]$, the trace of A and the complex length of $\Pi(A)$ are related by

$$(2.5.2) \quad \text{trace } A = \pm 2 \cosh(\text{Clength}(\Pi(A))/2).$$

(Note that the expression $2 \cosh(\text{Clength}(\Pi(A)))/2$ is defined only up to sign since $\text{Clength}(\Pi(A))$ is defined modulo $2\pi i$.)

The translation length of a loxodromic isometry g is denoted $\text{length } g$. It is equal to the real part of $\text{Clength } g$.

2.6. The cardinality of a finite set S will be denoted by $|S|$.

2.7. If E is a number field, its ring of integers will be denoted by \mathcal{O}_E .

If K and E are number fields, with $K \subset E$, we have $\mathcal{O}_K = \mathcal{O}_E \cap K$. In view of Definition 1.5, it follows that an element of \mathcal{O}_K is nifty (or swell) in \mathcal{O}_K if and only if it is nifty (or, respectively, swell) in \mathcal{O}_E .

2.8. Expressions in decimal notation are to be taken literally. For example, if x is a real number, to write $x = 0.39$ means that x is the rational number $39/100$. To write $x = 0.39\dots$ means that x lies in the interval $[39/100, 40/100]$. With this understanding, expressions in decimal notation may legitimately be used in proofs. For example, from $x = 0.395\dots$ we may deduce that $x^2 = 0.156\dots$; but from $x = 0.375\dots$ we may deduce only that $x^2 = 0.14\dots$

3. SOME PROPERTIES OF $\mathrm{SL}_2(\mathbb{F}_q)$

Lemma 3.1. *Let k be a finite field, let p denote its characteristic, let g be an element of $\mathrm{SL}_2(k)$, set $t = \mathrm{trace} g \in k$, and let m denote the order of g .*

- (1) *If $t = 0$ then m divides 4. More specifically, in this case we have $m = 4$ if $p \neq 2$, and $m \leq 2$ if $p = 2$.*
- (2) *If $t^2 = 2$ then m divides 8, and is equal to 8 if $p \neq 2$.*
- (3) *If $t = -1$ then m divides 3, and is equal to 3 if $p \neq 3$.*
- (4) *If $t = 1$ then m divides 6. More specifically, in this case we have $m = 6$ if p is not equal to 2 or 3; we have $m = 3$ if $p = 2$; and we have $m = 2$ or $m = 6$ if $p = 3$.*

Proof. The characteristic polynomial of g is $X^2 - tX + 1$. By the Cayley-Hamilton theorem we have

$$(3.1.1) \quad g^2 - tg + \mathrm{id} = 0.$$

If $t = 0$ then (3.1.1) gives $g^2 = -\mathrm{id}$. Hence $g^4 = \mathrm{id}$, and $g^2 = \mathrm{id}$ if and only if $p = 2$. This proves (1).

If $t^2 = 2$ then $g^4 + \mathrm{id} = (g^2 - tg + \mathrm{id})(g^2 + tg + \mathrm{id})$, which in view of (3.1.1) implies that $g^4 = -\mathrm{id}$. Hence $g^8 = \mathrm{id}$, but $g^4 \neq \mathrm{id}$ unless $p = 2$. This proves (2).

If $t = -1$ then $g^3 - \mathrm{id} = (g - \mathrm{id})(g^2 - tg + \mathrm{id})$, which in view of (3.1.1) implies that $g^3 = \mathrm{id}$. Since the identity has trace 2, we cannot have $g = \mathrm{id}$ unless $p = 3$. This proves (3).

Now suppose that $t = 1$. We have $g \neq \mathrm{id}$ in this case, since 2 is not congruent to 1 modulo any prime. On the other hand we have $g^3 + \mathrm{id} = (g + \mathrm{id})(g^2 - tg + \mathrm{id})$, which in view of (3.1.1) implies that $g^3 = -\mathrm{id}$. Hence $g^6 = \mathrm{id}$, and $g^3 = \mathrm{id}$ if and only if $p = 2$. Furthermore, if $g^2 = \mathrm{id}$, it follows from (3.1.1) and the assumption $t = 1$ that $g = 2\mathrm{id}$; taking traces of both sides we obtain $1 = 4 \in k$ and hence $p = 3$. This proves (4). \square

Definitions 3.2. Let k be a finite field, and let G be a subgroup of $\mathrm{SL}_2(k)$. We shall say that G is *potentially triangularizable* if there is a finite extension l of k such that G is conjugate in $\mathrm{GL}_2(l)$ to a group of upper triangular matrices in $\mathrm{SL}_2(l)$. A subgroup of $\mathrm{PSL}_2(k)$ will be said to be *potentially triangularizable* if its pre-image under the natural homomorphism $\mathrm{SL}_2(k) \rightarrow \mathrm{PSL}_2(k)$ is potentially triangularizable.

Proposition 3.3. *Let k be a finite field and let \bar{G} be a subgroup of $\mathrm{PSL}_2(k)$. Suppose that $H_1(\bar{G}; \mathbb{Z}_2)$ and $H_1(\bar{G}; \mathbb{Z}_3)$ are trivial, and that \bar{G} is not potentially triangularizable. Then at least one of the following alternatives holds:*

- (i) \bar{G} is conjugate in $\mathrm{PGL}_2(k)$ to $\mathrm{PSL}_2(k_0)$, for some subfield k_0 of k ; or
- (ii) \bar{G} is isomorphic to the alternating group A_5 .

Proof. It follows from the proof of [17, II.8.27] that for any finite subgroup \bar{G} of $\mathrm{PSL}_2(k)$, either (i) or (ii) holds, or G is potentially triangularizable, or G is isomorphic to \mathcal{A}_4 or \mathcal{S}_5 or a dihedral group D_{2n} . But $H_1(\mathcal{A}_4; \mathbb{Z}_3)$, $H_1(\mathcal{S}_5; \mathbb{Z}_2)$ and $H_1(D_n; \mathbb{Z}_2)$ are non-trivial. \square

Corollary 3.4. *Let k be a finite field and let G be a solvable subgroup of $\mathrm{SL}_2(k)$. Suppose that $H_1(G; \mathbb{Z}_2)$ and $H_1(G; \mathbb{Z}_3)$ are trivial. Then G is potentially triangularizable.*

Proof. Suppose that G is not potentially triangularizable. Let \bar{G} denote the image of G under the quotient map $\mathrm{SL}_2(k) \rightarrow \mathrm{PSL}_2(k)$. Then \bar{G} is not potentially triangularizable. Hence one of the alternatives (i) or (ii) of Proposition 3.3 holds. If (i) holds and $|k_0| > 3$ or if (ii) holds, then G_0 is a non-abelian simple group and the solvability of G is contradicted. If (i) holds and $|k_0| \leq 3$, then either $H_1(\bar{G}; \mathbb{Z}_2)$ or $H_1(\bar{G}; \mathbb{Z}_3)$ is non-trivial, and we have a contradiction to the hypothesis that $H_1(G; \mathbb{Z}_2)$ and $H_1(G; \mathbb{Z}_3)$ are trivial. \square

Notation 3.5. If G is a finite group and ℓ is a prime, I will denote by $\sigma_\ell(G)$ the dimension of the \mathbb{Z}_ℓ -vector space $H_1(T; \mathbb{Z}_\ell)$, where T is the ℓ -Sylow subgroup of Γ/N .

Lemma 3.6. *Let k be a finite field of odd characteristic, and let G be a subgroup of $\mathrm{SL}_2(k)$. Suppose that $H_1(G; \mathbb{Z}_2)$ and $H_1(G; \mathbb{Z}_3)$ are both trivial, and that G is not potentially triangularizable. Then G has the following properties:*

- (1) *The center Z of G has order 2.*
- (2) *G/Z is a non-abelian simple group.*
- (3) *The only normal subgroups of G are G , Z and the trivial subgroup.*
- (4) *Every exact sequence of the form*

$$\{\mathrm{id}\} \longrightarrow C \longrightarrow \tilde{G} \longrightarrow G \longrightarrow \{\mathrm{id}\},$$

where C is a cyclic group of order 2 and \tilde{G} is an arbitrary finite group, is split.

- (5) *For every prime ℓ we have $\sigma_\ell(\tilde{G}) = \sigma_\ell(G/Z)$.*
- (6) *If x is any element of G and if m denotes its order, then the order of xZ in G/Z is equal to m if m is odd, and to $m/2$ if m is even.*
- (7) *The order of G/Z is divisible by 6.*

Proof. Let $P : \mathrm{SL}_2(k) \rightarrow \mathrm{PSL}_2(k)$ denote the quotient map, and set $\bar{G} = P(G)$. By Lemma 3.3 we may assume that either $\bar{G} = \mathrm{PSL}_2(k_0)$ for some subfield k_0 of k , or $\bar{G} \cong \mathcal{A}_5$. In the case $\bar{G} = \mathrm{PSL}_2(k_0)$, let q denote the order of k_0 ; we have $|\bar{G}| = (q^3 - q)/2$. Hence $|\bar{G}|$ is divisible by 6 in this case. In the case $\bar{G} \cong \mathcal{A}_5$, we have $|\bar{G}| = 60$. Hence:

3.6.1. *$|\bar{G}|$ is divisible by 6.*

To prove the remaining assertions, we first observe that $-\mathrm{id}$ is the only element of order 2 in $\mathrm{SL}_2(k)$, and hence that:

3.6.2. *Every even-order subgroup of $\mathrm{SL}_2(k)$ contains $-\mathrm{id}$.*

Since \bar{G} has even order by Assertion (7), it follows from 3.6.2 that the order-2 subgroup $\langle -\mathrm{id} \rangle$ of $\mathrm{SL}_2(k)$ is contained in G . Note that $\langle -\mathrm{id} \rangle$ is central in $\mathrm{SL}_2(k)$ and is therefore contained in the center Z of G . Furthermore, $\langle -\mathrm{id} \rangle$ is the kernel of $P|_G$. Hence $G/\langle -\mathrm{id} \rangle \cong \bar{G}$.

Since $H_1(G; \mathbb{Z}_2)$ and $H_1(G; \mathbb{Z}_3)$ are both trivial, $H_1(\bar{G}; \mathbb{Z}_2)$ and $H_1(\bar{G}; \mathbb{Z}_3)$ are also trivial. In the case $\bar{G} = \mathrm{PSL}_2(k_0)$, it follows that $q \geq 4$, and hence that \bar{G} is simple and non-abelian. Since the

only other case is $\bar{G} \cong \mathcal{A}_5$, it follows that \bar{G} is always a simple non-abelian group. Since $\langle -\text{id} \rangle \leq Z$, and since $G/\langle -\text{id} \rangle \cong \bar{G}$ in particular has a trivial center, it follows that $\langle -\text{id} \rangle = Z$. This establishes Assertions (1) and (2).

Since we have shown that $G/Z \cong \bar{G}$, Assertion (7) now follows from 3.6.1.

Now let x be any element of G , and let m denote its order. If m is even, we have $Z = \langle \text{id} \rangle \leq \langle x \rangle$ by 3.6.2. If m is odd then $\langle x \rangle \cap Z = \{\text{id}\}$ since Z has order 2. This proves Assertion (6).

If N is any normal subgroup of G , then NZ is a normal subgroup of \bar{G} , which is simple. Hence either NZ is trivial, in which case N is either Z or the trivial subgroup, or $NZ = \bar{G}$. In the latter case, it follows from Assertion (7) that N has even order, and hence by 3.6.2 we have $Z = \langle -\text{id} \rangle \leq N$. Combining this with $NZ = \bar{G}$ we deduce that $N = G$ in this case. This proves Assertion (3).

For an odd prime ℓ , Assertion (5) follows from Assertion (1). To prove Assertion (5) for $\ell = 2$, we note that G/Z has an element of even order by Assertion (7), and hence by Assertion (6) that G has an element of order 4. If T denotes a 2-Sylow subgroup of G , it follows that T has an element of order 4. Since the generator $-\text{id}$ of Z is the unique element of order 2 in G , we deduce that $-\text{id}$ is a square in T . This implies that the natural homomorphism $H_1(T) \rightarrow H_1(T/Z)$ is an isomorphism. As T/Z is the 2-Sylow subgroup of G/Z , this shows that $\sigma_2(\bar{G}) = \sigma_2(G/Z)$, and Assertion (5) is established.

It remains to prove Assertion (4). Suppose that

$$(3.6.3) \quad \{\text{id}\} \longrightarrow C \longrightarrow \tilde{G} \xrightarrow{p} G \longrightarrow \{\text{id}\}$$

is an exact sequence, where \tilde{G} is a finite group and C is cyclic of order 2. Then $\tilde{Z} \doteq p^{-1}(Z)$ is normal in \tilde{G} ; furthermore, \tilde{Z} has order 4 and is therefore abelian. Hence the action of \tilde{G} on \tilde{Z} by conjugation defines an action of \tilde{G}/\tilde{Z} on \tilde{Z} , which is described by a homomorphism $\alpha : \tilde{G}/\tilde{Z} \rightarrow \text{Aut}(\tilde{Z})$. But \tilde{G}/\tilde{Z} is isomorphic to G/Z and is therefore simple and non-abelian by Assertion (2). Since $|\tilde{G}| = 4$, the group $\text{Aut}(\tilde{Z})$ is isomorphic to a subgroup of \mathcal{S}_3 and is therefore solvable. Hence α is the trivial homomorphism. This means that \tilde{Z} is central in \tilde{G} , so that \tilde{G} is a central extension of $\tilde{G}/\tilde{Z} \cong G/Z \cong \bar{G}$.

In the notation of [15], \bar{G} is isomorphic either to $\mathcal{A}_5 \cong A_1(4)$ or to $\text{PSL}_2(k_0) = A_1(|k_0|)$, where k_0 is a field of order at least 4. It therefore follows from [15, Theorems 6.1.2 and 6.1.4] that the Schur multiplier $H^2(\bar{G}; \mathbb{C}^*)$ is isomorphic to \mathbb{Z}_2 . Hence every perfect central extension of \bar{G} has order at most $2|\bar{G}|$ ([5, (33.8)]). Since $|\tilde{G}| = 4|\bar{G}|$, it follows that \tilde{G} is not perfect. Thus there exists a surjective homomorphism $\beta : \tilde{G} \rightarrow A$ for some non-trivial abelian group A . Set $\tilde{N} = \ker \beta$. Then $N \doteq \alpha(\tilde{N})$ is normal in G , and by Assertion (3) we have either $N = G$ or $|N| \leq 2$.

If $|N| \leq 2$ then $|\tilde{N}| \leq 4$, and so \tilde{N} is abelian. Since $\tilde{G}/\tilde{N} \cong A$ is also abelian, \tilde{G} is solvable, which is impossible since it admits a surjective to the simple non-abelian group \bar{G} . Hence $N = G$, i.e. p maps \tilde{N} onto G . But $|\tilde{G}| = 2|G|$, and so if $p|_{\tilde{N}} : \tilde{N} \rightarrow G$ had a non-trivial kernel, we would have $\tilde{N} = \tilde{G}$; this is impossible because \tilde{G}/\tilde{N} is isomorphic to the non-trivial group A . Hence p maps \tilde{N} isomorphically onto G . Thus the sequence (3.6.3) splits, and Assertion (4) is proved. \square

Proposition 3.7. *Let k be a finite field, and let $\bar{G} \leq \text{PSL}_2(k)$ be a subgroup. Suppose that \bar{G} is not potentially triangularizable, and that $H_1(\bar{G}; \mathbb{Z}_2)$ and $H_1(\bar{G}; \mathbb{Z}_3)$ are trivial. Then $\sigma_2(\bar{G}) \geq 2$.*

Proof. Let T denote the 2-Sylow subgroup of \bar{G} . We are required to show that $\dim_{\mathbb{Z}_2} H_1(T; \mathbb{Z}_2) \geq 2$. Since T is 2-nilpotent, it suffices to show that T is non-cyclic.

Since $H_1(\bar{G}; \mathbb{Z}_2) = 0$ and \bar{G} is not potentially triangularizable, it follows from Lemma 3.3 that \bar{G} is isomorphic either to $\mathrm{PSL}_2(k_0)$ for some finite field k_0 , or to the alternating group \mathcal{A}_5 . In the latter case, T is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Now suppose that $\bar{G} \cong \mathrm{PSL}_2(k_0)$ for some k_0 . If k_0 has characteristic 2 then T is isomorphic to the additive group of k_0 , which is non-cyclic unless $|k_0| = 2$. However, in the latter case \bar{G} is isomorphic to $\mathrm{PSL}_2(\mathbb{F}_2)$, and we have $H_1(\bar{G}; \mathbb{Z}_2) \neq 0$, a contradiction to the hypothesis.

There remains the case that k_0 has characteristic $p > 2$. In this case we set $q = |k_0|$, and we consider the sets

$$\mathcal{S} = \{x^2 : x \in k_0\} \subset k_0$$

and

$$\mathcal{T} = \{-1 - x^2 : x \in k_0\} \subset k_0.$$

We have $|\mathcal{S}| = |\mathcal{T}| = (q+1)/2$. Hence $|\mathcal{S}| + |\mathcal{T}| > q$, so that $\mathcal{S} \cap \mathcal{T} \neq \emptyset$. We fix an element $c \in \mathcal{S} \cap \mathcal{T}$, and we fix $a, b \in k_0$ such that $a^2 = c$ and $-1 - b^2 = c$. Then $a^2 + b^2 = -1$, and hence the matrix

$$M \doteq \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

belongs to $\mathrm{SL}_2(k_0)$. We set

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}_2(k_0).$$

The elements A and M of $\mathrm{SL}_2(k_0)$ have trace 0 and hence have order 4 by Lemma 3.1. By direct calculation we find that $AM = -MA$. $a, b \in k_0$. If $\pi : \mathrm{SL}_2(k_0) \rightarrow \mathrm{PSL}_2(k_0)$ denotes the natural homomorphism, it follows that $\pi(A)$ and $\pi(M)$ commute. On the other hand, A and M do not commute since $p \neq 2$. If we had $\pi(M) = \pi(A)^\epsilon$ for some $\epsilon \in \{1, -1\}$, we would have $M = \pm A^\epsilon$, and A and M would commute, a contradiction. Hence the subgroup of $\mathrm{PSL}_2(k_0)$ generated by $\pi(A)$ and $\pi(B)$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, and the 2-Sylow subgroup of $\mathrm{PSL}_2(k_0)$ is therefore non-cyclic. \square

4. CONGRUENCE KERNELS

Definition 4.1. A normal subgroup N of a group Γ will be termed *cosolvable* if Γ/N is solvable.

Notation 4.2. Let E be a number field and let \mathfrak{J} be an ideal in \mathcal{O}_E . I will denote by $R_{\mathfrak{J}}$ the ring $\mathcal{O}_E/\mathfrak{J}$. If \mathfrak{P} is a prime ideal then $R_{\mathfrak{P}}$ is a finite field which I will denote by $k_{\mathfrak{P}}$.

If \mathfrak{J} is any ideal in \mathcal{O}_E , I will denote by $\eta_{\mathfrak{J}} : \mathcal{O}_E \rightarrow R_{\mathfrak{J}}$ the quotient homomorphism. I will denote by $h_{\mathfrak{J}}$ the natural homomorphism $\mathrm{GL}_2(\mathcal{O}_E) \rightarrow \mathrm{GL}_2(R_{\mathfrak{J}})$, defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \eta_{\mathfrak{J}}(a) & \eta_{\mathfrak{J}}(b) \\ \eta_{\mathfrak{J}}(c) & \eta_{\mathfrak{J}}(d) \end{pmatrix}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_E)$.

Definition 4.3. Let Γ be a group and let p be a prime number. A subgroup N of Γ will be called a *characteristic- p congruence kernel* if there exist a number field E , an injective homomorphism $\rho : \Gamma \rightarrow \mathrm{SL}_2(\mathcal{O}_E) \leq \mathrm{GL}_2(\mathcal{O}_E)$, and a prime ideal \mathfrak{P} in \mathcal{O}_E , such that $\mathrm{char} k_{\mathfrak{P}} = p$ and $N = \ker(h_{\mathfrak{P}} \circ \rho)$.

A subgroup N of a group Γ will be called a *congruence kernel* if it is a characteristic- p congruence kernel of Γ for some prime p . I will call N an *odd-characteristic congruence kernel* if it is a characteristic- p congruence kernel for some odd prime p .

Note that a congruence kernel in Γ is a normal subgroup of finite index in Γ .

4.4. Let G be a group, and let p be a prime number. Recall that the *mod p lower central series* $(\lambda_n(G, p))_{n \geq 1}$ is the sequence of subgroups defined recursively by setting $\lambda_1(G, p) = G$, and defining $\lambda_{n+1}(G, p)$ to be generated by all p -th powers in $\lambda_n(G, p)$ and all commutators of the form $[x, y]$ with $x \in G$ and $y \in \lambda_n(G, p)$. We call $\lambda_2(G, p)$ the *mod p commutator subgroup* of G . Recall that G is p -nilpotent if and only if $\lambda_n(G, p) = \{\text{id}\}$ for some n , and that G is said to be *residually p -nilpotent* if $\bigcap_{n \geq 1} \lambda_n(G, p) = \{\text{id}\}$.

Proposition 4.5. *Let p be a prime, and suppose that N is a characteristic- p congruence kernel in some group Γ . Then N is residually p -nilpotent.*

Proof. It suffices to prove that if E is a number field and \mathfrak{P} is a prime ideal in \mathcal{O}_E , then the kernel of $h_{\mathfrak{P}} : \text{GL}_2(\mathcal{O}_E) \rightarrow \text{GL}_2(k_{\mathfrak{P}})$ is residually p -nilpotent, where $p = \text{char } k_{\mathfrak{P}}$.

For each $n \geq 1$, let G_n denote the kernel of $h_{\mathfrak{P}^n} : \text{GL}_2(\mathcal{O}_E) \rightarrow \text{GL}_2(R_{\mathfrak{P}^n})$. Then G_1/G_n is isomorphic to the kernel K_n of the natural homomorphism $\text{GL}_2(R_{\mathfrak{P}^n}) \rightarrow \text{GL}_2(k_{\mathfrak{P}})$.

If M_n denotes the ring of 2×2 matrices over $R_{\mathfrak{P}^n}$ then $K_n \subset \text{id} + \mathfrak{P}M_n$. On the other hand, for any $g \in \text{id} + \mathfrak{P}M_n$, we have $\det g \equiv 1 \pmod{\mathfrak{P}}$. Since $R_{\mathfrak{P}^n}$ is a local ring with maximal ideal $\mathfrak{P}R_{\mathfrak{P}^n}$ it follows that $\det g$ is invertible in $R_{\mathfrak{P}^n}$ and hence that $g \in \text{GL}_2(R_{\mathfrak{P}^n})$. This shows that $\text{id} + \mathfrak{P}M_n \subset \text{GL}_2(R_{\mathfrak{P}^n})$, from which it follows that $K_n = \text{id} + \mathfrak{P}M_n$. Hence $|K_n| = |\mathfrak{P}M_n|$.

The local ring $R_{\mathfrak{P}^n}$ has residue field $k_{\mathfrak{P}}$ of characteristic p ; hence $|R_{\mathfrak{P}^n}|$ is a power of p . Since $\mathfrak{P}M_n$ is a submodule of M_n , which is a rank-four free module over $R_{\mathfrak{P}^n}$, the order of $\mathfrak{P}M_n$ is also a power of p . This shows that K_n is a finite p -group, and is therefore p -nilpotent. Hence G_1/G_p is p -nilpotent, and so for some j_n we have

$$(4.5.1) \quad G_n \geq \lambda_{j_n}(G_1, p).$$

Finally, since $\bigcap_{n=1}^{\infty} \mathfrak{P}^n = \{0\}$, we have $\bigcap_{n=1}^{\infty} G_n = \{\text{id}\}$. Hence it follows from (4.5.1) that $\bigcap_{n=1}^{\infty} \lambda_{j_n}(G_1, p) = \{\text{id}\}$, which implies that G_1 is residually p -nilpotent, as required. (This argument is similar to the discussion on p. 87 of [14]). \square

Lemma 4.6. *Let Γ be a finitely generated subgroup of $\text{SL}_2(\mathcal{O}_E)$ which has no abelian subgroup of finite index. Let p be a prime, and let N be a characteristic- p congruence kernel in Γ . Then $\dim_{\mathbb{Z}_p} H_1(N; \mathbb{Z}_p) \geq 2$.*

Proof. According to Proposition 4.5, N is residually p -nilpotent. Let $N = N_1 \geq N_2 \geq \dots$ denote its mod p lower central series. Assume that the conclusion of the lemma is false, so that $\dim_{\mathbb{Z}_p} H_1(N; \mathbb{Z}_p) \leq 1$. Then for each $n \geq 1$, the group N/N_n is p -nilpotent, and $\dim_{\mathbb{Z}_p} H_1(N/N_n) \leq 1$. It follows that N/N_n is cyclic, and in particular abelian, for every n . Hence for any elements $x, y \in N$, the commutator $[x, y]$ has trivial image in N/N_n for every n ; that is, $[x, y]$ lies in $\bigcap_{n=1}^{\infty} N_n$, which is trivial by residual p -nilpotence. This shows that N is abelian. But N has finite index in Γ since Γ is finitely generated, and we have a contradiction to the hypothesis. \square

Notation 4.7. Let γ be an element of a group Γ , and let N be a finite-index normal subgroup of Γ . We will denote by $m_N(\gamma)$ the order of xN in Γ/N .

Lemma 4.8. *Let p be a prime, and let N be a characteristic- p congruence kernel in a group Γ . Let an element $\gamma_0 \in \Gamma$ be given, and set $m = m_N(\gamma_0)$. Suppose that*

- (1) $m > 1$, and $m \neq p$; and
(2) $H_1(\Gamma; \mathbb{Z}_s) = 0$ for $s = 2, 3$ and m .

Then N is not cosolvable.

Proof. Assume that N is cosolvable. Since N is a characteristic- p congruence kernel, Γ/N is isomorphic to a subgroup G of $\mathrm{SL}_2(k)$ for some field k of characteristic p . Since $H_1(\Gamma; \mathbb{Z}_2)$ and $H_1(\Gamma; \mathbb{Z}_3)$ are trivial, so are $H_1(G; \mathbb{Z}_2)$ and $H_1(G; \mathbb{Z}_3)$. The cosolvability of N implies that G is solvable. By Corollary 3.4 it follows that G is potentially triangularizable, and is therefore isomorphic to a group of upper triangular matrices in $\mathrm{SL}_2(k')$ for some finite field k' . Hence there is a short exact sequence

$$0 \longrightarrow U \xrightarrow{\alpha} G \xrightarrow{\beta} Q \longrightarrow 0,$$

where Q is isomorphic to a subgroup of the multiplicative group of k' , while U is isomorphic to a subgroup of the additive group of k' , which has exponent p . Since $H_1(\Gamma; \mathbb{Z}_m) = 0$ we have $H_1(G; \mathbb{Z}_m) = 0$, and hence the order of Q is relatively prime to m . But by the definition of $m = m_N(\gamma_0)$, the order of g_0 in G is m , and we must therefore have $\beta(g_0) = 0$. Hence g_0 is in the image of α , so that U has an element of order m . This implies that $m|p$, which contradicts the hypothesis. \square

5. CONSTRUCTING USEFUL QUOTIENTS

The main results of this section are Propositions 5.8 and 5.9, which constitute the main algebraic steps in the proof of Theorem 1.6.

Notation 5.1. Let N be a finite-index normal subgroup of a group Γ , and let ℓ be a prime. I will denote by $\nu_\ell(N)$ the number $\sigma_\ell(\Gamma/N)$. I will denote by $\kappa_\ell(N)$ the dimension of the \mathbb{Z}_ℓ -vector space $H_1(N; \mathbb{Z}_\ell)$.

Lemma 5.2. *Let N_1 and N_2 be finite-index normal subgroups of a group Γ , and let u be an element of Γ . Suppose that*

- N_1 is a non-cosolvable odd-characteristic congruence kernel;
- N_2 is either a cosolvable normal subgroup of odd index or a non-cosolvable odd-characteristic congruence kernel;
- $m_{N_1}(u) \neq m_{N_2}(u)$; and
- $H_1(\Gamma; \mathbb{Z}_2) = H_1(\Gamma; \mathbb{Z}_3) = 0$.

For $i = 1, 2$, set $\theta_i = m_{N_i}(u)$ if $m_{N_i}(u)$ is odd, and $\theta_i = m_{N_i}(u)/2$ if $m_{N_i}(u)$ is even. Then there exists a finite-index normal subgroup N in Γ such that

- (1) $\nu_\ell(N) = \nu_\ell(N_1) + \nu_\ell(N_2)$ for every prime ℓ , and
(2) $m_N(u)$ is the least common multiple of θ_1 and θ_2 .

Proof. For $i = 1, 2$ we set $G_i = \Gamma/N_i$ and $g_i = uN_i \in G_i$. We define a homomorphism $h : \Gamma \rightarrow G_1 \times G_2$ by setting $h(\gamma) = (\gamma N_1, \gamma N_2)$ for every $\gamma \in \Gamma$. We set $W = h(\Gamma) \leq G_1 \times G_2$. For $i = 1, 2$, if π_i denotes the projection from $G_1 \times G_2$, it is clear that $\pi_i|_W : W \rightarrow G_i$ is surjective.

Since N_1 is a characteristic- p_1 congruence subgroup of Γ , the group G_1 is isomorphic to a subgroup G'_1 of $\mathrm{SL}_2(k_1)$, for some finite field k_1 of characteristic p_1 . Since N_1 is not cosolvable, G'_1 is not potentially triangularizable.

For $s = 2$ and 3 , since $H_1(\Gamma; \mathbb{Z}_s) = 0$, we have $H_1(G_1; \mathbb{Z}_s) = 0$ and hence $H_1(G'_1; \mathbb{Z}_s) = 0$. Hence the conclusions of Lemma 3.6 hold with G_1 playing the role of G . In particular, according to conclusion (1) of Lemma 3.6, the center of G_1 , which we will denote by Z_1 , has order 2.

Set $J = W \cap (G_1 \times \{\text{id}\})$. Then J is the kernel of the surjective homomorphism $\pi_2|_W : W \rightarrow G_2$. Hence we have an exact sequence

$$(5.2.1) \quad \{\text{id}\} \longrightarrow J \longrightarrow W \xrightarrow{\pi_2|_W} G_2 \longrightarrow \{\text{id}\}$$

We claim:

5.2.2. *J is a normal subgroup of $G_1 \times \{\text{id}\}$.*

To prove 5.2.2, we consider an arbitrary element x of J and an arbitrary element z of $G_1 \times \{\text{id}\}$. Let us write $x = (\xi, \text{id})$ and $z = (\zeta, \text{id})$, where $x, z \in G_1$. Since $\pi_1|_W : W \rightarrow G_1$ is surjective, there is an element y of W such that $\pi_1(y) = \zeta$. We may write $y = (\zeta, v)$ for some $v \in G_2$. Since $x \in J \subset W$ and $y \in W$, we have $xyx^{-1} \in W$. But

$$\begin{aligned} xyx^{-1} &= (\zeta, v)(\xi, \text{id})(\zeta^{-1}, v^{-1}) \\ &= (\zeta\xi\zeta^{-1}, \text{id}) \\ &= zxz^{-1}, \end{aligned}$$

from which it follows that zxz^{-1} belongs to W and hence to J . This proves 5.2.2.

In view of 5.2.2 and conclusion (3) of Lemma 3.6, J must be equal to $G_1 \times \{\text{id}\}$ or $Z_1 \times \{\text{id}\}$ or $\{\text{id}\} \times \{\text{id}\}$. Hence:

5.2.3. *Either $J = G_1 \times \{\text{id}\}$ or $|J| \leq 2$.*

We claim that, in fact, we have

$$(5.2.4) \quad J = G_1 \times \{\text{id}\}.$$

I will prove (5.2.4) by assuming that $|J| \leq 2$ and deriving a contradiction; (5.2.4) will then follow in view of 5.2.3.

According to the hypothesis, N_2 is either cosolvable or an odd-characteristic congruence kernel. If $|J| \leq 2$, and if N_2 is cosolvable (so that G_2 is solvable), then the exactness of (5.2.1) implies that W is solvable. Since $\pi_1|_W : W \rightarrow G_1$ is surjective, it follows that G_1 is solvable. But according to conclusion (2) of Lemma 3.6, G_1 has a quotient which is a non-abelian simple group. This is the required contradiction in this case.

Now suppose that $|J| \leq 2$, and that N_2 is an odd-characteristic congruence kernel which is not cosolvable; thus G_2 is non-solvable and is isomorphic to a subgroup G'_2 of $\text{SL}_2(k_2)$ for some finite field k_2 . Since G_2 is non-solvable, G'_2 is not potentially triangularizable. For $s = 2, 3$, since $H_1(\Gamma; \mathbb{Z}_s) = 0$, we have $H_1(G_2; \mathbb{Z}_s) = 0$ and therefore $H_1(G'_2; \mathbb{Z}_s) = 0$. Hence the conclusions of Lemma 3.6 hold with G_2 playing the role of G .

If $|J| = 2$, we may apply conclusion (4) of Lemma 3.6, with W and G_2 playing the respective roles of \tilde{G} and G , to deduce that the exact sequence (5.2.1) splits. In particular we have $H_1(W; \mathbb{Z}_2) \neq 0$. This is a contradiction, since $W = h(\Gamma)$ and $H_1(\Gamma; \mathbb{Z}_2) = 0$ by hypothesis.

If $|J| = 1$, the exactness of 5.2.1 implies that $\pi_2|_W$ is an isomorphism from W to G_2 . Let $\omega : G_2 \rightarrow W$ denote the inverse isomorphism, and set $\beta = \pi_1 \circ \omega : G_2 \rightarrow G_1$. Since $\pi_1|_W : W \rightarrow G_1$

is surjective, β is also surjective. Note that by the definition of the homomorphism h we have $\pi_2(h(u)) = uN_2 = g_2$, and hence $\omega(g_2) = h(u)$. Hence

$$\beta(g_2) = \pi_1(\omega(g_2)) = \pi_1(h(u)) = uN_1 = g_1.$$

Since by hypothesis the orders $m_{N_1}(u)$ and $m_{N_2}(u)$ of g_1 and g_2 are not equal, β cannot be an isomorphism.

Let $L \neq \{\text{id}\}$ denote the kernel of β . Applying Conclusion (3) of Lemma 3.6, with G_2 playing the role of G , we find that either $L = G_2$ or L is the center of G_2 . If $L = G_2$ then G_1 is trivial; this is impossible because by hypothesis N_1 is not cosolvable, i.e. G_1 is not solvable. Finally, if L is the center of G_2 then Conclusion (2) of Lemma 3.6, applied with G_2 in the role of G , implies that $G_1 \cong G_2/L$ is a non-abelian simple group. But Conclusion (1) of Lemma 3.6, applied with G_1 in the role of G , implies that G_1 has a non-trivial center. Again we have the required contradiction. This proves 5.2.4.

We now claim:

5.2.5. $W = G_1 \times G_2$, i.e. h is surjective.

To prove 5.2.5, let an element $g = (g_1, g_2)$ of $G_1 \times G_2$ be given. Since $\pi_2|_W : W \rightarrow G_2$ is surjective, there is an element u of W such that $\pi_1(u) = g_2$. We may write $u = (\alpha, g_2)$ for some $\alpha \in G_1$. Then $gu^{-1} = (g_1\alpha^{-1}, \text{id}) \in G_1 \times \{\text{id}\}$, and it follows from 5.2.4 that $gu^{-1} \in J \subset W$. Since $u \in W$ it follows that $g \in W$, and 5.2.5 is established.

We now claim:

5.2.6. For each $i \in \{1, 2\}$ there exist a group \bar{G}_i and a homomorphism $P_i : G_i \rightarrow \bar{G}_i$ such that $P_i(g_i)$ has order θ_i , and such that $\sigma_\ell(\bar{G}_i) = \sigma_\ell(G_i)$ for every prime ℓ .

To prove 5.2.6, we distinguish two cases. By hypothesis, either N_i is a non-cosolvable congruence kernel, or $i = 2$ and N_2 is a cosolvable normal subgroup of odd index. If N_i is a non-cosolvable congruence kernel then for some finite field k_i , the group G_i is isomorphic to a subgroup G'_i of $\text{SL}_2(k_i)$ which is not potentially triangularizable. For $s = 2, 3$, since $H_1(\Gamma; \mathbb{Z}_s) = 0$, we have $H_1(G'_i; \mathbb{Z}_s) = 0$. Hence the conclusions of Lemma 3.6 hold with G_i playing the role of G . If we set $\bar{G}_i = G_i/Z_i$, where Z_i denotes the center of G_i , and define $P_i : G_i \rightarrow \bar{G}_i$ to be the quotient map, then the assertions of 5.2.6 follow from Assertions (5) and (6) of Lemma 3.6.

On the other hand, if $i = 2$ and N_2 is a cosolvable normal subgroup of odd index, we may set $\bar{G}_2 = G_2$, and define $P_2 : G_2 \rightarrow \bar{G}_2$ to be the identity map. In this case, the order $m_{N_2}(u)$ of $g_2 = P_2(g_2)$ is odd and hence equal to θ_2 ; and it is trivial that $\sigma_\ell(\bar{G}_2) = \sigma_\ell(G_2)$. Thus 5.2.6 is proved.

Now fix groups \bar{G}_1 and \bar{G}_2 , and homomorphisms P_1 and P_2 , having the properties stated in 5.2.6. Set $P = P_1 \times P_2 : G_1 \times G_2 \rightarrow \bar{G}_1 \times \bar{G}_2$. Since $h : \Gamma \rightarrow G_1 \times G_2$ is surjective, so is $P \circ h : \Gamma \rightarrow \bar{G}_1 \times \bar{G}_2$. Hence if we set $N = \ker(P \circ h)$, we have $\Gamma/N \cong \bar{G}_1 \times \bar{G}_2$.

If ℓ is a prime and \bar{T}_i denotes the ℓ -Sylow subgroup of \bar{G}_i , then $\bar{T}_1 \times \bar{T}_2$ is the ℓ -Sylow subgroup of $\bar{G}_1 \times \bar{G}_2$. Hence $\sigma_\ell(\bar{G}_1 \times \bar{G}_2) = \sigma_\ell(\bar{G}_1) + \sigma_\ell(\bar{G}_2)$. In view of 5.2.6 it follows that

$$\sigma_\ell(\bar{G}_1 \times \bar{G}_2) = \sigma_\ell(G_1) + \sigma_\ell(G_2),$$

and hence

$$\nu_\ell(N) = \sigma_\ell(\bar{G}_1) + \sigma_\ell(G_2) = \nu_\ell(N_1) + \nu_\ell(N_2).$$

This gives conclusion (1) of the lemma.

Finally, $m_N(u)$ is the order of $P \circ h(u)$ in $\bar{G}_1 \times \bar{G}_2$. Since $P \circ h(u) = (P_1(g_1), P_2(g_2))$, and since $P_i(g_i)$ has order θ_i in G_i by 5.2.6, the order of $P \circ h(u)$ in $\bar{G}_1 \times \bar{G}_2$ is the least common multiple of θ_1 and θ_2 . This gives conclusion (2) of the lemma. \square

Lemma 5.3. *Let N_1 and N_2 be odd-characteristic congruence kernels in a group Γ , and let u be an element of Γ . Suppose that*

- neither N_1 nor N_2 is cosolvable;
- $m_{N_1}(\gamma_0) \neq m_{N_2}(\gamma_0)$; and
- $H_1(\Gamma; \mathbb{Z}_2) = H_1(\Gamma; \mathbb{Z}_3) = 0$.

For $i = 1, 2$, set $\theta_i = m_{N_i}(u)$ if $m_{N_i}(u)$ is odd, and $\theta_i = m_{N_i}(u)/2$ if $m_{N_i}(u)$ is even. Then there exists a finite-index normal subgroup N in Γ such that

- (1) $\nu_2(N) \geq 4$, and
- (2) $m_N(u)$ is the least common multiple of θ_1 and θ_2 .

Proof. The hypotheses of the lemma immediately imply those of Lemma 5.2. Hence there exists a finite-index normal subgroup N of Γ such that conclusions (1) and (2) of Lemma 5.2 hold. Conclusion (2) of Lemma 5.2 is conclusion (2) of the present lemma.

For $i = 1, 2$, the group Γ/N_i is isomorphic to a subgroup G_i of $\mathrm{SL}_2(k_i)$ for some finite field k_i . Since N_i is not cosolvable, G_i is not potentially triangularizable. Since $H_1(\Gamma; \mathbb{Z}_2) = 0$, we have $H_1(G_i; \mathbb{Z}_2) = 0$. Hence it follows from Proposition 3.7 that $\nu_2(N_i) = \sigma_2(G_i) \geq 2$ for $i = 1, 2$. Conclusion (1) of Lemma 5.2 gives

$$\nu_2(N) = \nu_2(N_1) + \nu_2(N_2) \geq 4,$$

so that conclusion (1) of the present lemma holds. \square

Lemma 5.4. *Let Γ be an infinite, finitely generated group, let p be a prime, and let $d \geq 2$ be an integer. Let N be a characteristic- p congruence kernel in Γ . Suppose that N is cosolvable. Set $G = \Gamma/N$. Set*

$$S = \{2, 3, p\} \cup \{p^r - 1 : 1 \leq r < d\},$$

and suppose that $H_1(\Gamma, \mathbb{Z}_s) = 0$ for every $s \in S$. Then either

- (i) the p -Sylow subgroup T of G is an elementary abelian p -group of rank at least d , normal in G , and G/T is a cyclic group whose order is not divisible by 2, 3 or p ; or
- (ii) G is a cyclic group whose order is not divisible by 2, 3 or p , and $\kappa_p(N) \geq d$.

Proof. Since N is a characteristic- p congruence kernel, Γ/N is isomorphic to a subgroup G of $\mathrm{SL}_2(k)$ for some finite field k of characteristic p . Since N is cosolvable, G is solvable. Furthermore, since $H_1(\Gamma, \mathbb{Z}_s) = 0$ for every $s \in S$, we have $H_1(G; \mathbb{Z}_s) = 0$ and $H_1(\bar{G}; \mathbb{Z}_s) = 0$ for every $s \in S$. Hence Corollary 3.4 implies that G is potentially triangularizable. It follows that there exist a finite extension k' of k , a unipotent subgroup $T \triangleleft G$, isomorphic to a subgroup of the additive group of k' , such that $Q \doteq G/T$ is isomorphic to a subgroup of the multiplicative group of k' . In particular, T is an elementary abelian p -group, Q is cyclic, and p does not divide the order of Q . Hence T is the p -Sylow subgroup of G .

Since Q is a homomorphic image of G and hence of Γ , we have $H_1(Q; \mathbb{Z}_s) = 0$ for every $s \in S$. In particular $H_1(Q; \mathbb{Z}_p) = 0$, and $H_1(Q; \mathbb{Z}_{p^r-1}) = 0$ for every strictly positive integer $r < d$. Since Q is a finite cyclic group, it follows that:

5.4.1. $|Q|$ is relatively prime to every integer in the set S .

We define a group \tilde{G} and a subgroup W of \tilde{G} as follows. If the group T is non-trivial, we set $\tilde{G} = G$ and $W = T$. If T is trivial, we define \tilde{G} to be Γ/N' , where N' denotes the mod p commutator subgroup (see 4.4) of N , and we set $W = N/N' \leq \tilde{G}$. In both cases, we claim:

5.4.2. The group \tilde{G} is finite and W is a non-trivial elementary abelian p -group. Furthermore, W is normal in \tilde{G} , and $\tilde{G}/W \cong Q$.

To prove 5.4.2, we first consider the case in which T is non-trivial. In this case $\tilde{G} = G$ is isomorphic to a subgroup of $\mathrm{SL}_2(k')$ and is therefore finite, and we have $W = T$. By the discussion above, T is an elementary abelian p -group and is normal in G . Since we are in the case where $T \neq \{\mathrm{id}\}$ we have $W \neq \{\mathrm{id}\}$. Furthermore, we have $\tilde{G}/W = G/T = Q$ by definition. Thus 5.4.2 is true in this case.

Now consider the case in which T is trivial. Since N has finite index in the finitely generated group Γ , it is itself finitely generated; hence the mod p commutator subgroup N' has finite index in N and therefore in Γ , and so $\tilde{G} = \Gamma/N'$ is finite. Since $W = N/N'$ is finite and is the quotient of N by its terms of the mod- p commutator subgroup, W is an elementary p -group. Furthermore, N is a non-trivial group because Γ is infinite, and N is residually p -nilpotent by Proposition 4.5. It follows that the mod p commutator subgroup N' is a proper subgroup of N , so that W is non-trivial in this case as well. The normality of $W = N/N'$ in $\tilde{G} = \Gamma/N'$ follows from the normality of N in Γ . Furthermore, we have $\tilde{G}/W = (\Gamma/N')/(N/N') \cong \Gamma/N = G$, and $G \cong Q$ since T is trivial. Thus 5.4.2 is proved.

We let r denote the rank of the elementary abelian p -group W . By 5.4.2 we have $r > 0$. We claim that

$$(5.4.3) \quad r \geq d.$$

To prove (5.4.3) we first note that since W is abelian, the action of \tilde{G} by conjugation on $W \triangleleft \tilde{G}$ induces an action of \tilde{G}/W on W . This action is described by a homomorphism $\alpha : \tilde{G}/W \rightarrow \mathrm{Aut}(W)$. Since $\mathrm{Aut}(W) \cong \mathrm{GL}_r(\mathbb{F}_p)$, we have

$$(5.4.4) \quad |\mathrm{Aut}(W)| = (p^r - 1)(p^r - p) \cdots (p^r - p^{r-1}).$$

Assume that $r < d$. Then it follows from (5.4.4) that $|\mathrm{Aut}(W)|$ is a product of integers in the set S . It therefore follows from 5.4.1 that $|\tilde{G}/W| = |Q|$ is relatively prime to $|\mathrm{Aut}(W)|$. Hence the homomorphism α must be trivial. This means that the action of \tilde{G}/W on W is trivial, i.e. that W is a central subgroup of \tilde{G} . Since \tilde{G}/W is isomorphic to Q by 5.4.2, and is therefore cyclic, it follows that \tilde{G} is abelian. But since $r > 0$ by 5.4.2, the prime p divides $|W|$ and hence divides \tilde{G} . As \tilde{G} is a homomorphic image of Γ it now follows that $H_1(\Gamma, \mathbb{Z}_p) \neq 0$. Since $p \in S$, this contradicts the hypothesis. Thus (5.4.3) is proved.

To establish the conclusion of the lemma, we first consider the case in which T is non-trivial. In this case we have $\tilde{G} = G$ and $W = T$. We have observed that T is the p -Sylow subgroup of G . By 5.4.2 and (5.4.3), T is an elementary abelian p -group of rank at least d , normal in G , and $G/T \cong Q$.

Furthermore, we have observed that Q is cyclic, and by 5.4.1 its order is not divisible by 2, 3 or p . Thus Alternative (i) of the conclusion holds in this case.

Now consider the case in which T is trivial. In this case we have $G \cong G/T = Q$. Thus G is cyclic, and by 5.4.1 its order is not divisible by 2, 3 or p . In this case we have $W = N/N' \cong H_1(N; \mathbb{Z}_p)$. Hence the rank r of the elementary abelian p -group W is equal to $\kappa_p(N)$, and in view of (5.4.3) it follows that $\kappa_p(N) \geq d$. Thus Alternative (i) of the conclusion holds in this case. \square

Remark 5.5. If Alternative (i) of Lemma 5.4 holds, then it follows from the definitions that $\nu_p(N) = \sigma_p(G) \geq d$.

Remark 5.6. If Alternative (ii) of Lemma 5.4 holds then $\nu_p(N) \leq 1$. In view of Remark 5.5 it follows that Alternatives (i) and (ii) of Lemma 5.4 are mutually exclusive.

Lemma 5.7. *Let Γ be a finitely generated group which has no abelian subgroup of finite index. Suppose that $H_1(\Gamma, \mathbb{Z}_s) = 0$ for $s = 2, 3, 7$. Then for every characteristic-2 congruence kernel N in Γ , we have either (i) $\nu_2(N) \geq 4$ or (ii) $\kappa_2(N) \geq 4$.*

Proof. Suppose that N is a characteristic-2 congruence kernel in Γ . We first consider the case in which Γ is cosolvable. In this case we apply Lemma 5.4 with $p = 2$ and $d = 4$. In the notation of Lemma 5.4 we have $S = \{1, 2, 3, 7\}$. By the hypothesis of the present lemma we have $H_1(\Gamma; \mathbb{Z}_s) = 0$ for $s = 2, 3, 7$; tautologically we have $H_1(\Gamma; \mathbb{Z}_1) = H_1(\Gamma; 0) = 0$. Hence it follows from Lemma 5.4 and Remark 5.5 that either $\nu_2(N) \geq 4$ or $\kappa_2(N) \geq 4$.

Now suppose that N is not cosolvable. Then for some finite field k of characteristic 2, the group $G \doteq \Gamma/N$ is isomorphic to a subgroup of $\mathrm{SL}_2(k)$ which is not potentially triangularizable. According to Lemma 3.3, G is isomorphic either to $\mathrm{SL}_2(\mathbb{F}_{2^r})$ for some $r \geq 1$, or to the alternating group \mathcal{A}_5 . Since \mathcal{A}_5 is isomorphic to $\mathrm{SL}_2(\mathbb{F}_4)$, the group G is in fact isomorphic to $\mathrm{SL}_2(\mathbb{F}_{2^r})$ for some $r \geq 1$. The 2-Sylow subgroup of G is an elementary abelian 2-group of rank r . In particular we have $\nu_2(N) = \sigma_2(G) = r$. If $r = 1$ then $H_1(G; \mathbb{Z}_2) \neq 0$, which is impossible since $H_1(\Gamma; \mathbb{Z}_2) = 0$ by hypothesis. If $r \geq 4$ then alternative (i) of the present lemma holds. Hence we may assume that r is equal to 2 or 3. It follows that G is simple, and that the order $(2^r)^3 - (2^r)$ of G is equal to 60 or 504.

We set $V = H_1(N; \mathbb{Z}_2)$ and $d = \dim_{\mathbb{Z}_2} V$. It follows from Lemma 4.6 that $d \geq 2$. If $d \geq 4$ then by definition we have $\kappa_2(N) \geq 4$, which is alternative (ii) of the lemma. I will now assume that d is equal to 2 or 3, and obtain a contradiction.

We regard V as an elementary 2-group of rank d . We let \mathcal{L} denote the kernel of the natural homomorphism from N to V . Then \mathcal{L} is a characteristic subgroup of $N \triangleleft \Gamma$ and is therefore normal in Γ . We set $R = \Gamma/\mathcal{L}$. There is a canonical isomorphic identification of $V \cong N/\mathcal{L}$ with a subgroup of R , and we have $R/V = (\Gamma/\mathcal{L})/(N/\mathcal{L}) \cong \Gamma/\mathcal{L} \cong G$.

Since N is normal in Γ , the subgroup V is normal in R . Since V is abelian, the action of R on V by conjugation induces an action of $R/V \cong G$ on V . The action is described by a homomorphism $\alpha : R/V \rightarrow \mathrm{Aut}(V)$. Since $R/V \cong G$ is simple, α is either injective or trivial. We have $\mathrm{Aut}(V) \cong \mathrm{GL}_d(\mathbb{F}_2)$. Since d is equal to 2 or 3, the order of $\mathrm{Aut}(V)$ is either $(2^2 - 1)(2^2 - 2) = 6$ or $(2^3 - 1)(2^3 - 2)(2^3 - 2^2) = 168$. Hence $|G|$ does not divide the order of $\mathrm{Aut}(V)$, and α cannot be injective.

Hence α is the trivial homomorphism. This means that V is central in R , so that R is a central extension of $R/V \cong G$. In the notation of [15] we have $G \cong \mathrm{SL}_2(\mathbb{F}_{2^r}) = A_1(2^r)$ and $r \in \{2, 3\}$. It

therefore follows from [15, Theorems 6.1.2 and 6.1.4] that the Schur multiplier $H^2(\bar{G}; \mathbb{C}^*)$ is isomorphic to \mathbb{Z}_2 . Hence every perfect central extension of \bar{G} has order at most $2|\bar{G}|$ ([5, (33.8)]). Since $|R| = 2^r|\bar{G}| \geq 4|\bar{G}|$, it follows that R is not perfect. Thus there exists a surjective homomorphism $\beta : R \rightarrow A$ for some non-trivial abelian group A .

If $\beta|V$ is the trivial homomorphism then β factors through a homomorphism of $R/V \cong G$ onto A . This is impossible since G is simple and non-abelian. If $\beta|V$ is non-trivial, then since $|V|$ is a power of 2, the order of A is even. This implies that R admits a homomorphism onto \mathbb{Z}_2 , which is impossible since R is a quotient of Γ and $H_1(\Gamma; \mathbb{Z}_2) = 0$. Thus we have the required contradiction. \square

Lemma 5.8. *Let Γ be a subgroup of $\mathrm{SL}_2(\mathcal{O}_E)$ for some number field E . Suppose that $H_1(\Gamma; \mathbb{Z}_s) = 0$ for $s = 2, 3$ and 7 , and that Γ has no abelian subgroup of finite index. Let z be an element of Γ , and suppose that trace z is swell in \mathcal{O}_E . Then Γ has a finite-index normal subgroup N such that either*

- (i) $\nu_2(N) \geq 4$ and $z^4 \in N$, or
- (ii) $\kappa_2(N) \geq 4$ and $z^2 \in N$.

Proof. Set $\tau = \text{trace } z$. The hypothesis that τ is swell means that τ and $\tau^2 - 2$ are non-units in \mathcal{O}_E . Hence there are prime ideals \mathfrak{p}_1 and \mathfrak{p}_2 in \mathcal{O}_E such that $\tau \in \mathfrak{p}_1$ and $\tau^2 - 2 \in \mathfrak{p}_2$. For $i = 1, 2$, we set $k_i = k_{\mathfrak{p}_i}$, $p_i = \text{char } k_i$, $h_i = h_{\mathfrak{p}_i}$ and $N_i = \ker h_i$. By definition N_i is a characteristic- p_i congruence kernel. We set $g_i = h_i(z)$ and $t_i = \text{trace } g_i$. We set $m_i = m_{N_i}(z)$, and note that m_i is the order of g_i in $\mathrm{SL}_2(k_i)$. Note also that

$$t_1 = \eta_1(\tau) = 0$$

and that

$$t_2^2 = \eta_2(\tau^2) = 2.$$

First suppose that for some $j \in \{1, 2\}$ we have $p_j = 2$. If $j = 1$ we have $t_j = 0$, and if $j = 2$ we have $t_j^2 = 2 = 0$. Hence in any case we have $t_j = 0$. By assertion (1) of Lemma 3.1 it follows that $m_j \leq 2$. Hence $z^2 \in N_j$ (and in particular $z^4 \in N_j$). On the other hand, by Lemma 5.7, we have either $\nu_2(N) \geq 4$ or $\kappa_2(N) \geq 4$. Thus one of the alternatives (i) or (ii) of the present lemma holds in this case, with $N = N_j$.

Now suppose that p_1 and p_2 are both strictly greater than 2, so that N_1 and N_2 are odd-characteristic congruence kernels. In this case it follows from assertions (1) and (2) of Lemma 3.1 that $m_1 = 4$ and $m_2 = 8$. Hence for $i = 1, 2$ we have $m_i > 1$ and $m_i \neq p_i$ (since m_i is composite). Note also that since $H_1(\Gamma, \mathbb{Z}_2) = 0$, we have $H_1(\Gamma, \mathbb{Z}_{m_i}) = 0$ for $i = 1, 2$. Since we also have $H_1(\Gamma; \mathbb{Z}_s) = 0$ for $s = 2$ and 3 , it now follows from Lemma 4.8 that neither N_1 nor N_2 is cosolvable. Since in addition we have $m_1 \neq m_2$, the hypotheses of Lemma 5.3 hold. In the notation of that lemma we have $\theta_1 = 2$ and $\theta_2 = 4$. Hence Lemma 5.3 gives a finite-index normal subgroup N of Γ such that $\nu_2(N) \geq 4$ and $m_N(u) = 4$. In particular $z^4 \in N$. Hence alternative (i) of the present lemma holds in this case. \square

Lemma 5.9. *Let Γ be a subgroup of $\mathrm{SL}_2(\mathcal{O}_E)$ for some number field E . Suppose that $H_1(\Gamma; \mathbb{Z}_s) = 0$ for $s = 2, 3$ and 7 , and that Γ has no abelian subgroup of finite index. Let z be an element of Γ . Suppose that trace z is nifty but not swell in \mathcal{O}_E . Then for some $\ell \in \{2, 3\}$, there is a finite-index normal subgroup N of Γ such that $\max(\nu_\ell(N), \kappa_\ell(N)) \geq 4$ and $z^3 \in N$.*

Proof. Set $\tau = \text{trace } z$. The hypothesis that τ is nifty but not swell implies that $\tau - 1$ and $\tau + 1$ are non-units in \mathcal{O}_E . Hence there are prime ideals \mathfrak{p}_1 and \mathfrak{p}_2 in \mathcal{O}_E such that $\tau - 1 \in \mathfrak{p}_1$ and $\tau + 1 \in \mathfrak{p}_2$. For $i = 1, 2$, we set $k_i = k_{\mathfrak{p}_i}$, $p_i = \text{char } k_i$, $h_i = h_{\mathfrak{p}_i}$ and $N_i = \ker h_i$. By definition N_i is a characteristic- p_i congruence kernel. We set $g_i = h_i(z)$ and $t_i = \text{trace } g_i$. We set $m_i = m_{N_i}(z)$, and note that m_i is the order of g_i in $\text{SL}_2(k_i)$. Note also that

$$t_1 = \eta_1(\tau) = 1$$

and that

$$t_2 = \eta_2(\tau) = -1.$$

First consider the case that for some $j \in \{1, 2\}$ we have $p_j = 2$. Then by Lemma 5.7 we have either $\nu_2(N_j) \geq 4$ or $\kappa_2(N_j) \geq 4$. By assertions (3) and (4) of Lemma 3.1, m_j is equal to 1 or 3, and hence $z^3 \in N_j$. Thus the conclusion of the present lemma holds in this case, with $N = N_j$ and $\ell = 2$.

We now turn to the case in which p_1 and p_2 are both strictly greater than 2, so that N_1 and N_2 are odd-characteristic congruence kernels. According to Assertion (4) of Lemma 3.1, we have $m_1 \in \{2, 6\}$. According to Assertion (3) of Lemma 3.1, we have $m_2 \in \{1, 3\}$.

Since by hypothesis we have $H_1(\Gamma, \mathbb{Z}_2) = H_1(\Gamma, \mathbb{Z}_3) = 0$, it now follows for $i = 1, 2$ that $H_1(\Gamma, \mathbb{Z}_{m_i}) = 0$. Furthermore, we have $m_1 > 1$; and since p_1 is an odd prime, we have $m_1 \neq p_1$. It therefore follows from Lemma 4.8 that:

5.9.1. N_1 is not cosolvable.

We distinguish two subcases, depending on whether N_2 is cosolvable. First consider the subcase in which N_2 is not cosolvable. Since we have observed that $m_1 \in \{2, 6\}$ and $m_2 \in \{1, 3\}$, we have $m_1 \neq m_2$. Combining these observations with 5.9.1, we see that all the hypotheses of Lemma 5.3 hold in this subcase. In the notation of Lemma 5.3 we have $\theta_1 \in \{1, 3\}$ and $\theta_2 \in \{1, 3\}$. Hence Lemma 5.3 gives a finite-index normal subgroup N of Γ such that $\nu_2(N) \geq 4$ and $m_N(z) \in \{1, 3\}$. This implies the conclusion of the present lemma, with $\ell = 2$.

The rest of the proof is devoted to the subcase in which N_2 is cosolvable. In this subcase we claim:

$$(5.9.2) \quad p_2 = 3.$$

To prove this, assume that $p_2 \neq 3$. Since we also have $p_2 \neq 2$, it follows from Lemma 3.1 that $m_2 = 3$, so that $m_2 > 1$ and $m_2 \neq p_2$. Hence Lemma 4.8 implies that N_2 is not cosolvable, a contradiction. This proves 5.9.2.

We now apply Lemma 5.4 with $p = p_2 = 3$ and $d = 3$, and with N_2 playing the role of N . In the notation of Lemma 5.4, we have $S = \{2, 3, 8\}$. By the hypothesis of the present lemma, $H_1(\Gamma; \mathbb{Z}_s)$ is trivial for $s = 2$ and for $s = 3$. Since $H_1(\Gamma; \mathbb{Z}_2)$ is trivial, so is $H_1(\Gamma; \mathbb{Z}_8)$. Hence if we set $G_2 = \Gamma/N_2$, Lemma 5.4 implies:

5.9.3. *Either (i) the 3-Sylow subgroup T of G_2 is an elementary abelian 3-group of rank at least 3, normal in G_2 , and G_2/T is a cyclic group whose order is not divisible by 2 or 3; or (ii) G_2 is a cyclic group whose order is not divisible by 2 or 3, and $\kappa_3(N_2) \geq 3$.*

By Remark 5.6, Alternatives (i) and (ii) of 5.9.3 are mutually exclusive.

We now define a subgroup N_2^* of Γ as follows. If Alternative (i) of 5.9.3 holds, we set $N_2^* = N_2$. If Alternative (ii) of 5.9.3 holds, we define N_2^* to be the mod-3 commutator subgroup of N_2 . In either case, we set $G_2^* = \Gamma/N_2^*$, and let T^* denote the 3-Sylow subgroup of G_2^* . We claim:

5.9.4. T^* is an elementary abelian 3-group of rank at least 3, normal as a subgroup of G_2^* , and G_2^*/T^* is a cyclic group of odd order.

If Alternative (i) of 5.9.3 holds, 5.9.4 is immediate. If Alternative (ii) of 5.9.3 holds, then $K \doteq N_2/N_2^*$ is a normal subgroup of $G_2^* = \Gamma/N_2^*$, and K is an elementary abelian 3-group since N_2^* is the mod-3 commutator subgroup of N_2 . Since $G_2^*/K \cong \Gamma/N_2 = G_2$ is a cyclic group whose order is not divisible by 2 or 3, we have $K = T^*$. By the definition of $\kappa_3(N_2)$, the rank of $T^* = K$ is $\kappa_3(N_2) \geq 3$. This completes the proof of 5.9.4.

Next, we claim that

$$(5.9.5) \quad m_{N_2^*}(z) \in \{1, 3\}.$$

If Alternative (i) of 5.9.3 holds, we have $m_{N_2^*} = m_2$. As we have observed that $m_2 \in \{1, 3\}$, (5.9.5) holds in this case. In the case where Alternative (ii) of 5.9.3 holds, $m_2 = m_{N_2}(z) \in \{1, 3\}$ is the order of zN_2 in Γ/N_2 , which is a cyclic group whose order is not divisible by 3. Hence $m_2 = 1$, i.e. $z \in N_2$. It follows that N_2^*z belongs to the elementary abelian 3-group N_2/N_2^* , and therefore has order 1 or 3. Thus (5.9.5) is proved in all cases.

It follows from 5.9.4 that G_2^* is solvable and has odd order; thus $N_2^* \triangleleft \Gamma$ is cosolvable and has odd index. According to 5.9.1, the odd-characteristic congruence kernel N_1 is not cosolvable. By 5.9.5 we have $m_{N_2^*}(z) \in \{1, 3\}$, and we observed above that $m_{N_1} = m_1 \in \{2, 6\}$. In particular we have $m_{N_2^*}(z) \neq m_{N_1}(z)$. Thus the hypotheses of Lemma 5.2 hold with N_2^* playing the role of N_2 in the statement of that lemma. In the notation of Lemma 5.2, we have $\theta_1 = m_{N_1}(z) \in \{1, 3\}$ and $\theta_2 = m_{N_2^*}(z) \in \{1, 3\}$. Lemma 5.2 therefore gives a finite-index normal subgroup N of Γ such that $m_N(z) \in \{1, 3\}$ and such that $\nu_\ell(N) = \nu_\ell(N_1) + \nu_\ell(N_2^*)$ for every prime ℓ . Taking $\ell = 3$, we find from 5.9.4 that

$$\nu_3(N_2^*) = \sigma_3(G_2^*) = \dim_{\mathbb{Z}_3}(T^*) \geq 3.$$

By Assertion (7) of Lemma 3.6, the group $G_1 \doteq \Gamma/N_1$ has order divisible by 3, and hence $\nu_3(N_1) = \sigma_3(G_1) \geq 1$. Thus we have

$$\nu_3(N) = \nu_3(N_1) + \nu_3(N_2^*) \geq 4.$$

On the other hand, since $m_N(z) \in \{1, 3\}$, we have $z^3 \in N$. Thus the conclusion of the lemma holds in this case, with $\ell = 3$. \square

6. INDEPENDENCE CRITERIA IN 3-MANIFOLD GROUPS

The main results of this section are Propositions 6.3 and 6.4, which provide the transition between the results of Section 5 and our geometric theorems.

Lemma 6.1. *Suppose that Γ_0 is a torsion-free cocompact discrete subgroup of $\text{Isom}_+(\mathbb{H}^3)$, and that x and γ are non-commuting elements of Γ_0 . Then no non-trivial power of $\gamma x \gamma^{-1}$ commutes with any non-trivial power of x .*

Proof. Since Γ_0 is discrete, torsion-free and cocompact it is purely loxodromic. Hence Γ_0 is torsion-free and the centralizer of every non-trivial element of Γ_0 is cyclic. In particular any two commuting elements of Γ have a common non-trivial power.

If x_0 denotes a generator of the centralizer of x , then x_0 is clearly a primitive element of Γ having x as a power.

Suppose that $\gamma x^m \gamma^{-1}$ commutes with x^n for some $m, n \neq 0$. If $z \neq 1$ is a common power of $\gamma x^m \gamma^{-1}$ and x^n , the centralizer C of z contains both $\gamma x_0 \gamma^{-1}$ and x_0 . Since $\gamma x_0 \gamma^{-1}$ and x_0 are primitive and C is cyclic, each of the elements $\gamma x_0 \gamma^{-1}$ and x_0 is a generator of C , and hence $\gamma x_0 \gamma^{-1} = x_0^{\pm 1}$. In particular γ^2 commutes with x_0 . Thus the centralizer of γ^2 , which is also cyclic, contains both γ and x_0 ; since x_0 is primitive this means that $\gamma \in \langle x_0 \rangle$, and hence γ commutes with x , a contradiction. \square

Lemma 6.2. *Let Γ be a discrete, cocompact subgroup of $\text{Isom}_+(\mathbb{H}^3)$. Let \mathcal{T} be a subgroup of Γ such that $\dim_{\mathbb{Z}_\ell} H_1(\mathcal{T}; \mathbb{Z}_\ell) \geq 4$ for some prime ℓ . Suppose that u and v are non-commuting elements of Γ , and that m and n are positive integers such that u^m and $vu^n v^{-1}$ lie in \mathcal{T} . Then u^m and $vu^n v^{-1}$ are independent elements (2.1) of Γ .*

Proof. It follows from Lemma 6.1 that u^m and $vu^n v^{-1}$ do not commute.

We claim:

6.2.1. *The group $F = \langle u^m, vu^n v^{-1} \rangle$ is not cocompact.*

To prove this, note that since F is discrete, torsion-free and cocompact we have $\mathcal{T} \cong \pi_1(M)$ for some closed, orientable hyperbolic 3-manifold M . The hypothesis then implies that $H_1(M; \mathbb{Z}_\ell) \geq 4$ for some prime ℓ . It then follows from [25, Proposition 1.1] that any two-generator subgroup of $\pi_1(M)$ has infinite index in $\pi_1(M)$. Hence F has infinite index in \mathcal{T} and is therefore not cocompact. This proves 6.2.1.

It follows from [4, Theorem 7.1] that any two-generator non-cocompact purely loxodromic subgroup of $\text{Isom}_+(\mathbb{H}^3)$ is free. Hence by 6.2.1, F is free. But since u^m and $vu^n v^{-1}$ do not commute, F is non-abelian. Hence it is free of rank 2, and so u^m and $vu^n v^{-1}$ are independent. \square

Proposition 6.3. *Let $\Gamma \leq \text{SL}_2(\mathbb{C})$ be a cocompact torsion-free lattice having integral traces, such that $H_1(\Gamma; \mathbb{Z}_s) = 0$ for $s = 2, 3$ and 7. Let K denote the trace field of Γ . Let u and v be non-commuting elements of Γ . Suppose that u is a power of an element of Γ whose trace is swell in \mathcal{O}_K . Then either*

- (1) u and $vu^4 v^{-1}$ are independent elements of Γ , or
- (2) u^2 and $vu^2 v^{-1}$ are independent elements of Γ .

Proof. Let us write $u = z^k$, where k is a positive integer and z is an element of Γ such that trace z is swell in \mathcal{O}_K . Then trace z is swell in \mathcal{O}_E by 2.7. Hence Lemma 5.8 gives a finite-index normal subgroup N of Γ such that either (i) $\nu_2(N) \geq 4$ and $z^4 \in N$, or (ii) $\kappa_2(N) \geq 4$ and $z^2 \in N$. Set $G = N/\Gamma$, let $\pi : \Gamma \rightarrow G$ denote the quotient homomorphism, and set $g = \pi(u)$.

First suppose that (i) holds. Let T denote the 2-Sylow subgroup of N . Since $u^4 = (z^4)^k \in N$, we have $g^4 = e$; hence we may suppose T to be chosen within its conjugacy class so that $g \in T$. We have $\dim(H_1(T); \mathbb{Z}_2) = \sigma_2(G) = \nu_2(N) \geq 4$.

Now set $\tilde{T} = \pi^{-1}(T)$. Then $\pi|_{\tilde{T}} : \tilde{T} \rightarrow T$ is surjective, and hence $\dim(H_1(\tilde{T}); \mathbb{Z}_2) \geq 4$. Since $\pi(u) = g \in T$, we have $u \in \tilde{T}$. Since N is normal and contains u^4 , we have $vu^4 v^{-1} \in N \leq \tilde{T}$. It now follows from the case $m = 1, n = 4$ of Lemma 6.2, with $\mathcal{T} = \tilde{T}$, that u and $vu^4 v^{-1}$ are independent in Γ .

Now suppose that (ii) holds. We have $\dim(H_1(N); \mathbb{Z}_2) = \kappa_2(N) \geq 4$. Since N is normal and contains $u^2 = (z^2)^k$, it also contains $vu^2 v^{-1}$. We now apply the case $m = n = 2$ of Lemma 6.2, with $\mathcal{T} = N$, to deduce that u^2 and $vu^2 v^{-1}$ are independent in Γ . \square

Proposition 6.4. *Let $\Gamma \leq \mathrm{SL}_2(\mathbb{C})$ be a cocompact torsion-free lattice having integral traces, such that $H_1(\Gamma; \mathbb{Z}_2)$ and $H_1(\Gamma; \mathbb{Z}_3)$ are trivial. Let K denote the trace field of Γ . Let u and v be non-commuting elements of Γ , and suppose that u is a power of element of Γ whose trace is nifty but not swell in \mathcal{O}_K . Then u^3 and vu^3v^{-1} are independent elements of Γ .*

Proof. Let us write $u = z^k$, where k is a positive integer and z is an element of Γ such that trace z is nifty but not swell \mathcal{O}_K . It then follows from 2.7 that trace z is nifty but not swell in \mathcal{O}_E . Hence Lemma 5.9 gives a finite-index normal subgroup N of Γ and a prime $\ell \in \{2, 3\}$ such that $z^3 \in N$ and $\max(\nu_\ell(N), \kappa_\ell(N)) \geq 4$. Thus either $\nu_\ell(N) \geq 4$ or $\kappa_\ell(N) \geq 4$. Then N contains $u^3 = (z^3)^k$. Since N is normal, it also contains vu^3v^{-1} .

Set $G = N/\Gamma$, let $\pi : \Gamma \rightarrow G$ denote the quotient homomorphism, and set $g = \pi(u)$.

Let T denote the ℓ -Sylow subgroup of N . We have $\dim(H_1(T); \mathbb{Z}_2) = \sigma_2(G) = \nu_2(N) \geq 4$.

Set $\tilde{T} = \pi^{-1}(T)$. Then $\pi|_{\tilde{T}} : \tilde{T} \rightarrow T$ is surjective, and hence

$$(6.4.1) \quad \dim(H_1(\tilde{T}; \mathbb{Z}_2)) \geq \dim(H_1(T); \mathbb{Z}_2) = \sigma(G) = \nu_\ell(N) \geq 4.$$

On the other hand, we have

$$(6.4.2) \quad \dim(H_1(N; \mathbb{Z}_2)) = \nu_\ell(N) \geq 4.$$

We define a finite-index normal subgroup \mathcal{T} of Γ as follows. If $\nu_\ell(N) \geq 4$ we set $\mathcal{T} = \tilde{T}$. If $\nu_\ell(N) < 4$, in which case $\kappa_\ell(N) \geq 4$, we set $\mathcal{T} = \tilde{T}$. It then follows from (6.4.1) and (6.4.2) that in either case we have $\dim(H_1(\mathcal{T}; \mathbb{Z}_2)) \geq 4$. By definition we also have $N \geq \mathcal{T}$ in either case, and hence u^3 and vu^3v^{-1} belong to \mathcal{T} . It now follows from the case $m = n = 3$ of Lemma 6.2 that u^3 and vu^3v^{-1} are independent in Γ . \square

7. DISTANCES ON A SPHERE

Proposition 7.1. *If Q_1, \dots, Q_n are points on S^2 , we have*

$$\sum_{1 \leq i < j \leq n} \cos d_s(Q_i, Q_j) \geq -n/2.$$

Proof. We regard S^2 as the unit sphere in \mathbf{E}^3 , and we let $v_i \in \mathbf{R}^3$ denote the position vector of Q_i . We have

$$\begin{aligned} 0 &\leq \left\langle \sum_{i=1}^n v_i, \sum_{i=1}^n v_i \right\rangle \\ &= \sum_{i=1}^n \|v_i\|^2 + \sum_{i \neq j} \langle v_i, v_j \rangle \\ &= n + 2 \sum_{1 \leq i < j \leq n} \langle v_i, v_j \rangle \\ &= 2(2 + \sum_{1 \leq i < j \leq n} \cos d_s(Q_i, Q_j)), \end{aligned}$$

from which the conclusion follows. \square

Corollary 7.2. *If I is a four-element index set and $(P_i)_{i \in I}$ is an indexed family of points on S^2 , there exist three distinct indices $p, q, q' \in I$ such that*

$$\cos(d_s(P_p, P_q)) + \cos(d_s(P_p, P_{q'})) \geq -2/3.$$

Proof. We may assume that $I = \{1, 2, 3, 4\}$. Let \mathcal{T} denote the set of all ordered pairs of the form (p, S) , where p is an index in $\{1, 2, 3, 4\}$ and S is a two-element subset of $\{1, 2, 3, 4\}$ that does not contain p . We have $|\mathcal{T}| = 12$. For each element (p, S) of \mathcal{T} , let us set $\alpha_{(p,S)} = \cos(d_s(P_p, P_q)) + \cos(d_s(P_p, P_{q'}))$, where q and q' are the elements of S . For each pair (i, j) with $1 \leq i < j \leq 4$, there are exactly four elements (p, S) of \mathcal{T} such that one of the indices i, j is equal to p and the other belongs to S . Hence

$$(7.2.1) \quad \sum_{(p,S) \in \mathcal{T}} \alpha_{(p,S)} = 4 \sum_{1 \leq i < j \leq 4} \cos(d_s(P_i, P_j)) \geq -8,$$

where the final inequality follows from Proposition 7.1. Since there are 12 terms on the left-hand side of (7.2.1), we must have $\alpha_{(p,S)} \geq -2/3$ for some $(p, S) \in \mathcal{T}$. \square

8. NIFTY TRACES AND DISPLACEMENTS

In this section we give the proof of Theorem 1.6, after a good many technical preliminaries.

We set

$$\mathcal{D} = \{(T, \mu, h) : 0 < T < 2, 0 < h < \mu\} \subset \mathbf{R}^3.$$

We define a real-valued function Φ with domain \mathcal{D} by

$$\Phi(T, \mu, h) = \max \left(2 \operatorname{arccosh}((1 + T/2) \sinh^2 \mu + \cosh h), \operatorname{arccosh}(\cosh \mu \cosh(\mu - h)) + 2\mu \right).$$

Lemma 8.1. *Let $(T, \mu, h) \in \mathcal{D}$ be given. Let $A_1B_1C_1$ and $A_2B_2C_2$ be hyperbolic triangles such that $d(A_i, C_i) \leq \mu$ and $d(B_i, C_i) \leq \mu$ for $i = 1, 2$. Suppose that*

$$(8.1.1) \quad \cos(\angle(A_1, C_1, B_1)) + \cos(\angle(A_2, C_2, B_2)) \geq -T.$$

Then

$$d(A_1, B_1) + d(A_2, B_2) \leq \Phi(T, \mu, h).$$

Proof. For $i = 1, 2$ we set $c_i = d(B_i, A_i)$, $b_i = d(C_i, A_i)$, $a_i = d(C_i, B_i)$, and $\gamma_i = \angle(B_i, C_i, A_i)$. By hypothesis we have

$$(8.1.2) \quad a_i \leq \mu, \quad b_i \leq \mu$$

for $i = 1, 2$, and

$$(8.1.3) \quad \cos \gamma_1 + \cos \gamma_2 \geq -T.$$

The hyperbolic law of cosines gives

$$(8.1.4) \quad \cosh c_i = \cosh b_i \cosh a_i - \sinh b_i \sinh a_i \cos \gamma_i.$$

We claim that for each $i \in \{1, 2\}$ we have

$$(8.1.5) \quad \cosh c_i \leq \max((1 - \cos \gamma_i) \sinh^2 \mu + \cosh h, \cosh \mu \cosh(\mu - h)).$$

To prove 8.1.5 for a give $i \in \{1, 2\}$, we first consider the special case in which $\gamma_i \geq \pi/2$. In this case we have $\cos \gamma_i \leq 0$; hence (8.1.4), with (8.1.2), gives

$$\begin{aligned} \cosh c_i &\leq \cosh^2 \mu - \sinh^2 \mu \cos \gamma_i \\ &= (1 - \cos \gamma_i) \sinh^2 \mu + 1 \\ &\leq (1 - \cos \gamma_i) \sinh^2 \mu + \cosh h, \end{aligned}$$

so that (8.1.5) holds. Next consider the special case in which $\min(a_i, b_i) \geq \mu - h$. In this case, in view of (8.1.2), both a_i and b_i lie in the interval $[\mu - h, \mu]$. Hence $|a_i - b_i| \leq h$. Using (8.1.4), we find that

$$\begin{aligned} \cosh c_i &= \cosh(a_i - b_i) + \sinh a_i \sinh b_i (1 - \cos \gamma_i) \\ &\leq \cosh h + (\sinh^2 \mu)(1 - \cos \gamma_i), \end{aligned}$$

so that again (8.1.5) holds. There remains only the case in which $\gamma_i < \pi/2$ and $\min(a_i, b_i) < \mu - h$. Then $\cos \gamma_i > 0$, and by symmetry we may assume that $a_i \leq \mu - h$. By (8.1.2) we have $b_i \leq \mu$. Using (8.1.4) we deduce that

$$\begin{aligned} \cosh c_i &= \cosh b_i \cosh a_i - \sinh b_i \sinh a_i \cos \gamma_i \\ &< \cosh \mu \cosh(\mu - h), \end{aligned}$$

which again implies (8.1.5). Thus (8.1.5) is proved in all cases.

Since (8.1.5) holds for each $i \in \{1, 2\}$ we have either

$$(8.1.6) \quad \cosh c_i \leq (1 - \cos \gamma_i) \sinh^2 \mu + \cosh h \quad \text{for each } i \in \{0, 1\},$$

or

$$(8.1.7) \quad \cosh c_i \leq \cosh \mu \cosh(\mu - h) \quad \text{for some } i \in \{0, 1\}.$$

If (8.1.6) holds then, using (8.1.3) we find that

$$\begin{aligned} \cosh c_1 + \cosh c_2 &\leq (1 - \cos \gamma_1) \sinh^2 \mu + \cosh h + (1 - \cos \gamma_2) \sinh^2 \mu + \cosh h \\ &\leq (2 - T) \sinh^2 \mu + 2 \cosh h. \end{aligned}$$

Since \cosh is a convex function, we have

$$\begin{aligned} \cosh \left(\frac{c_1 + c_2}{2} \right) &\leq \frac{1}{2} (\cosh c_1 + \cosh c_2) \\ &\leq \frac{1}{2} ((2 - T) \sinh^2 \mu + 2 \cosh h), \end{aligned}$$

so that

$$c_1 + c_2 \leq 2 \operatorname{arccosh}((1 - T/2) \sinh^2 \mu + \cosh h) \leq \Phi(T, \mu, h).$$

This is the conclusion of the lemma in this case.

It remains to prove that if (8.1.7) holds then we still have $c_1 + c_2 \leq \Phi(T, \mu, h)$. By symmetry we may assume that $\cosh c_1 \leq \cosh \mu \cosh(\mu - h)$, so that

$$c_1 \leq \operatorname{arccosh}(\cosh \mu \cosh(\mu - h)).$$

By the triangle inequality and (8.1.2) we have

$$c_2 \leq a_2 + b_2 \leq 2\mu.$$

Hence

$$c_1 + c_2 \leq \operatorname{arccosh}(\cosh \mu \cosh(\mu - h)) + 2\mu \leq \Phi(T, \mu, h).$$

□

Lemma 8.2. *Let $(T, \mu, h) \in \mathcal{D}$ and $n > 4$ be given. Let P be a point of \mathbb{H}^3 , and let ξ_1, \dots, ξ_n be isometries of \mathbb{H}^3 such that $d(P, \xi_i \cdot P) \leq \mu$ for $i = 1, \dots, n$. Suppose that there exist indices $k, l \in \{1, \dots, n-1\}$, with $l \geq k+2$, such that*

$$(8.2.1) \quad \cos(\angle(\xi_k^{-1} \cdot P, P, \xi_{k+1} \cdot P)) + \cos(\angle(\xi_l^{-1} \cdot P, P, \xi_{l+1} \cdot P)) \geq -T.$$

Set $\beta = \xi_1 \xi_2 \cdots \xi_n$. Then

$$d(P, \beta \cdot P) \leq \Phi(T, \mu, h) + (n-4)\mu.$$

Proof. For $i = 1, \dots, n$ set $P_i = \xi_i \cdot P$ and $D_i = d(P, P_i)$, so that $D_i \leq \mu$. Set $\beta_j = \xi_1 \cdot \xi_2 \cdots \xi_j$ for $0 \leq j \leq n$, so that $\beta_0 = 1$ and $\beta_n = \beta$. Set $Q_j = \beta_j \cdot P$, and note that $Q_j = \beta_{j-1} \cdot P_j$. For $0 \leq m < m' \leq n$, the triangle inequality gives

$$\begin{aligned} d(Q_m, Q_{m'}) &\leq \sum_{j=m+1}^{m'} d(Q_{j-1}, Q_j) \\ &= \sum_{j=m+1}^{m'} d(\beta_{j-1} \cdot P, \beta_{j-1} \cdot P_j) \\ &= \sum_{j=m+1}^{m'} D_j, \end{aligned}$$

and hence

$$(8.2.2) \quad d(Q_m, Q_{m'}) \leq (m' - m)\mu.$$

Note that (8.2.2) is trivial when $m = m'$, and is therefore true whenever $0 \leq m < m' \leq n$. Hence

$$(8.2.3) \quad \begin{aligned} d(P, \beta \cdot P) &\leq d(Q_0, Q_{k-1}) + d(Q_{k-1}, Q_{k+1}) + d(Q_{k+1}, Q_{l-1}) + d(Q_{l-1}, Q_{l+1}) + d(Q_{l+1}, Q_n) \\ &\leq (n-4)\mu + d(Q_{k-1}, Q_{k+1}) + d(Q_{l-1}, Q_{l+1}). \end{aligned}$$

Next we note that if i is equal to either k or l , we have

$$(8.2.4) \quad d(Q_{i-1}, Q_{i+1}) = d(\beta_{i-1} \cdot P, \beta_{i-1} \xi_i \xi_{i+1} \cdot P) = d(\xi_i^{-1} \cdot P, \xi_{i+1} \cdot P).$$

We have $d(\xi_i^{-1} \cdot P, P) = D_i \leq \mu$, and $d(\xi_{i+1} \cdot P, P) = D_{i+1} \leq \mu$ for $i = k, l$. By hypothesis we have

$$\cos(\angle(\xi_k^{-1} \cdot P, P, \xi_{k+1} \cdot P)) + \cos(\angle(\xi_l^{-1} \cdot P, P, \xi_{l+1} \cdot P)) \geq -T.$$

Hence we may apply Lemma 8.1, with $A_1 = A_2 = P$, $B_1 = \xi_k^{-1} \cdot P$, $B_2 = \xi_l^{-1} \cdot P$, $C_1 = \xi_{k+1} \cdot P$ and, $C_2 = \xi_{l+1} \cdot P$, to conclude that

$$(8.2.5) \quad d(\xi_k^{-1} \cdot P, \xi_{k+1} \cdot P) + d(\xi_l^{-1} \cdot P, \xi_{l+1} \cdot P) \leq \Phi(T, \mu, h).$$

From (8.2.3), (8.2.4) and (8.2.5) it follows that

$$\begin{aligned} d(P, \beta \cdot P) &\leq (n-4)\mu + d(Q_{k-1}, Q_{k+1}) + d(Q_{l-1}, Q_{l+1}) \\ &= (n-4)\mu + d(\xi_k^{-1} \cdot P, \xi_{k+1} \cdot P) + d(\xi_l^{-1} \cdot P, \xi_{l+1} \cdot P) \leq (n-4)\mu + \Phi(T, \mu, h), \end{aligned}$$

which is the conclusion of the lemma. \square

We will also need the following variant of Lemma 8.2.

Lemma 8.3. *Let $T \in [-1, 1]$ and $\mu > 0$ be given. Let P be a point of \mathbb{H}^3 , and let ξ_1, \dots, ξ_n be isometries of \mathbb{H}^3 such that $d(P, \xi_i \cdot P) \leq \mu$ for $i = 1, \dots, n$. Suppose that there is an index $k \in \{1, \dots, n-1\}$ such that $\cos(\angle(\xi_k^{-1} \cdot P, P, \xi_{k+1} \cdot P)) \geq -T$. Set $\beta = \xi_1 \xi_2 \cdots \xi_n$. Then*

$$d(P, \beta \cdot P) \leq \operatorname{arccosh}(\cosh^2 \mu + T \sinh^2 \mu) + (n-2)\mu.$$

Proof. We set $\beta_j = \xi_1 \cdot \xi_2 \cdots \xi_j$ for $0 \leq j \leq n$, so that $\beta_0 = 1$ and $\beta_n = \beta$. By the triangle inequality we have

$$(8.3.1) \quad d(P, \beta \cdot P) \leq \left(\sum_{i=0}^{k-2} d(\beta_i \cdot P, \beta_{i+1} \cdot P) \right) + d(\beta_{k-1} \cdot P, \beta_{k+1} \cdot P) + \left(\sum_{i=k+1}^{n-1} d(\beta_i \cdot P, \beta_{i+1} \cdot P) \right),$$

where the sums $\sum_{i=0}^{k-2} d(\beta_i \cdot P, \beta_{i+1} \cdot P)$ and $\sum_{i=k+1}^{n-1} d(\beta_i \cdot P, \beta_{i+1} \cdot P)$ are interpreted as being 0 in the respective cases $k = 1$ and $k = n - 1$. For $i = 0, \dots, n - 1$ we have

$$(8.3.2) \quad d(\beta_i \cdot P, \beta_{i+1} \cdot P) = d(\beta_i \cdot P, \beta_i \xi_{i+1} \cdot P) \leq \mu.$$

Furthermore,

$$(8.3.3) \quad \begin{aligned} d(\beta_{k-1} \cdot P, \beta_{k+1} \cdot P) &= d(\beta_{k-1} \cdot P, \beta_{k-1} \xi_i \xi_{i+1} \cdot P) \\ &= d(P, \xi_i \xi_{i+1} \cdot P) \\ &= d(\xi_i^{-1} \cdot P, \xi_{i+1} \cdot P). \end{aligned}$$

Applying the hyperbolic law of cosines to the triangle with vertices $\xi_i^{-1} \cdot P$, P and $\xi_{i+1} \cdot P$, we find that $\cosh d(\xi_i^{-1} \cdot P, \xi_{i+1} \cdot P)$ is equal to

$$\cosh d(\xi_i^{-1} \cdot P, P) \cosh d(P, \xi_{i+1} \cdot P) - \sinh d(\xi_i^{-1} \cdot P, P) \sinh d(P, \xi_{i+1} \cdot P) \cos(\angle(\xi_i^{-1} \cdot P, P, \xi_{i+1} \cdot P)),$$

and hence that

$$(8.3.4) \quad \cosh d(\xi_i^{-1} \cdot P, P) \leq \cosh^2 \mu + \sinh^2 \mu + T \sinh^2 \mu.$$

The conclusion of the lemma follows immediately from (8.3.1), (8.3.2), (8.3.3) and (8.3.4). \square

8.4. We now turn to the proof of Theorem 1.6, which was stated in the introduction.

Proof of Theorem 1.6. We consider the unit sphere S in the tangent space to \mathbb{H}^3 . We let I denote the set $\{x, x^{-1}, y, y^{-1}\} \subset \Gamma$. For each $t \in I$ let r_t denote the ray from P to $t \cdot P$, and let $Q_t \in S$ denote the unit tangent vector to the ray r_t . Then for any two distinct indices $s, t \in I$, we have

$$(8.4.1) \quad d_s(Q_s, Q_t) = \angle(s \cdot P, P, t \cdot P).$$

We claim:

8.4.2. *There exist non-commuting elements u and v of I that at least one of the following alternatives holds:*

- (i) $\cos(d_s(Q_{v^{-1}}, Q_u)) + \cos(d_s(Q_{v^{-1}}, Q_{u^{-1}})) \geq -2/3$, or
- (ii) $\cos(d_s(Q_{v^{-1}}, Q_u)) + \cos(d_s(Q_{u^{-1}}, Q_u)) \geq -2/3$.

To prove this, first note that according to Corollary 7.2, there exist three distinct indices $\xi, \eta, \eta' \in I$ such that

$$\cos(d_s(Q_\xi, Q_\eta)) + \cos(d_s(Q_\xi, Q_{\eta'})) \geq -2/3.$$

Because ξ, η, η' are distinct elements of I , two of them must be inverses of each other. By symmetry we may assume that either $\eta' = \eta^{-1}$ or $\eta' = \xi^{-1}$. If $\eta' = \eta^{-1}$, let us set $u = \eta$ and $v = \xi^{-1}$. Then we have

$$\cos(d_s(Q_{v^{-1}}, Q_u)) + \cos(d_s(Q_{v^{-1}}, Q_{u^{-1}})) \geq -2/3,$$

which gives Alternative (i) of 8.4.2. If $\eta' = \xi^{-1}$, let us set $u = \xi$ and $v = \eta^{-1}$. Then we have

$$\cos(d_s(Q_u, Q_{v^{-1}})) + \cos(d_s(Q_u, Q_{u^{-1}})) \geq -2/3,$$

which gives Alternative (ii). Thus 8.4.2 is established.

We now fix non-commuting elements u and v of the set $\{x, x^{-1}, y, y^{-1}\} \subset \Gamma$ such that one of the alternatives (i), (ii) of 8.4.2 holds. Note that $\text{trace } u \in \{\text{trace } x, \text{trace } y\}$. In particular,

8.4.3. *The element u is a power of some element z of Γ whose trace is nifty.*

We now set $\mu = 0.3925$ if $\text{trace } z$ is swell, and $\mu = 0.3$ otherwise. We are required to prove that $\max(d(P, x \cdot P), d(P, y \cdot P)) > \mu$. For the rest of the argument I will assume that $\max(d(P, x \cdot P), d(P, y \cdot P)) \leq \mu$, and eventually this will produce a contradiction.

Since u and v are non-commuting elements of $\{x, x^{-1}, y, y^{-1}\}$, we have

$$(8.4.4) \quad \max(d(P, u \cdot P), d(P, v \cdot P)) = \max(d(P, x \cdot P), d(P, y \cdot P)) \leq \mu.$$

We claim:

8.4.5. *For every integer $m \geq 3$ and every real number h with $0 < h < \mu$, we have*

$$d(P, vu^m v^{-1} \cdot P) \leq \Phi(2/3, \mu, h) + (m - 2)\mu.$$

The proof of 8.4.5 is an application of Lemma 8.2. Given any integer $m \geq 3$, let us set $n = m + 2$, $\xi_1 = v$, $\xi_i = u$ for $i = 2, \dots, m + 1$ and $\xi_{m+2} = v^{-1}$. It follows from (8.4.4) that $d(P, \xi_i \cdot P) \leq \mu$ for $i = 1, \dots, m + 2$. According to Lemma 8.2, the inequality asserted in 8.4.5 will hold if there are indices $k, l \in \{1, \dots, m + 1\}$, with $l \geq k + 2$, such that (8.2.1) holds with $T = 2/3$.

If Alternative (i) of 8.4.2 holds, then by 8.4.1 we have

$$\begin{aligned} -2/3 &\leq \cos(\angle(v^{-1} \cdot P, P, u \cdot P)) + \cos(\angle(u^{-1} \cdot P, P, v^{-1} \cdot P)) \\ &= \cos(\angle(\xi_1^{-1} \cdot P, P, \xi_2 \cdot P)) + \cos(\angle(\xi_{m+1}^{-1} \cdot P, P, \xi_{m+2} \cdot P)). \end{aligned}$$

Thus (8.2.1) holds with $T = 2/3$, $k = 1$ and $l = m + 1$. We have $k - l = m \geq 3$ in this case.

If Alternative (ii) of 8.4.2 holds, then by 8.4.1 we have

$$\begin{aligned} -2/3 &\leq \cos(\angle(v^{-1} \cdot P, P, u \cdot P)) + \cos(\angle(u^{-1} \cdot P, P, u \cdot P)) \\ &= \cos(\angle(\xi_1^{-1} \cdot P, P, \xi_2 \cdot P)) + \cos(\angle(\xi_m^{-1} \cdot P, P, \xi_{m+1} \cdot P)). \end{aligned}$$

Thus (8.2.1) holds with $T = 2/3$, $k = 1$ and $l = m$. We have $k - l = m - 1 \geq 2$ in this case. This completes the proof of 8.4.5.

There is a counterpart of 8.4.5 for the case $m = 2$. We claim that

$$(8.4.6) \quad d(P, vu^2 v^{-1} \cdot P) \leq \text{arccosh}(\cosh^2 \mu + \frac{1}{2} \sinh^2 \mu) + 2\mu.$$

To prove 8.4.6, we first note that by Proposition 7.1 we have

$$\cos(d_s(Q_u, Q_{u^{-1}})) + \cos(d_s(Q_u, Q_{v^{-1}})) + \cos(d_s(Q_{u^{-1}}, Q_{v^{-1}})) \geq -3/2.$$

Hence at least one of the terms $\cos(d_s(Q_u, Q_{u^{-1}}))$, $\cos(d_s(Q_u, Q_{v^{-1}}))$ or $\cos(d_s(Q_{u^{-1}}, Q_{v^{-1}}))$ is bounded below by $-1/2$. If we set $\xi_1 = v$, $\xi_2 = \xi_3 = u$ and $\xi_4 = v^{-1}$, this means that $\cos(d_s(Q_{\xi_k}, Q_{\xi_{k+1}^{-1}})) \geq -1/2$ for some $k \in \{1, 2, 3\}$. In view of (8.4.1), this in turn means that

$$\cos(\angle(\xi_k \cdot P, P, \xi_{k+1}^{-1} \cdot P)) \geq -1/2$$

for some $k \in \{1, 2, 3\}$. Thus the hypothesis of Lemma 8.3 holds with $n = 4$ and $T = 1/2$. The inequality (8.4.6) is now simply the conclusion of Lemma 8.3.

The rest of the proof is divided into cases depending on whether trace z is swell or not. We first consider the case in which trace u is swell. In this case we have $\mu = 0.3925$ by definition. Since trace z is swell, u is a power of z , and u and v do not commute, it follows from Proposition 6.3 that either u and vu^4v^{-1} are independent elements of Γ , or u^2 and vu^2v^{-1} are independent elements of Γ .

Consider the subcase in which u and vu^4v^{-1} are independent. By (8.4.4) we have $d(P, u \cdot P) \leq \mu = 0.3925$. Applying 8.4.5, with $m = 4$ and $h = 0.07$, we obtain

$$d(P, vu^4v^{-1} \cdot P) \leq \Phi(2/3, 0.3925, 0.07) + 0.3925 \times 2 = 2.084 \dots$$

Applying Theorem 2.2 with $X = u$ and $Y = vu^4v^{-1}$, we find

$$\begin{aligned} 1/2 &\geq 1/(1 + \exp(d(P, u \cdot P))) + 1/(1 + \exp(d(P, vu^4v^{-1} \cdot P))) \\ &> 1/(1 + \exp(0.3925)) + 1/(1 + \exp(2.09)) \\ &= 0.513 \dots, \end{aligned}$$

a contradiction.

Now consider the subcase in which u^2 and vu^2v^{-1} are independent. From (8.4.4) it follows that $d(P, u^2 \cdot P) \leq 2\mu = 0.785$, and according to 8.4.6 we have

$$\begin{aligned} d(P, vu^2v^{-1} \cdot P) &\leq \operatorname{arccosh}(\cosh^2 \mu + \frac{1}{2} \sinh^2 \mu) + 2\mu \\ &= \operatorname{arccosh}(\cosh^2(0.3925) + \frac{1}{2} \sinh^2(0.3925)) + 0.785 \\ &= 1.469007 \dots \end{aligned}$$

Applying Theorem 2.2 with $X = u^3$ and $Y = vu^3v^{-1}$, we find

$$\begin{aligned} 1/2 &\geq 1/(1 + \exp(d(P, u^2 \cdot P))) + 1/(1 + \exp(d(P, vu^2v^{-1} \cdot P))) \\ &> 1/(1 + e^{2 \times 0.3925}) + 1/(1 + e^{1.46901}) \\ &= 0.5003 \dots, \end{aligned}$$

and again we have a contradiction.

We now turn to the case in which trace z is not swell. In this case we have $\mu = 0.3$ by definition. By 8.4.3, u is a power of z and trace z is nifty. Since trace z is not swell, and since u and v do not commute, it follows from Proposition 6.4 that u^3 and vu^3v^{-1} are independent elements of Γ .

From (8.4.4) it follows that $d(P, u^3 \cdot P) \leq 3\mu = 0.9$. Applying 8.4.5 with $m = 3$ and $h = 0.06$, we obtain

$$d(P, vu^3v^{-1} \cdot P) \leq \Phi(2/3, 0.3, 0.06) + 0.3 = 1.29 \dots$$

Applying Theorem 2.2 with $X = u^3$ and $Y = vu^3v^{-1}$, we find

$$\begin{aligned} 1/2 &\geq 1/(1 + \exp(d(P, u^3 \cdot P))) + 1/(1 + \exp(d(P, vu^3v^{-1} \cdot P))) \\ &< 1/(1 + e^{0.9}) + 1/(1 + e^{1.3}) \\ &= 0.503\dots, \end{aligned}$$

and we have a contradiction in this case as well. \square

Corollary 8.5. *Let $\Gamma \leq \mathrm{SL}_2(\mathbb{C})$ be a non-elementary torsion-free discrete group having integral traces, and suppose that $H_1(\Gamma; \mathbb{Z}_p) = 0$ for $p = 2, 3$ and 7 . Let K denote the trace field of Γ . Suppose that*

$$(8.5.1) \quad \text{trace } \gamma \in \mathcal{O}_K \text{ is nifty for every } \gamma \in \Gamma - \{1\} \text{ with } \text{length}(\Pi(\gamma)) \leq 0.3.$$

Then 0.3 is a Margulis number for $\Pi(\Gamma)$.

Furthermore, if we assume that

$$(8.5.2) \quad \text{trace } \gamma \in \mathcal{O}_K \text{ is swell for every } \gamma \in \Gamma - \{1\} \text{ with } \text{length}(\Pi(\gamma)) \leq 0.3925,$$

then 0.3925 is a Margulis number for $\Pi(\Gamma)$.

Proof. If (8.5.2) holds, we set $\mu = 0.3925$. If (8.5.1) holds but (8.5.2) does not, we set $\mu = 0.3$. If x and y are non-commuting elements of Γ we must show that

$$\max(d(P, x \cdot P), d(P, y \cdot P)) > \mu.$$

This is trivial if either $\Pi(x)$ or $\Pi(y)$ has translation length greater than μ . We therefore assume that $\text{length}(\Pi(x))$ and $\text{length}(\Pi(y))$ are at most μ . If (8.5.2) holds, so that $\mu = 0.3925$, it follows that trace x and trace y are swell, and the second assertion of Theorem 1.6 implies that

$$\max(d(P, x \cdot P), d(P, y \cdot P)) > 0.3925 = \mu.$$

If (8.5.1) holds (8.5.2) does not, so that $\mu = 0.3$, it follows that trace x and trace y are nifty, and the first assertion of Theorem 1.6 implies that

$$\max(d(P, x \cdot P), d(P, y \cdot P)) > 0.3 = \mu.$$

\square

9. THE CASE OF A QUADRATIC TRACE FIELD

The main result of this section is Theorem 9.3, which will easily imply Theorem 1.1 of the introduction.

Proposition 9.1. *Let μ be a positive real number, and let A be an element of $\mathrm{SL}_2(\mathbb{C})$ such that $\text{length}(\Pi(A)) \leq \mu$. Set $\text{trace } A = \xi + i\eta$. Then*

$$(9.1.1) \quad \left(\frac{\xi}{2 \cosh(\mu/2)} \right)^2 + \left(\frac{\eta}{2 \sinh(\mu/2)} \right)^2 \leq 1.$$

Proof. Set $l + i\theta = \text{Clength}(\Pi(A))$, and set $z = \exp((l + i\theta)/2)$. By (2.5.2), we have

$$\xi + i\eta = \pm 2 \cosh(\text{Clength}(\Pi(A))/2) = z + \frac{1}{z} = z + \frac{\bar{z}}{|z|^2} = z + e^{-l}\bar{z}.$$

Hence if x and y denote the real and imaginary parts of z , we have $\xi = x(1+e^{-l})$ and $\eta = y(1-e^{-l})$, so that

$$\left(\frac{\xi}{1+e^{-l}}\right)^2 + \left(\frac{\eta}{1-e^{-l}}\right)^2 = x^2 + y^2 = |z|^2 = e^l.$$

This gives

$$\left(\frac{\xi}{2 \cosh(l/2)}\right)^2 + \left(\frac{\eta}{2 \sinh(l/2)}\right)^2 = 1,$$

from which the conclusion follows. \square

Lemma 9.2. *If $\Gamma \leq \mathrm{SL}_2(\mathbb{C})$ be a cocompact discrete group, the trace field of K cannot be a subfield of \mathbf{R} .*

Proof. Since Γ is cocompact, it is non-elementary. If $K \subset \mathbf{R}$, it follows from [6, proof of Theorem 5.2.1] that Γ is conjugate to a subgroup of $\mathrm{SL}_2(\mathbf{R})$; but this contradicts cocompactness. \square

Theorem 9.3. *Let $\Gamma \leq \mathrm{SL}_2(\mathbb{C})$ be a cocompact discrete group having integral traces. Suppose that $H_1(\Gamma; \mathbb{Z}_p) = 0$ for $p = 2, 3$ and 7 , and that the trace field of Γ is a quadratic number field. Then 0.3925 is a Margulis number for $\Pi(\Gamma)$.*

Proof. Let K denote the trace field of Γ . It follows from Lemma 9.2 that K is an imaginary quadratic field. By Corollary 8.5, it suffices to prove that if γ is a non-trivial element of Γ with $\mathrm{length}(\Pi(\gamma)) \leq 0.3925$, then $\tau \doteq \mathrm{trace} \gamma \in \mathcal{O}_K$ is swell.

Let ξ and η denote the real and imaginary parts of τ . Since $\mathrm{length}(\Pi(\gamma)) \leq 0.3925$, the inequality (9.1.1) holds with $\mu = 0.3925$. In particular we have

$$(9.3.1) \quad |\eta| \leq 2 \sinh(0.3925/2) = 0.395 \dots$$

Suppose that τ is not swell. By definition this means that either τ is a unit of \mathcal{O}_K or that $\tau^2 - 2$ is a unit.

First suppose that τ is a unit. Since K is an imaginary quadratic field, we have $\tau = \pm 1$, $\tau = \pm i$, or $\tau = (1 \pm i\sqrt{3})/2$. Since γ is loxodromic we cannot have $\tau = \pm 1$. If $\tau = \pm i$, or $\tau = (1 \pm i\sqrt{3})/2$, then $|\eta| \geq 1/2$, and (9.3.1) is contradicted.

There remains the possibility that $\tau^2 - 2$ is a unit. In this case we have $\tau^2 - 2 = \pm 1$, $\tau^2 - 2 = \pm i$, or $\tau^2 - 2 = (1 \pm i\sqrt{3})/2$. If $\tau^2 - 2 = \pm 1$ then either $\tau = \pm\sqrt{3}$ or $\tau = \pm i$. Since K is an imaginary field, it cannot contain $\sqrt{3}$. If $\tau = \pm i$ then τ is a unit in K , and we have already ruled out this case. If $\tau^2 - 2 = \pm i$, so that $\tau = \pm\sqrt{2 \pm i}$, or if $\tau^2 - 2 = (1 \pm i\sqrt{3})/2$, so that $\tau = \pm((5 \pm i\sqrt{3})/2)^{1/2}$, then $|\tau|^2$ is irrational, which is a contradiction since τ is an element of an imaginary quadratic field. \square

9.4. We conclude this section with the

Proof of Theorem 1.1. Suppose that M satisfies the hypothesis of Theorem 1.1. Let us write $M = \mathbb{H}^3/\Gamma_0$ for some torsion-free discrete subgroup Γ_0 of $\mathrm{Isom}_+(\mathbb{H}^3)$. According to 2.4, Γ_0 is the isomorphic image under Π of a cocompact (and torsion-free) subgroup Γ of $\mathrm{SL}_2(\mathbb{C})$. The hypotheses of Theorem 1.1 now immediately imply those of Theorem 9.3. The latter theorem therefore implies that 0.395 is a Margulis number for $P(\Gamma) = \Gamma_0$ and hence for M . \square

10. THE CASE OF A CUBIC TRACE FIELD

The main result of this section is Theorem 10.4, which will easily imply Theorem 1.2 of the introduction.

Lemma 10.1. *Let $K \subset \mathbb{C}$ be a degree-3 extension of \mathbb{Q} such that $K \not\subset \mathbf{R}$, and let ρ be an element of \mathcal{O}_K . Suppose that both ρ and $\rho - 1$ are units in \mathcal{O}_K . Then either (i) $|\operatorname{Im}(\rho)| > 0.36$, or (ii) there is a unit $\xi \in \mathcal{O}_K$ with minimal polynomial $X^3 + X^2 + 1$, such that either $\rho = \xi^{-4}$ or $\rho = \xi^{-5} = \xi^{-4} + 1$.*

Proof. If $\rho \in \mathbb{Q}$, then since ρ and $\rho - 1$ are units in \mathcal{O}_K , we have $\{\rho, \rho - 1\} \subset \{1, -1\}$, which is impossible. Hence $\rho \notin \mathbb{Q}$. Since K has degree 3 over \mathbb{Q} it follows that $K = \mathbb{Q}(\rho)$ and hence that $\rho \notin \mathbf{R}$. In particular, if we regard K as an abstract number field, it has a complex place. Since K is cubic it cannot have more than one complex place. In particular the number of complex places is odd, and hence by [26, Lemma 2.2], K has negative discriminant.

The hypothesis of the lemma implies that $\rho_1 \doteq \rho$ and $\rho_2 \doteq 1 - \rho$ are units. Since ρ_1 and ρ_2 satisfy the unit equation $\rho_1 + \rho_2 = 1$, and lie in the cubic field K of negative discriminant, it follows from [20, Théorème 2] that either

- (1) there is a unit $\xi \in \mathcal{O}_K$ with minimal polynomial $X^3 + X^2 - 1$ such that the unordered pair $\{\rho_1, \rho_2\}$ is one of the pairs on the following list:

$$\{\xi^2, \xi^3\}, \{\xi, \xi^5\}, \{-\xi^{-1}, \xi^{-3}\}, \{-\xi, \xi^{-2}\}, \{-\xi^4, \xi^{-1}\}, \{-\xi^{-4}, \xi^{-5}\}; \text{ or}$$

- (2) there is a unit $\eta \in \mathcal{O}_K$ with minimal polynomial $X^3 + X - 1$ such that the unordered pair $\{\rho_1, \rho_2\}$ is one of the pairs on the following list:

$$\{\eta, \eta^3\}, \{-\eta^{-2}, \eta^{-3}\}, \{-\eta^2, \eta^{-1}\}.$$

In case (1), since $\rho \notin \mathbf{R}$, the root ξ of $X^3 + X^2 - 1$ is imaginary. Hence $\xi = -(0.877\dots) \pm i(0.744\dots)$. Calculating powers of ξ we find that for $-3 \leq m \leq -1$ and for $1 \leq m \leq 5$ we have $|\operatorname{Im}(\xi^m)| > 0.5$, whereas $\operatorname{Im}(\xi^{-4}) = 0.18\dots$. Note that since ξ is a root of $X^3 + X^2 - 1$, we have $\xi^{-5} = \xi^{-4} + 1$. The conclusion of the lemma follows in this case.

In case (2), since $\rho \notin \mathbf{R}$, the root η of $X^3 + X - 1$ is imaginary. Hence $\eta = -(0.34116\dots) + i(1.16154\dots)$. Calculating powers of η we find that for $m = -1$ and for $1 \leq m \leq 3$ we have $|\operatorname{Im}(\eta^m)| > 0.7$, whereas $\operatorname{Im}(\eta^{-2}) = 0.368\dots$. Note that since η is a root of $X^3 + X - 1$, we have $\xi^{-3} = x^{-2} + 1$ and hence $\operatorname{Im}(\eta^{-3}) = 0.368\dots$. Thus the conclusion of the lemma holds in this case as well. \square

The following result is analogous to Lemma 10.1.

Lemma 10.2. *Let K be a degree-3 extension of \mathbb{Q} with $K \not\subset \mathbf{R}$, and let ρ be an element of \mathcal{O}_K . Suppose that both $\rho^2 - 2$ and $\rho - 1$ are units in \mathcal{O}_K . Then either $\operatorname{Im} \rho > 0.36$, or the minimal polynomial of ρ over \mathbb{Q} is $X^3 + 3X^2 - 14X + 11$.*

The proof of Lemma 10.1 was based on the results of [20]. As I am not aware of any similar sources that can be quoted in proving Lemma 10.2, I have provided a proof from scratch. This proof, which involves some rather laborious calculations, will be found in an appendix to this paper, Section 12.

Lemma 10.3. *Let K be a degree-3 extension of \mathbb{Q} with $K \not\subset \mathbf{R}$, and let τ be an element of \mathcal{O}_K . Then one of the following alternatives holds:*

- (i) τ is nifty; or
- (ii) $\text{Im}(\tau) > 0.36$; or
- (iii) one of the elements τ , $-\tau$, $1 + \tau$ or $1 - \tau$ of \mathcal{O}_K has minimal polynomial $X^3 + 2X^2 - 3X + 1$;
or
- (iv) one of the elements τ or $-\tau$ of \mathcal{O}_K has minimal polynomial $X^3 + 3X^2 - 14X + 11$.

Proof. Suppose that Alternative (i) does not hold. Then in particular either $\tau - 1$ or $\tau + 1$ is a unit in \mathcal{O}_K . Hence there is some $\epsilon \in \{1, -1\}$ such that $\epsilon\tau - 1$ is a unit. We set $\rho = \epsilon\tau$, so that $\rho - 1$ is a unit.

The assumption that Alternative (i) does not hold also implies that either τ is a unit in \mathcal{O}_K , or $\tau^2 - 2$ is a unit in \mathcal{O}_K .

We first consider the case in which τ is a unit. Then ρ and $\rho - 1$ are units. Lemma 10.1 then implies that either $|\text{Im}(\rho)| > 0.36$, or one of the elements ρ or $\rho - 1$ has the form ξ^{-4} , where ξ is a unit in \mathcal{O}_K having minimal polynomial $X^3 + X^2 - 1$. If $|\text{Im}(\rho)| > 0.36$, then since $\rho = \pm\tau$, Alternative (ii) of the present lemma holds.

If either ρ or $\rho - 1$ has the form ξ^{-4} , where $\xi^3 + \xi^2 - 1 = 0$, we have $\xi^6 = (1 - \xi^2)^2$ and hence $\xi^6 + 2\xi^2 = \xi^4 + 1$. Squaring both sides of the latter equality and simplifying, we obtain $\xi^{12} + 3x^8 + 2x^4 - 1 = 0$. Hence $-\xi^{-4}$ is a root of the polynomial $X^3 + 2X^2 - 3X + 1$, which is irreducible by the rational root test and is therefore the minimal polynomial of $-\xi^{-4}$. The latter element is equal to either $\pm\tau$ or $1 \pm \tau$, and hence Alternative (iii) of the present lemma holds.

We now turn to the case in which $\tau^2 - 2$ is a unit in \mathcal{O}_K . In this case $\rho^2 - 1$ and $\rho - 1$ are units. By Lemma 10.2, either $\text{Im} \rho > 0.36$, or the minimal polynomial of ρ is $X^3 + 3X^2 - 14X + 11$. Since $\rho = \pm\tau$, one of the alternatives (ii) or (iv) of the present lemma holds. \square

Theorem 10.4. *Let $\Gamma \leq \text{SL}_2(\mathbb{C})$ be a discrete cocompact group having integral traces, such that $H_1(\Gamma; \mathbb{Z}_s) = 0$ for $s = 2, 3, 7$. Suppose that the trace field of Γ is a cubic field. Then 0.3 is a Margulis number for $\Pi(\Gamma)$.*

Proof. Let K denote the trace field of Γ . According to Lemma 9.2 we have $K \not\subset \mathbf{R}$.

If x and y are non-commuting elements of Γ we must show that

$$(10.4.1) \quad \max(d(P, x \cdot P), d(P, y \cdot P)) > 0.3.$$

This is trivial if either $\Pi(x)$ or $\Pi(y)$ has real translation length greater than 0.3. We therefore assume that $\text{length}(\Pi(x))$ and $\text{length}(\Pi(y))$ are both at most 0.3.

If trace x and trace y are nifty, the first assertion of Theorem 1.6 implies that (10.4.1) holds. Hence we may assume that trace x and trace y are not both nifty, and by symmetry we may assume that $\tau \doteq \text{trace } x$ is not nifty.

Thus τ does not satisfy Alternative (i) of Lemma 10.3. Since $\text{length}(\Pi(x)) \leq 0.3$, it follows from Proposition 9.1 that $|\text{Im}(\tau)| \leq 2 \sinh(0.3/2) = 0.301\dots$, so that τ does not satisfy Alternative (ii) of Lemma 10.3 either. Hence τ must satisfy one of the alternatives (iii) or (iv) of Lemma 10.3. It follows that for some $\rho \in \{\tau, -\tau\}$, either the minimal polynomial of ρ or $\rho + 1$ is $X^3 + 2X^2 - 3X + 1$, or the minimal polynomial of $1 - \rho$ is $X^3 + 3X^2 - 14X + 11$. If $1 - \rho$ has minimal polynomial $X^3 + 2X^2 - 3X - 1$, then ρ has minimal polynomial $-((1 - X)^3 + 2(1 - X)^2 - 3(1 - X) + 1) =$

$X^3 - 5X^2 + 4X - 1$. Hence the minimal polynomial of ρ is one of the polynomials

$$(10.4.2) \quad X^3 - 5X^2 + 4X - 1, \quad X^3 + 2X^2 - 3X + 1, \quad \text{or} \quad X^3 + 3X^2 - 14X + 11.$$

We set $\rho_0 = \rho$, and for each $r \geq 1$, we set $\rho_r = \text{trace}(x^{2^r})$. In particular and for any $r \geq 0$ we have $\text{trace}(x^{2^r}) = \pm \rho_r$, and hence

$$(10.4.3) \quad \rho_{r+1} = \text{trace}((x^r)^2) = (\text{trace}(x^r))^2 - 2 = \rho_r^2 - 2.$$

We claim:

10.4.4. *For each $r \geq 0$, the element ρ_r is primitive in K .*

This is clear for $r = 0$ because the three possibilities for the minimal polynomial of ρ_0 listed in (10.4.2) are all cubic. If ρ_r is primitive in K for a given r , then since $\rho_{r+1} = \rho_r^2 - 2$ by (10.4.3), and since K is a degree-3 extension K of \mathbb{Q} , the element ρ_{r+1} is also primitive in K . This proves 10.4.4.

We let $f_r(X)$ denote the minimal polynomial of ρ_r , which by 10.4.4 has degree 3. Then f_0 is one of the three polynomials listed in (10.4.2). For each $r \geq 0$ we write $f_r(X) = X^3 + b_r X^2 + c_r X + d_r$.

Note that if $r \geq 1$ is an integer such that $|d_r| \neq 1$ and $|d_{r+1}| \neq 1$, then the minimal polynomials f_r and f_{r+1} of ρ_r and ρ_{r+1} both have constant terms $\neq \pm 1$, and hence ρ_r and $\rho_r + 1$ are non-units in \mathcal{O}_K . Since $\rho_r = \text{trace}(x^{2^r})$ by definition, and since $\rho_{r+1} = \rho_r^2 - 2$ by (10.4.3), this means by definition that $\text{trace}(x^{2^r})$ is swell. This shows:

10.4.5. *If $r \geq 1$ is an integer such that $|d_r| \neq 1$ and $|d_{r+1}| \neq 1$, then $\text{trace}(x^{2^r})$ is swell.*

For any $r \geq 0$, since $0 = f_r(\rho_r) = \rho_r^3 + b_r \rho_r^2 + c_r \rho_r + d_r$, we have $\rho_r^3 + c_r \rho_r = -(b_r \rho_r^2 + d_r)$, and hence $(\rho_r^3 + c_r \rho_r)^2 = (b_r \rho_r^2 + d_r)^2$. This gives

$$\rho_r^6 + (2c_r - b_r^2)\rho_r^4 + (c_r^2 - 2b_r d_r)\rho_r^2 - d_r^2 = 0.$$

Since $\rho_r^2 = \rho_{r+1} + 2$ by (10.4.3), it follows that

$$(\rho_{r+1} + 2)^3 + (2c_r - b_r^2)(\rho_{r+1} + 2)^2 + (c_r^2 - 2b_r d_r)(\rho_{r+1} + 2) - d_r^2 = 0,$$

so that ρ_{r+1} is a root of the cubic polynomial

$$X^3 + (2c_r - b_r^2 + 6)X^2 + (c_r^2 + 8c_r - 4b_r^2 - 2b_r d_r + 12)X + (2c_r^2 + 8c_r - 4b_r d_r - 4b_r^2 - d_r^2 + 8),$$

which must therefore be its minimal polynomial $f_{r+1}(X)$. This shows that

$$(10.4.6) \quad \begin{aligned} b_{r+1} &= 2c_r - b_r^2 + 6, \\ c_{r+1} &= c_r^2 + 8c_r - 4b_r^2 - 2b_r d_r + 12, \quad \text{and} \\ d_{r+1} &= 2c_r^2 + 8c_r - 4b_r d_r - 4b_r^2 - d_r^2 + 8. \end{aligned}$$

We now claim:

10.4.7. *If $d(P, x \cdot P) \leq 0.3$, then there is an integer $k > 0$ such that $\text{trace}(x^k)$ is swell and $d(P, x^k \cdot P) < 0.401$.*

In proving 10.4.7 I will take $\text{Clength}(\Pi(x))$ (2.5) to be normalized so that its imaginary part lies in $(-\pi, \pi]$. For any positive integer m , the complex length of $\Pi(x^m)$ (not necessarily normalized) is $m \text{Clength}(\Pi(x))$. I will denote by s the distance from s to the axis of $\Pi(x)$, and for each $m \geq 1$ I will set $D_m = d(P, \Pi(x)^m \cdot P)$. The hypothesis of 10.4.7 implies that $D_1 \leq 0.3$. By the discussion in 2.5, we have $s = \omega(m \text{Clength}(\Pi(x)), D_m)$ for every $m \geq 1$.

The proof of 10.4.7 is divided into three cases corresponding to the possibilities for the polynomial f_0 listed in (10.4.2). First suppose that $f_0(X) = X^3 - 5X^2 + 4X - 1$. From successive applications of (10.4.6) we find that $d_1 = -49$ and $d_2 = 4487$. Hence by 10.4.5, $\text{trace}(x^2)$ is swell. On the other hand, since ρ is an imaginary root of $X^3 - 5X^2 + 4X - 1$, we have $\rho = (0.4602\dots) \pm i(0.182582\dots)$. By (2.5.2) we then find that $\text{Clength}(\Pi(x)) = (0.1872\dots) \pm (0.8528\dots)i\pi$. Hence, computing values of ω from the defining formula (2.5.1), and following the conventions described in 2.8, we find that

$$s = \omega(\text{Clength}(\Pi(x)), D_1) \leq \omega(\text{Clength}(\Pi(x)), 0.3) = 0.120\dots,$$

and that

$$\omega(2 \text{Clength}(\Pi(x)), 0.395) = 0.13\dots > s = \omega(2 \text{Clength}(\Pi(x)), D_2).$$

Next suppose that $f_0(X) = X^3 + 2X^2 - 3X + 1$. From successive applications of (10.4.6) we find that $d_1 = -23$ and $d_2 = 53$. Hence by 10.4.5, $\text{trace}(x^2)$ is swell. On the other hand, since ρ is an imaginary root of $X^3 + 2X^2 - 3X + 1$, we have $\rho = (0.539797\dots) \pm i(0.182582\dots)$. By (2.5.2) we then find that $\text{Clength}(\Pi(x)) = (0.18927\dots) \pm (0.8268\dots)i\pi$. Hence

$$s = \omega(\text{Clength}(\Pi(x)), D) \leq \omega(\text{Clength}(\Pi(x)), 0.3) = 0.1205\dots,$$

and

$$\omega(2 \text{Clength}(\Pi(x)), 0.401) = 0.121\dots > s = \omega(2 \text{Clength}(\Pi(x)), D_2).$$

It follows that $D_2 < 0.401$. Thus 10.4.7 holds with $k = 2$ in this case.

The remaining case of 10.4.7 is the one in which $f_0(X) = X^3 + 3X^2 - 14X + 11$. In this case, from successive applications of (10.4.6) we find that $d_2 = -2569$ and $d_3 = 6578647$. Hence by 10.4.5, $\text{trace}(x^4)$ is swell. On the other hand, since ρ is an imaginary root of $X^3 + 3X^2 - 14X + 11$, we have $\rho = (1.38068\dots) \pm i(0.05457\dots)$. By (2.5.2) we then find that $\text{Clength}(\Pi(x)) = (0.0753\dots) \pm (0.5153\dots)i\pi$. Hence

$$s = \omega(\text{Clength}(\Pi(x)), D) \leq \omega(\text{Clength}(\Pi(x)), 0.3) = 0.19\dots,$$

and

$$\omega(4 \text{Clength}(\Pi(x)), 0.32) = 0.29\dots > s = \omega(4 \text{Clength}(\Pi(x)), D_4).$$

It follows that $D_4 < 0.32$. Thus 10.4.7 holds with $k = 4$ in this case, and is now established in all cases.

We now assume that $\max(d(P, x \cdot P), d(P, y \cdot P)) \leq 0.3$. We will show that this leads to a contradiction, thereby completing the proof of the theorem. Since in particular $d(P, x \cdot P) \leq 0.3$, it follows from 10.4.7 that we may fix an integer $k > 0$ such that $\text{trace}(x^k)$ is swell and $d(P, x^k \cdot P) < 0.401$.

It follows from Lemma 6.1 that x^k and yx^ky^{-1} do not commute. In particular, x^k and y do not commute.

Since $\text{trace}(x^k)$ is swell and y does not commute with x^k , it follows from Proposition 6.3 that either x^k and $yx^{4k}y^{-1}$ are independent elements of Γ , or x^{2k} and $yx^{2k}y^{-1}$ are independent elements of Γ .

If x^k and $yx^{4k}y^{-1}$ are independent elements of Γ , then by Theorem 2.2 we have

$$(10.4.8) \quad \frac{1}{1 + \exp(d(P, x^k \cdot P))} + \frac{1}{1 + \exp(d(P, yx^{4k}y^{-1} \cdot P))} \leq 1/2.$$

On the other hand, since $d(P, x^k \cdot P) < 0.401$ and $d(p, y \cdot P) \leq 0.3$, the left side of (10.4.8) is bounded below by

$$\frac{1}{1 + \exp(0.401)} + \frac{1}{1 + \exp(2 \times 0.3 + 4 \times 0.401)} = 0.5004 \dots$$

Thus we have the required contradiction in this case.

If x^{2k} and $yx^{2k}y^{-1}$ are independent elements of Γ , then by Theorem 2.2 we have

$$(10.4.9) \quad \frac{1}{1 + \exp(d(P, x^{2k} \cdot P))} + \frac{1}{1 + \exp(d(P, yx^{2k}y^{-1} \cdot P))} \leq 1/2.$$

On the other hand, since $d(P, x^k \cdot P) < 0.401$ and $d(p, y \cdot P) \leq 0.3$, the left side of (10.4.9) is bounded below by

$$\frac{1}{1 + \exp(2 \times 0.401)} + \frac{1}{1 + \exp(2 \times 0.3 + 2 \times 0.401)} = 0.507 \dots$$

Thus we have the required contradiction in this case as well. \square

10.5. We conclude with the

Proof of Theorem 1.2. Suppose that M satisfies the hypothesis of Theorem 1.2. Let us write $M = \mathbb{H}^3/\Gamma_0$ for some torsion-free cocompact discrete subgroup Γ_0 of $\text{Isom}_+(\mathbb{H}^3)$. According to 2.4, Γ_0 is the isomorphic image under Π of a cocompact (and torsion-free) subgroup Γ of $\text{SL}_2(\mathbb{C})$. The hypotheses of Theorem 1.2 now immediately imply those of Theorem 10.4. The latter theorem therefore implies that 0.3 is a Margulis number for $P(\Gamma) = \Gamma_0$ and hence for M . \square

11. FINITENESS OF NON-NIFTY ELEMENTS AND APPLICATIONS

In this section I will prove Theorems 1.3 and 1.4, which were stated in the introduction. These will be established by combining Propositions 6.3 and 6.4 and some of the geometric ideas of Section 8 with some deep results from number theory. Theorem 11.2 summarizes the number-theoretic information that I will need.

Proposition 11.1. *Let L be a number field and let c be an element of \mathcal{O}_L . Then there are at most finitely many ordered pairs (u_1, u_2) of units in \mathcal{O}_L such that*

$$(11.1.1) \quad u_1 + u_2 = c.$$

Proof. This is easily deduced from [16, Theorem D.8.1], a result due to Siegel and Mahler. We let L play the role of K in [16, Theorem D.8.1], and we choose the finite set S of absolute values in such a way that $1/c \in R_S$; this can be done by taking S to consist of all archimedean absolute values, together with all the non-archimedean absolute values w such that $|c|_w > 1$. Now if u_1 and u_2 are units in \mathcal{O}_L satisfying (11.1.1), then $v_1 = u_1/c$ and $v_2 = u_2/c$ are units in R_S whose sum is 1. According to [16, Theorem D.8.1], there are only finitely many possibilities for v_1 and v_2 , and hence for $u_1 = cv_1$ and $u_2 = cv_2$. \square

Theorem 11.2. *If K is an arbitrary number field, \mathcal{O}_K contains at most finitely many non-nifty elements.*

Proof. If $\tau \in \mathcal{O}_K$ is not nifty, in particular it is not swell. By definition this means that either τ is a unit or $\tau^2 - 2$ is a unit. Hence it suffices to prove that (a) there are at most finitely many non-nifty units in \mathcal{O}_K , and (b) there are at most finitely many elements $\tau \in \mathcal{O}_K$ such that $\tau^2 - 2$ is a unit.

To prove (a), suppose that τ is a non-nifty unit. Since τ is non-nifty, the definition implies that either $\tau - 1$ or $\tau + 1$ is a unit. If $\tau - 1$ is a unit, then $u_1 \doteq \tau$ and $u_2 \doteq 1 - \tau$ are units satisfying (11.1.1) with $c = 1$. If $\tau + 1$ is a unit, then $u_1 \doteq -\tau$ and $u_2 \doteq 1 + \tau$ are units in \mathcal{O}_K satisfying (11.1.1) with $c = 1$. Hence (a) follows from the case of Proposition 11.1 in which $L = K$ and $c = 1$.

To prove (b), suppose that τ is an element of \mathcal{O}_K such that $\tau^2 - 2$ is a unit. Consider the number field $L \doteq K[\sqrt{2}] \supset K$. In \mathcal{O}_L we have $\tau^2 - 2 = (\tau - \sqrt{2})(\tau + \sqrt{2})$, and hence $\tau - \sqrt{2}$ and $\tau + \sqrt{2}$ are units. Hence $u_1 \doteq \sqrt{2} - \tau$ and $u_2 \doteq \sqrt{2} + \tau$ are units satisfying (11.1.1) with $c = 2\sqrt{2}$. Hence (b) follows from the case of Proposition 11.1 in which $L = K[\sqrt{2}]$ and $c = 2\sqrt{2}$. \square

Remark 11.3. Using the results of [8] one can give a bound on the number of non-nifty elements of \mathcal{O}_K which is exponential in the degree of K .

The proof of Theorem 1.3 will combine Theorem 11.2 with the following result, the proof of which will be extracted from [12].

Proposition 11.4. *Let M be a closed, orientable hyperbolic 3-manifold, and let μ be a positive real number. Let us write $M = \mathbb{H}^3/\Gamma_0$ for some torsion-free cocompact discrete subgroup Γ_0 of $\text{Isom}_+(\mathbb{H}^3)$, and let $q: \mathbb{H}^3 \rightarrow M$ denote the quotient map. Let C be a closed geodesic in M of length $l < \mu$, and let $X \leq \Gamma_0$ denote the stabilizer of $q^{-1}(C)$. Suppose that for every $\gamma \in \Gamma_0 - X$, for every $x \in X - \{1\}$ and for every $P \in \mathbb{H}^3$, we have $\max(d(P, x \cdot P), d(P, \gamma x \gamma^{-1} \cdot P)) \geq \mu$. Then there is an embedded tube about C having radius R , where R is defined by*

$$(11.4.1) \quad \sinh^2 R = \frac{\cosh \mu - \cosh \sqrt{\frac{4\pi l}{\sqrt{3}}}}{\cosh \sqrt{\frac{4\pi l}{\sqrt{3}}} - 1}.$$

Proof. The group X is cyclic and is generated by an element x_0 with translation length l . Let $L = l + i\theta$ denote its complex length. It is a lemma due to Zagier [19, p. 1045] that if l and θ are real numbers with $0 < l < \pi\sqrt{3}$ then there exists an integer $n \geq 1$ such that

$$(11.4.2) \quad \cosh nl - \cos n\theta \leq \cosh \sqrt{\frac{4\pi l}{\sqrt{3}}} - 1.$$

Let us set $x = x_0^n$, and let R be defined by (1.3.1), and let Z denote the radius- R neighborhood of A . We claim:

11.4.3. *For every $P \in Z$ we have $d(P, x \cdot P) < \mu$.*

To prove 11.4.3, let $P \in Z$ be given and set $D = \text{dist}(P, x \cdot P)$. Since A is the axis of x , we may apply 2.5, with x playing the role of γ in 2.5, to deduce that the distance from P to A is $\omega(\text{Clength } \Pi(x), D)$. Hence

$$R > \omega(\text{Clength } \Pi(x), D) = \omega(nl + i n\theta, D),$$

which in view of (2.5.1) means that

$$(11.4.4) \quad \sinh^2 R > \frac{\cosh D - \cosh nl}{\cosh nl - \cos n\theta}.$$

It follows from (11.4.2) that in the right hand side of (11.4.4), the denominator is at most $\cosh(\sqrt{4\pi l/\sqrt{3}}) - 1$, while the numerator is at least $\cosh D - \cosh(\sqrt{4\pi l/\sqrt{3}})$. Hence

$$\sinh^2 R > \frac{\cosh D - \cosh \sqrt{\frac{4\pi l}{\sqrt{3}}}}{\cosh \sqrt{\frac{4\pi l}{\sqrt{3}}} - 1},$$

which with (1.3.1) implies that $D < \mu$. This proves 11.4.3.

We are required to prove that there is an embedded tube about C having radius R . This is equivalent to showing that for every $\gamma \in \Gamma_0 - X$ we have $\gamma \cdot Z \cap Z = \emptyset$. Suppose to the contrary that $\gamma \cdot Z \cap Z$ contains a point P . Since $P \in Z$, it follows from 11.4.3 that $d(P, x \cdot P) < \mu$. Since $\gamma^{-1} \cdot P \in Z$, it follows from 11.4.3 that $d(x \cdot \gamma^{-1} \cdot P, \gamma^{-1} \cdot P) < \mu$. Hence we have $d(P, \gamma x \gamma^{-1} \cdot P) < \mu$. On the other hand, the hypothesis implies that $\max(d(P, x \cdot P), d(P, y \cdot P)) \geq \mu$. This contradiction completes the proof. \square

11.5. I will now turn to the proof of Theorem 1.3, which was stated in the introduction.

Proof of Theorem 1.3. According to Theorem 11.2, K contains at most finitely many non-nifty elements. Let τ_1, \dots, τ_N be all the non-nifty elements of $K \subset \mathbb{C}$ that do not lie in the interval $[-2, 2] \subset \mathbf{R} \subset \mathbb{C}$; here N is a non-negative integer. For each integer j with $1 \leq j \leq N$, let S_j denote the set of all complex numbers L with strictly positive real part such that $2 \cosh(L/2) = \pm \tau_j$. For each j we have $S_j \neq \emptyset$ since $\tau_j \notin [-2, 2] \subset \mathbf{R}$; and all elements of S_j have the same real part, say $l_j > 0$. We set $\epsilon = \min(0.3, l_1, \dots, l_N)$. (In particular, if $N = 0$ we have $\epsilon = 0.3$.)

Let M be any closed hyperbolic 3-manifold which has trace field K , has integral traces, and satisfies $H_1(M; \mathbb{Z}_s) = 0$ for $s = 2, 3, 7$. Let C be a primitive closed geodesic in M having length $l < \epsilon$. Let us write $M = \mathbb{H}^3/\Gamma_0$ for some torsion-free cocompact discrete subgroup Γ_0 of $\text{Isom}_+(\mathbb{H}^3)$. According to 2.4, Γ_0 is the isomorphic image under Π of a cocompact (and torsion-free) subgroup Γ of $\text{SL}_2(\mathbb{C})$. Since Γ_0 is torsion-free and cocompact it is purely loxodromic, and hence $\text{trace } \gamma \notin [-2, 2] \subset \mathbf{R}$ for any $\gamma \in \Gamma - \{1\}$. It follows that any non-nifty element of \mathcal{O}_K which is the trace of an element of $\Gamma - \{1\}$ must be among the τ_j .

For some primitive element x_0 of Γ we have $C = A/\langle \Pi(x_0) \rangle$, where A denotes the axis of $\Pi(x_0)$. Set $\tau = \text{trace } x_0$ and $L = C\text{length}(\Pi(x_0))$. According to (2.5.2) we have $\tau = \pm 2 \cosh(L/2)$. On the other hand, we have $\text{Re } L = \text{length}(\Pi(x_0)) = l < \epsilon$. Hence for any j with $1 \leq j \leq N$ we have $\text{Re } L < l_j$. If τ were non-nifty we would have $\tau = \tau_j$ for some j ; since $2 \cosh(L/2) = \pm \tau$, this would imply that $L \in S_j$ and hence that $\text{Re } L = l_j$, a contradiction. Thus:

11.5.1. *The trace of x_0 is nifty.*

The conclusion of Theorem 1.3 asserts that there is an embedded tube about C having radius R , where R is defined by (11.4.1) with $\mu = 0.3$. According to Proposition 11.4, it suffices to prove that if γ is any element of $\Gamma - \langle x_0 \rangle$, we have

$$(11.5.2) \quad \max(d(P, x \cdot P), d(P, \gamma x \gamma^{-1} \cdot P)) \geq (\log 3)/3$$

for every non-trivial element x of $\langle x_0 \rangle$. Note that since γ does not belong to $\langle x_0 \rangle$, it does not commute with x_0 .

We first prove (11.5.2) in the case where trace x_0 is swell. In this case, for each $x \in \langle x_0 \rangle$ it follows from Proposition 6.3 that either (i) x and $\gamma x^4 \gamma^{-1}$ are independent elements of Γ , or (ii) x^2 and $\gamma x^2 \gamma^{-1}$ are independent elements of Γ . In subcase (i), Theorem 2.2 gives

$$1/(1 + \exp(d(P, x \cdot P))) + 1/(1 + \exp(d(P, \gamma x^4 \cdot P))) \leq 1/2.$$

If (11.5.2) is false for some $x \in \langle x_0 \rangle$, we have $d(P, x \cdot P) < (\log 3)/3$ and $d(P, \gamma x^4 \gamma^{-1} \cdot P) < 4(\log 3)/3$. Hence

$$1/2 > 1/(1 + \exp((\log 3)/3)) + 1/(1 + \exp(4(\log 3)/3)) = 0.59\dots,$$

a contradiction. In subcase (ii), Theorem 2.2 gives

$$1/(1 + \exp(d(P, x^2 \cdot P))) + 1/(1 + \exp(d(P, \gamma x^2 \cdot P))) \leq 1/2.$$

If (11.5.2) is false for some $x \in \langle x_0 \rangle$, we have $d(P, x^2 \cdot P) < 2(\log 3)/3$ and $d(P, \gamma x^2 \gamma^{-1} \cdot P) < 2(\log 3)/3$. Hence

$$1/2 > 1/(1 + \exp(2(\log 3)/3)) + 1/(1 + \exp(2(\log 3)/3)) = 0.64\dots,$$

a contradiction.

It remains to prove (11.5.2) in the case where trace x_0 is not swell. According to 11.5.1, trace x_0 is nifty. It therefore follows from Proposition 6.4 that for each $x \in \langle x_0 \rangle$ the elements x^3 and $\gamma x^3 \gamma^{-1}$ are independent elements in Γ . If (11.5.2) is false for some $x \in \langle x_0 \rangle$, we have $d(P, x^3 \cdot P) < \log 3$ and $d(P, \gamma x^3 \gamma^{-1} \cdot P) < \log 3$. Hence

$$1/2 > 1/(1 + \exp(\log 3)) + 1/(1 + \exp(2(\log 3)/3)) = 1/2,$$

a contradiction. □

The following lemma will be needed for the proof of Theorem 1.4.

Lemma 11.6. *Let Γ be a cocompact, torsion-free subgroup of $\mathrm{SL}_2(\mathbb{C})$ having integral traces, and suppose that $H_1(\Gamma; \mathbb{Z}_p) = 0$ for $p = 2, 3$ and 7 . Let K denote the trace field of Γ . Suppose that for every pair of elements $x, y \in \Gamma$, either $\langle x, y \rangle$ has infinite index in Γ , or at least one of the elements x, y or xy has nifty trace in \mathcal{O}_K . Then 0.183 is a Margulis number for $\Pi(\Gamma) \leq \mathrm{Isom}_+(\mathbb{H}^3)$.*

Proof. We must show that if x and y are non-commuting elements of Γ , then for every point $P \in \mathbb{H}^3$ we have $\max(d(P, x \cdot P), d(P, y \cdot P)) > 0.183$. By hypothesis, either $\langle x, y \rangle$ has infinite index in Γ , or at least one of the elements x, y or xy has nifty trace in \mathcal{O}_K . If $\langle x, y \rangle$ has infinite index in Γ , then $\langle x, y, \rangle$ is a two-generator non-cocompact purely loxodromic subgroup of $\mathrm{Isom}_+(\mathbb{H}^3)$, and is therefore free. Since x and y do not commute, $\langle x, y, \rangle$ has rank 2, and so x and y are independent. By Theorem 2.2, it follows that $1/(1 + \exp(d(P, x \cdot P))) + 1/(1 + \exp(d(P, y \cdot P))) \leq 1/2$, and hence that $\max(d(P, x \cdot P), d(P, y \cdot P)) \geq \log 3$. We may therefore assume that at least one of the elements x, y or xy has nifty trace in \mathcal{O}_K . By symmetry we may assume that either x or xy has nifty trace in \mathcal{O}_K .

We define elements $u, v \in \Gamma$ as follows. If trace x is nifty we set $u = x$ and $v = y$. If trace x is not nifty (so that trace (xy) is nifty) we set $u = xy$ and $v = x^{-1}$. In either case, trace u is nifty. Furthermore, in either case, since x and y do not commute, u and v do not commute.

If trace u is swell, it follows from Proposition 6.3 that either u and vu^4v^{-1} are independent elements of Γ , or u^2 and vu^2v^{-1} are independent elements of Γ . If trace u is nifty but not swell, it follows

from Proposition 6.4 that u^3 and vu^3v^{-1} are independent elements of Γ . Thus in any case there is a pair (X, Y) in the list

$$(11.6.1) \quad (x, yx^4y^{-1}), (x^2, yx^2y^{-1}), (x^3, yx^3y^{-1}), (xy, (yx)^4), ((xy)^2, (yx)^2), ((xy)^3, (yx)^3)$$

such that X and Y are independent elements of Γ .

For each of the pairs (X, Y) in the list (11.6.1), X and Y are defined as words in x and y . In each entry in the list except $(xy, (yx)^4)$, the lengths of the words defining X and Y are both at most 6. In the case of the entry $(xy, (yx)^4)$, the lengths are 2 and 8. Hence if we set $\mu = \max(d(P, x \cdot P), d(P, y \cdot P))$, then in each case we have either

$$(11.6.2) \quad \max(d(P, X \cdot P), d(P, Y \cdot P)) \leq 6\mu$$

or

$$(11.6.3) \quad d(P, X \cdot P) \leq 2\mu \quad \text{and} \quad d(P, Y \cdot P) \leq 8\mu.$$

On the other hand, since X and Y are independent, Theorem 2.2 gives

$$\frac{1}{1 + \exp(d(P, X \cdot P))} + \frac{1}{1 + \exp(d(P, Y \cdot P))} \leq 1/2.$$

Hence if (11.6.2) holds we have

$$(11.6.4) \quad \frac{2}{1 + \exp(6\mu)} \leq 1/2,$$

and if (11.6.3) holds we have

$$(11.6.5) \quad \frac{1}{1 + \exp(2\mu)} + \frac{1}{1 + \exp(8\mu)} \leq 1/2.$$

The left hand sides of (11.6.4) and (11.6.5) are monotonically decreasing in μ , and respectively take the values 0.5002... and 0.59... when $\mu = 0.183$. Hence $\mu > 0.183$, as required. \square

11.7. I will now turn to the proof of Theorem 1.4, which was stated in the introduction.

Proof of Theorem 1.4. Suppose that there is an infinite sequence $(M_j)_{j \geq 0}$ of pairwise non-isometric closed, orientable hyperbolic 3-manifolds satisfying conditions (1)–(3) of the statement of the theorem. For each j write $M_j = \mathbb{H}^3/\Gamma_0^{(j)}$ for some torsion-free cocompact discrete subgroup $\Gamma_0^{(j)}$ of $\text{Isom}_+(\mathbb{H}^3)$. According to 2.4, $\Gamma_0^{(j)}$ is the isomorphic image under Π of a cocompact (and torsion-free) subgroup $\Gamma^{(j)}$ of $\text{SL}_2(\mathbb{C})$. Then it follows from Lemma 11.6 that for each j there exist elements $x_j, y_j \in \Gamma^{(j)}$ such that the traces of x_j, y_j and $x_j y_j$ are all non-nifty, and the subgroup $\Delta^{(j)} = \langle x_j, y_j \rangle$ has finite index in $\Gamma^{(j)}$. Set $\widetilde{M}_j = \mathbb{H}^3/\Delta^{(j)}$, so that \widetilde{M}_j is a finite-sheeted covering of M_j .

Since by Theorem 11.2 there are only finitely many non-nifty elements in \mathcal{O}_K , we may assume after passing to a subsequence that the sequences $(\text{trace } x_j)_{j \geq 0}$, $(\text{trace } y_j)_{j \geq 0}$ and $(\text{trace}(x_j y_j))_{j \geq 0}$ are constant. Thus, given any $j \geq 0$, we have $\text{trace } x_j = \text{trace } x_0$, $\text{trace } y_j = \text{trace } y_0$ and $\text{trace}(x_j y_j) = \text{trace}(x_0 y_0)$. It then follows from [24, Proposition 4.4.2] that for every word W in two letters we have $\text{trace } W(x_j, y_j) = \text{trace } W(x_0, y_0)$. According to [10, Proposition 1.5.2], this implies that for some $A_j \in \text{SL}_2(\mathbb{C})$ we have $x_j = Ax_0 A^{-1}$ and $y_j = Ay_0 A^{-1}$. In particular $\Delta_j = A\Delta_0 A^{-1}$, so that \widetilde{M}_j is isometric to \widetilde{M}_0 for each j .

Set D denote the diameter of \widetilde{M}_0 . Then since each M_j has a finite-sheeted covering isometric to \widetilde{M}_0 , each M_j has diameter at most D .

Let v denote the infimum of the volumes of all closed hyperbolic 3-manifolds; we have $v > 0$, for example by [19, Theorem 1]. Since each M_j has volume at least v , diameter at most D , constant curvature -1 and dimension 3, it follows from the main theorem of [22] that the M_j represent only finitely many diffeomorphism types. By the Mostow rigidity theorem, they represent only finitely many isometry types. This is a contradiction. \square

12. APPENDIX: PROOF OF LEMMA 10.2

We begin with a few preliminary lemmas.

Lemma 12.1. *Let K be a degree-3 extension of \mathbb{Q} , and let ρ be an element of \mathcal{O}_K . Suppose that both $\rho^2 - 2$ and $\rho - 1$ are units in \mathcal{O}_K . Let $f \in \mathbb{Q}[X]$ denote the minimal polynomial of ρ over \mathbb{Q} . Then there exist $r, s \in \mathbb{Z}$ satisfying $r^2 - 2s^2 = \pm 1$, such that either*

- (i) $f(X) = X^3 + (s - 2)X^2 - (r + s + 2)X + (r + 4)$, or
- (ii) $f(X) = X^3 + sX^2 - (r + s + 2)X + r$.

Proof. If $\rho \in \mathbb{Q}$, then since $\rho^2 - 2$ and $\rho - 1$ are units in \mathcal{O}_K , we have both $\rho - 1 = \pm 1$ and $\rho^2 - 2 = \pm 1$, which is impossible. Hence ρ is a primitive element of \mathbb{Q} , and f has degree 3. Since $\rho \in \mathcal{O}_K$ we have $f \in \mathbb{Z}[X]$. We write $f(X) = X^3 + bX^2 + cX + d$, where $b, c, d \in \mathbb{Z}$.

Set $F(X) = f(X + 1) \in \mathbb{Z}[X]$. Then F is the minimal polynomial of $\rho - 1$ over \mathbb{Q} . Since $\rho - 1$ is a unit in \mathcal{O}_K , the constant term of F is ± 1 . Thus $1 + b + c + d = f(1) = F(0) = \pm 1$. Hence:

$$(12.1.1) \quad \text{Either } b + c + d = 0 \text{ or } b + c + d = 2.$$

Next, note that since $f(\rho) = 0$, we have $\rho^3 + c\rho = -(b\rho^2 + d)$, and hence

$$0 = (\rho^3 + c\rho)^2 - (b\rho^2 + d)^2 = \rho^6 + (2c - b^2)\rho^4 + (c^2 - 2bd)\rho^2 - d^2.$$

Thus if we set

$$g(X) = X^3 + (2c - b^2)X^2 + (c^2 - 2bd)X - d^2 \in \mathbb{Z}[X],$$

then $g(\rho^2) = 0$. Now since ρ has degree 3 we cannot have $\rho^2 \in \mathbb{Q}$. Hence ρ^2 is itself a primitive element of \mathbb{Q} , and g is therefore the minimal polynomial of ρ^2 . If we set $G(X) = g(X + 2) \in \mathbb{Z}[X]$, it follows that G is the minimal polynomial of $\rho^2 - 2$ over \mathbb{Q} . Since $\rho^2 - 2$ is a unit in \mathcal{O}_K , the constant term of G is ± 1 . Thus

$$8 + 4(2c - b^2) + 2(c^2 - 2bd) - d^2 = g(2) = G(0) = \pm 1.$$

Hence:

$$(12.1.2) \quad 4(2c - b^2) + 2(c^2 - 2bd) - d^2 = -8 \pm 1.$$

According to (12.1.1) we have either $c = -(b + d)$ or $c = -(b + d + 2)$. Let us first consider the case in which $c = -(b + d)$. In this case, we have

$$\begin{aligned} 4(2c - b^2) + 2(c^2 - 2bd) - d^2 &= -8(b + d) - 4b^2 + 2(b + d)^2 - 4bd - d^2 \\ &= (d - 4)^2 - 2(b + 2)^2 - 8. \end{aligned}$$

Hence according to (12.1.2), we have

$$(d - 4)^2 - 2(b + 2)^2 = \pm 1.$$

If we set $r = d - 4$ and $s = b + 2$, it now follows that $r^2 - 2s^2 = \pm 1$ and that $f(X) = X^3 + bX^2 + cX + 1 = X^3 + (s - 2)X^2 - (r + s + 2)X + (r + 4)$. Thus Alternative (i) of the conclusion of the lemma holds in this case.

We now turn to the case in which $c = -(b + d + 2)$. In this case, we have

$$\begin{aligned} 4(2c - b^2) + 2(c^2 - 2bd) - d^2 &= -8(b + d + 1) - 4b^2 + 2(b + d + 2)^2 - 4bd - d^2 \\ &= d^2 - 2b^2 - 8 \end{aligned}$$

Hence according to (12.1.2), we have

$$d^2 - 2b^2 = \pm 1.$$

If we set $r = d$ and $s = b$, it now follows that $r^2 - 2s^2 = \pm 1$ and that $f(X) = X^3 + bX^2 + cX + d = X^3 + sX^2 - (r + s + 2)X + r$. Thus Alternative (ii) of the conclusion of the lemma holds in this case. \square

Lemma 12.2. *Let G denote the function defined for $x \neq 0$ and for all real y by*

$$(12.2.1) \quad G(x, y) = (1 + y)^2 + \frac{4}{x}(1 + y)^3 - 4y - \frac{27}{x^2}y^2 - \frac{18}{x}y(1 + y).$$

Then for every $x < 0$ and every y with $1/2 \leq |y| \leq 2$, we have $G(x, y) \geq (y - 1)^2 - 27y^2/x^2$. Furthermore, for every x with $x \geq 68$ and for every y with $1.31 \leq |y| \leq 2$, we have $G(x, y) > 0$.

Proof. The definition of $G(x, y)$ may be rewritten as

$$(12.2.2) \quad G(x, y) = (y - 1)^2 - \frac{4}{x}(1 + y)(y - \frac{1}{2})(2 - y) - \frac{27}{x^2}y^2.$$

For $-2 \leq y \leq -1/2$, and for $1/2 \leq y \leq 2$, it is clear that

$$(12.2.3) \quad (1 + y)(y - \frac{1}{2})(2 - y) \geq 0.$$

The first assertion of the lemma follows immediately from (12.2.2) and (12.2.3).

Now suppose that $x \geq 68$ and $1.31 \leq |y| \leq 2$. In this case it follows from (12.2.2) and (12.2.3) that

$$\begin{aligned} (12.2.4) \quad G(x, y) &\geq G(68, y) \\ &= \frac{1}{17}(y^3 + (\frac{31}{2} - \frac{27}{4 \times 68})y^2 - \frac{71}{2}y + 18) \\ &> H(y)/17, \end{aligned}$$

where $H(y)$ is the one-variable polynomial function defined by

$$H(y) = y^3 + (15.4)y^2 - (35.5)y + 18.$$

The critical points of H occur at -11.31 and $1.04\dots$. The former is a local maximum and the latter is a local minimum. In particular, H is monotone decreasing on the interval $[-2, -1.31]$ and monotone increasing on the interval $[1.31, 2]$. Hence for $-2 \leq y \leq -1.31$ we have $H(y) \geq H(-1.31) = 88.6\dots$, and for $1.31 \leq y \leq 2$ we have $H(y) \geq H(1.31) = 0.171\dots$. In particular we have $H(y) > 0$ whenever $1.31 \leq |y| \leq 2$. In view of (12.2.4), this establishes the second assertion of the lemma. \square

Lemma 12.3. *Let $f \in \mathbb{Z}[X]$ be given. Suppose that there exist $r, s \in \mathbb{Z}$ satisfying $r^2 - 2s^2 = \pm 1$, $|s| \geq 70$, and $|r| \geq 99$, such that either*

$$(i) \quad f(X) = X^3 + (s - 2)X^2 - (r + s + 2)X + (r + 4), \text{ or}$$

$$(ii) f(X) = X^3 + sX^2 - (r + s + 2)X + r.$$

Then the discriminant of f is positive.

Proof. Since $r^2 - 2s^2 = \pm 1$, we have $|r/s| = \sqrt{1 \pm s^{-2}}$. Since $|s| \geq 70$, it follows that

$$(12.3.1) \quad 1.41 < \sqrt{2 - 1/4900} \leq |r/s| \leq \sqrt{2 + 1/4900} < 1.42.$$

Let us write $f(X) = X^3 + bX^2 + cX + d$. If Alternative (i) of the hypothesis holds we have $b = s - 2$, $d = r + 4$ and $c = -(b + d)$. If Alternative (ii) holds we have $b = s$, $d = r$ and $c = -(b + d + 2)$.

We set $\lambda = d/b$. We claim that

$$(12.3.2) \quad 1.31 < |\lambda| < 1.6.$$

If Alternative (ii) holds then $\lambda = r/s$, and (12.3.2) follows from (12.3.1). Now suppose that Alternative (i) holds. In this case we have

$$(12.3.3) \quad \lambda = \frac{d}{b} = \frac{r+4}{s-2} = \frac{r}{s} \left(1 + \frac{4}{r}\right) \left(1 - \frac{2}{s}\right)^{-1}.$$

Since $|s| \geq 70$ and $|r| \geq 99$, it follows from (12.3.1) and (12.3.3) that

$$1.41 \left(1 + \frac{2}{70}\right)^{-1} \left(1 - \frac{4}{99}\right) < |\lambda| < 1.42 \left(1 - \frac{2}{70}\right)^{-1} \left(1 + \frac{4}{99}\right),$$

which implies (12.3.2). Thus (12.3.2) holds in all cases.

Let Δ denote the discriminant of f . We have

$$(12.3.4) \quad \Delta = b^2c^2 - 4c^3 - 4b^3d - 27d^2.$$

In the case where Alternative (i) of the hypothesis holds, (12.3.4) gives

$$\Delta = b^2(b+d)^2 + 4(b+d)^3 - 4b^3d - 27d^2 - 18bd(b+d),$$

and hence

$$(12.3.5) \quad \frac{\Delta}{b^4} = G(b, \lambda),$$

where G is the function defined by (12.2.1). Since $s \geq 70$ by the hypothesis, we have $|b| = |s - 2| \geq |s| - 2 \geq 68$ in this case. Furthermore, by (12.3.2) we have $1.31 < |\lambda| < 1.6$. Hence Lemma 12.2 implies that $G(b, \lambda) > 0$, which by (12.3.5) shows that $\Delta > 0$ in this case.

In the case where Alternative (ii) of the hypothesis holds, (12.3.4) gives

$$\Delta = b^2(b+d+2)^2 + 4(b+d+2)^3 - 4b^3d - 27d^2 - 18bd(b+d+2),$$

and hence

$$(12.3.6) \quad \frac{\Delta}{b^4} = (1 + \lambda + 2/b)^2 + \frac{4}{b}(1 + \lambda + 2/b)^3 - 4\lambda - \frac{27}{b^2}\lambda^2 - \frac{18}{b}\lambda(1 + \lambda + \frac{2}{b}).$$

Comparing (12.3.6) with the definition of the function G in (12.2.1), we find that

$$(12.3.7) \quad \frac{\Delta}{b^4} - G(b, \lambda + \frac{2}{b}) = \frac{8}{b} + \frac{27}{b^2}(\frac{4\lambda}{b} + \frac{4}{b^2}) + \frac{36}{b^2}(1 + \lambda + \frac{2}{b}).$$

Since $\lambda = d/b = r/s$ in this case, it follows from (12.3.1) that

$$(12.3.8) \quad 1.41 < |\lambda| < 1.42.$$

In particular, since $|b| = |s| \geq 70$ in this case, it follows that

$$(12.3.9) \quad 1.38 < |\lambda + 2/b| < 1.45.$$

Since in addition to (12.3.9) we have $|b| \geq 70 > 68$, it follows from Lemma 12.2 that

$$(12.3.10) \quad G(b, \lambda + 2/b) > 0 \text{ if } b \geq 0,$$

and that

$$(12.3.11) \quad G(b, \lambda + 2/b) \geq (\lambda - 1)^2 - 27\lambda^2/b^2 \text{ if } b < 0.$$

In the subcase where $b \geq 0$, so that $b \geq 70$, we find from (12.3.7) and (12.3.2) that

$$\begin{aligned} \frac{\Delta}{b^4} - G(b, \lambda + \frac{2}{b}) &> \frac{8}{b} + (\frac{108}{b^3} + \frac{36}{b^2})\lambda \\ &> \frac{1}{b}(8 - (108/70^3 + 36/70^2)(1.6)) \\ &= (7.98\dots)/b > 0. \end{aligned}$$

Since $G(b, \lambda + 2/b) > 0$ by (12.3.10), it now follows that $\Delta > 0$ in this subcase.

In the subcase where $b < 0$, (12.3.7) and (12.3.8) give

$$\begin{aligned} \frac{\Delta}{b^4} - G(b, \lambda + \frac{2}{b}) &> -\frac{8}{70} - \frac{108}{70^3} - \frac{36}{70^2}(1 + 1.42 + \frac{2}{70}) \\ &> -0.134. \end{aligned}$$

But in this subcase, (12.3.11) and (12.3.8) give

$$G(b, \lambda + 2/b) \geq (0.41)^2 - 27(1.41^2/70^2) > 0.15.$$

Hence we have $\Delta > 0$ in this subcase as well. \square

We are now ready to give the

Proof of Lemma 10.2. Let $f \in \mathbb{Q}[X]$ denote the minimal polynomial of ρ over \mathbb{Q} . According to Lemma 12.1, there exist $r, s \in \mathbb{Z}$ satisfying $r^2 - 2s^2 = \pm 1$, such that either (i) $f(X) = f_{r,s,0}(X) \doteq X^3 + (s-2)X^2 - (r+s+2)X + (r+4)$, or (ii) $f(X) = f_{r,s,2}(X) \doteq X^3 + sX^2 - (r+s+2)X + r$.

In particular f has degree 3, and hence $K = \mathbb{Q}(\rho)$. Since $K \not\subset \mathbf{R}$, we have $\rho \notin \mathbf{R}$. Thus ρ and $\bar{\rho}$ are distinct roots of f . Since f has degree 3 it must also have a real root σ .

Let Δ denote the discriminant of f . By definition we have

$$(12.3.12) \quad \Delta = (\rho - \sigma)^2(\bar{\rho} - \sigma)^2(\rho - \bar{\rho})^2 = -4 \operatorname{Im}(\rho)^2 \cdot |\rho - \sigma|^4 < 0.$$

The equation $r^2 - 2s^2 = \pm 1$ satisfied by r and s is a special case of Pell's equation. Solving it by the method given in [21, Section 7.8], we find that either (r, s) is one of the eighteen pairs

$$(12.3.13) \quad (\pm 1, 0), (\pm 3, \pm 2), (\pm 7, \pm 5), (\pm 17, \pm 12), (\pm 41, \pm 29),$$

or else we have $|r| \geq 70$ and $|s| \geq 99$. In the latter case, Lemma 12.3 asserts that $\Delta > 0$, which contradicts (12.3.12).

Hence f must be of the form $f_{r,s,0}$ or $f_{r,s,2}$ where (r, s) is one of the pairs listed in (12.3.13). Of these thirty-six possibilities for f , twenty-nine have positive discriminant and are therefore ruled out by (12.3.12). (The discriminant may be calculated by the formula (12.3.4), taking $a = s - 2$, $b = r + s + 2$ and $c = r + 4$ when $f = f_{r,s,0}$, and $a = s$, $b = r + s + 2$ and $c = r$ when $f = f_{r,s,2}$.)

The remaining possibilities for f are $f_{-1,0,0}$, $f_{-3,2,0}$, $f_{7,5,0}$, $f_{-1,0,2}$, $f_{-3,2,0}$, $f_{-3,-2,0}$, and $f_{-7,-5,2}$. The imaginary roots of f are equal to

$$\begin{aligned} & -(1.573\dots) \pm i(.368\dots) \text{ if } f = f_{-1,0,0}, \\ & -(.662\dots) \pm i(.562\dots) \text{ if } f = f_{-3,2,0}, \\ & +(1.380\dots) \pm i(.054\dots) \text{ if } f = f_{7,5,0}, \\ & -(.662\dots) \pm i(.562\dots) \text{ if } f = f_{-1,0,2}, \\ & -(1.539\dots) \pm i(.368\dots) \text{ if } f = f_{-3,2,2}, \\ & +(.303\dots) \pm i(1.435\dots) \text{ if } f = f_{-3,-2,2}, \quad \text{and} \\ & +(1.784\dots) \pm i(1.307\dots) \text{ if } f = f_{-7,-5,2}. \end{aligned}$$

Since the imaginary roots of all of these polynomials other than $f_{7,5,0}(X) = X^3 + 3X^2 - 14X + 11$ have imaginary parts of absolute value greater than 0.36, the conclusion of the lemma follows. \square

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