Writing Problems for Algebra

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Chicago Symposium, March 2009

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- Solving
- Manipulating
- Graphing
- Modeling

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What is algebra about?

- Solving ?
- Manipulating ?
- Graphing ?
- Modeling ?
- "It's where you find the x"
- What are the concepts of algebra?

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Reading Expressions

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- $$\frac{\sigma}{\sqrt{n}}$$

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Don't just do something, stand there!

- $$\int \left(\frac{y^2-1}{y}\right)^2 dy, \qquad \int \frac{t^2+t}{\sqrt{t+1}}\, dt, \qquad \int \frac{1-t}{t^2-2t}\, dt$$
- $$\int \sin^4 x\, dx, \qquad \int \sin w \cos^4 w\, dw$$
- What is the "outside function" in
$$h(x) = (f(x))^3?$$

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- The skill needed for these problems is not only manipulation, but contemplation.

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Picturing calculations

- $$\lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$
- $$\frac{d}{dx}\left(\arctan\left(\frac{1}{x}\right)\right)$$
- $$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

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Algebra skills

- Manipulative skill
- Observational skill
- Strategic competence
- Flexibility with the use of symbols
- Perception of structure
- A sense of purpose

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Do the equations have a solution? Explain how you know without solving them.

1. $\frac{x+3}{2x+5} = 1$
2. $\frac{2x+3}{2x+5} = 1$
3. $\frac{x+3}{2x+6} = 1$

Without solving them, say whether the equations in 1–6 have a positive solution, a negative solution, the solution zero, or no solution. Give a reason for your answer.

1. $7x = 5$
2. $3x + 5 = 7$
3. $3x + 7 = 5$
4. $5 - 3x = 7$
5. $3 - 5x = 7$

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In the following problems, the solution to the equation depends on the constant a. Assuming a is positive, what is the effect of increasing a on the value of the solution? Does the solution increase, decrease, or remain unchanged? Give a reason for your answer that can be understood without solving the equation.

1. $x - a = 0$.
2. $ax = 1$.
3. $ax = a$.
4. $\dfrac{x}{a} = 1$.

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A person's monthly income is $\$I$, her monthly rent is $\$R$, and her monthly food expense is $\$F$. In 1–4, say whether the two expressions have the same value. If not, say which is larger, or that there is not enough information to decide. Briefly explain your reasoning in terms of income and expenses in each case.

1. $I - R - F$ and $I - (R + F)$
2. $12(R + F)$ and $12R + 12F$
3. $I - R - F + 100$ and $I - R - (F + 100)$
4. $\dfrac{R + F}{I}$ and $\dfrac{I - R - F}{I}$

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If the tickets for a concert cost \$$p$ each, the number of people who will attend is $2500 - 80p$. Which of the following best describes the meaning of the 80 in this expression?

(i) The price of an individual ticket.

(ii) The slope of the graph of attendance against ticket price.

(iii) The price at which no one will go to the concert.

(iv) The number of people who will decide not to go if the price is raised by one dollar.

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Prices are increasing at 5% per year. What is wrong with the following statements? Correct the formula in the statement.

1. A \$6 item costs $\$(6 \cdot 1.05)^7$ in 7 years' time.
2. A \$3 item costs $\$3(0.05)^{10}$ in ten years' time.
3. The percent increase in prices over a 25-year period is $(1.05)^{25}$.
4. If time t is measured in months, then the price of a \$100 item at the end of one year is $\$100(1.05)^{12t}$.
5. If the rate at which prices increase is doubled, then the price of a \$20 object in 7 years' time is $\$20(2.10)^7$.
6. Prices change by $10 \cdot 5\% = 50\%$ over a decade.
7. Prices change by $(5/12)\%$ in one month.

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1. After a container of ice-cream has been sitting in a room for t minutes, its temperature in degrees Fahrenheit is

$$a - b2^{-t} + b,$$

where a and b are positive constants. Write this expression in a form that

- shows that the temperature is always greater than a.
- shows that the temperature is always less than $a + b$.

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increasing or decreasing as a function of R_2? $\dfrac{1}{R_1} + \dfrac{1}{R_2}$

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To convert from miles to kilometers, Abby takes the number of miles, doubles it, then subtracts 20% from the result. Renato first divides the number of miles by 5, and then multiplies the result by 8.

(a) Write an algebraic expression for each method.

(b) Use your answer to part (a) to decide if the two methods give the same answer.

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A peanut, dropped at time $t = 0$ from an upper floor of the Empire State Building, has height in feet above the ground t seconds later given by

$$h(t) = -16t^2 + 1024.$$

What does the factored form

$$h(t) = -16(t - 8)(t + 8)$$

tell us about when the peanut hits the ground?

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