DNR-Based Instruction in Mathematics as a Conceptual Framework[1]

Guershon Harel
University of California, San Diego

Lester (2005) addresses a crucial weakness of the current scientific culture in mathematics education research (MER)—the lack of attention to theory and philosophy. He indicates several major problems that contribute to this weakness, one of which is the widespread misunderstanding among researchers of what it means to adopt a theoretical or conceptual stance toward one's work. He offers a model to think about educational research in MER. The model is an adaptation of Stokes' (1997) "dynamic" model for thinking about scientific and technological research, which blends two motives: "the quest for *fundamental understanding* and *considerations of use*" (p. 465). According to this model, the essential goals of MER are to understand fundamental problems that concern the learning and teaching of mathematics and to utilize this understanding to investigate existing products and develop new ones that would potentially advance the quality of mathematics education.

Another weakness, not addressed by Lester, is that attention to mathematical content is peripheral in many current frameworks and studies in mathematics education. Perhaps the most significant contribution of mathematics education research in the last three decades is the progress our field has made in understanding the special nature of the learning—and therefore the teaching—of mathematical concepts and ideas (Thompson, 1998). The body of literature on whole number concepts and operations, rational numbers and proportional reasoning, algebra, problem solving, proof, geometric and spatial thinking produced since the 70s and into the 90s has given mathematics education research the identity as a research domain, a domain that is distinct from other related domains, such as psychology, sociology, and ethnography. In contrast, many current studies, rigorous and important in their own right as they might be, are adscititious to mathematics and the special nature of the learning and teaching of mathematics. Often, upon reading a report on such a study, one is left with the impression that the report would remain intact if each mention of "mathematics" in it is replaced by a corresponding mention of a different academic subject such as history, biology, or physics. There is a risk that, if this trend continues, MER will likely lose its identity. As Schoenfeld (2000) points out, the ultimate purpose of MER is to understand the nature of *mathematical* thinking, teaching, and learning and to use such understanding to improve *mathematics* instruction at all grade levels. A key term in Schoenfeld's statement is *mathematics*: It is the *mathematics, its unique constructs, its history, and its epistemology* that makes *mathematics education* a discipline in its own right.

DNR-based instruction in mathematics is a conceptual framework that is consistent with Lester's model, in that (a) it is a research-based framework, (b) it attempts to understand fundamental problems in mathematics learning and teaching, and (c) it utilizes this understanding to develop new potentially effective curricular products. This paper discusses the origins of *DNR,* outlines its structure and goals, and demonstrates its application in mathematics instruction. The paper is organized around five sections. The first section outlines the research

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studies from which *DNR* was conceived. The second section describes an actual mathematical lesson guided by the *DNR* framework. The goal of this section is to give the reader a background image of this framework and of its possible application in mathematics instruction. The analysis of the lesson in terms of the *DNR* framework will be presented in the fourth section, after discussing the framework in the third section. *DNR* has been discussed at length elsewhere (Harel, 2001, 2007, 2008a; 2008b; 2008c), and so the third section only presents essential elements of *DNR*, those that are needed to demonstrate its consistency with the above three characteristics. The paper concludes, in the fifth section, with a few reflection comments

1. Research-Based Framework

My research interest has revolved around two areas, the Multiplicative Conceptual Field (MCF) and Advanced Mathematical Thinking (AMT). Initially, my work in AMT focused on the learning and teaching of linear algebra. The goals of this work include: understanding students' difficulties with key linear algebra concepts; identifying essential characteristics of existing teaching approaches in linear algebra and examining their relative efficacy; and offering alternative experimentally-based approaches. Gradually, my interest in these two areas expanded to students' conceptions of justification and proof in mathematics. This was likely the result of attempts to build consistent models for students' justifications for their actions and responses to mathematical tasks. With funding from the National Science Foundation, I conducted a research project, called PUPA, whose aim was to investigate students' *p*roof *u*nderstanding, *p*roduction, and *a*ppreciation. One of the main products of the PUPA project was a taxonomy of students' conceptions of proof, called *proof schemes,* which refer to how an individual (or community) assures oneself or convinces others about the truth of a mathematical assertion. This project also included the design and implementation of instructional treatments aimed at helping students gradually modify their proof schemes toward those held and practiced in the mathematics community. This work was guided by systematic observations of students' mathematical behaviors during a series of teaching experiments, by analyses of the development of proof in the history of mathematics, and by other related research findings—my own and others'—reported in the literature (e.g., Schoenfeld, Balachef, Chazan, Hanna, Fischbein & Kedem, and Boero). Additional critical inputs in the formation of these instructional treatments were findings from my earlier studies in AMT and MCF; they included: the learning and teaching of linear algebra at the high school level and at the college level (e.g., Harel, 1985, 1989); the concept of proof held by elementary school teachers (e.g., Martin & Harel, 1989); the development of MCF concepts with middle school students (e.g., Harel, Behr, Post, & Lesh, 1992); and the mathematical knowledge of prospective and inservice elementary school teachers, with a particular focus on their knowledge of MCF concepts (e.g., Harel & Behr, 1995, Post, Harel, Behr, and Lesh, 1991). The results of these instructional treatments were largely successful in that students gradually developed mathematically mature ways of thinking, manifested in their ability to formulate conjectures and use logical deduction to reach conclusions. Together with this conceptual change, the students developed solid understanding of the subject matter taught (see, for example, Harel & Sowder, 1998, Harel, 2001, Sowder & Harel, 2003).

This success was a source of encouragement to return to my earlier effort to identify and formulate the most basic foundations, the guiding principles, of the instructional treatments I have employed in various teaching experiments over the years—an effort that began in the mid eighties (see Harel, 1985, 1989, 1990) and continued until late nineties with the conclusion of the PUPA study. The result of this effort was a conceptual framework I labeled *DNR-based instruction* (or *DNR,* for short) because of the centrality of three instructional principles in the framework: *Duality, Necessity,* and *Repeated Reasoning.* The framework implemented in the studies of the eighties was gradually refined in subsequent studies, a process that stabilized at the end of the PUPA study but is continuing to this day.

Together with the PUPA study, I began in 1997 a series of teaching experiments to study the effectiveness of *DNR-based instruction* in professional development courses for inservice mathematics teachers. The main goal of these teaching experiments was to advance *teachers*' *knowledge base*. Observations from these experiments suggest that here too the instructional treatments employed have brought about a significant change in teachers' *knowledge base*, particularly in their knowledge of mathematics, in their use of justification and proof, and in their understanding of how students learn.

In 2003 I embarked on an NSF-funded project, which aimed at systematically examining the effect of *DNR-based instruction* on the *teaching practices* of algebra teachers and on the achievement of their students. More specifically, the *DNR* project addressed the question: Will a *DNR*-based instruction be effective in developing the *knowledge base* of algebra teachers? By and large the answer to this question was found to be affirmative. Teachers made significant progress in their understanding of mathematics, how students learn mathematics, and how to teach mathematics according to how students learn it. Three observations from this study are worth highlighting: First, the development of teachers' knowledge concerning how students learn mathematics and how to teach it accordingly seems to be conditioned by the development of and reflection on their own mathematical knowledge. Second, institutional constraints (e.g., demand to cover a large number of topics and excessive attention to standardized testing) are major inhibitors for a successful implementation of *DNR-based instruction* by the teachers who participated in the *DNR* project. Third, even intensive professional development spanning a two-year period is not sufficient to prepare teachers to be autonomous in altering their current curricula to be consistent with *DNR*.

In all, the formation of *DNR-based instruction in mathematics* has been impacted by various experiences, formal and informal. The formal experiences comprise a series of teaching experiments in elementary, secondary, and undergraduate mathematics courses, as well as teaching experiments in professional development courses for teachers in each of these levels. One of the most valuable lessons from all these experiences is the realization that indeed, as Piaget claimed, learning is adaptation—it is a process alternating between assimilation and accommodation directed toward a (temporary) equilibrium, a balance between the structure of the mind and the environment. This view of learning is central in the conceptual framework presented here, and is present in all of my reports on these teaching experiments and teaching experiences.

2. *DNR*-Based Lesson

This section outlines an actual mathematical lesson guided by the *DNR* framework. The goal is to give the reader a concrete background image of this framework and of its application in mathematics instruction. The lesson was conducted several times, both with in-service secondary teachers in professional development institutes and with prospective secondary teachers in an elective class in their major. In the discussion of this lesson, all learners are referred to as students. The lesson will be described as a sequence of four segments of students' responses and teacher's actions. Each segment is further divided into fragments to allow for reference in the analysis that follows. For ease of reference, the fragments are numbered independently of the segments in which they occur. The first segment, Segment 0, describes the problem around which the lesson was organized. For this section to have its intended effect, the reader is strongly encouraged to solve this problem before proceeding to read the subsequent segments.

Segment 0: The Problem

1. The students were asked to work in small groups or individually (their choice) on the following problem:

   A farmer owns a rectangular piece of land. The land is divided into four rectangular pieces, known as Region A, Region B, Region C, and Region D, as in Figure 1.

A	B
C	D

Figure 1: The Rectangular Land

   One day the farmer's daughter, Nancy, asked him, what is the area of our land? The father replied: I will only tell you that the area of Region B is 200 m^2 larger than the area of Region A; the area of Region C is 400 m^2 larger than the area of Region B; and the area of Region D is 800 m^2 larger than area of Region C. What answer to her question will Nancy derive from her father's statement?

Segment I: **Students' Initial Conclusion**

2. All students translated the farmer's statement into a system of equations similar to the following (where A, B, C, and D represent the areas of Regions A, B, C, and D, respectively).

$$\begin{cases} B = A + 200 \\ C = B + 400 \\ D = C + 800 \end{cases}$$

3. Some constructed the following fourth equation by adding the previous three equations and then tried to solve the system by substitution or elimination of variables.

$$B + C + D = (A + 200) + (B + 400) + (C + 800)$$

4. Their declared intention was to have a fourth equation to correspond to the four unknowns, A, B, C, and D. They manipulated the four equations in hope to determine a unique value of each unknown.
5. The teacher asked the different groups to present their solutions. After the first presentation, the class' conclusion was rather uniform: Namely, Nancy cannot determine the areas of the land on the basis of her father's statement because "there isn't enough information." A further discussion, led by the teacher, of the implication of this conclusion led to the following consensus:

   $Total\ Area = 4A + 2200$. Hence, there are infinitely many values for the area of the land, each is dependent on the choice of a value of A.

Segment II: **Necessitating an Examination of the Initial Conclusion**

6. Building on this shared understanding, the teacher presented the students with a new task:

   Offer *two* values for the total area. For each value, construct a figure that illustrates your solution.

7. At first, the students just offered two values for the total area, which they obtained by substituting two random values for A in the function, $Total\ Area = 4A + 2200$. After some classroom discussion, the students realized that this is not enough: one must show that the areas of the four regions entailed from a choice of A must be so that the given geometric configuration of the regions, A, B, C, and D, is preserved.
8. The students seemed confident that this can be achieved easily. Their plan was: Select any (positive) value for the length of A and determine the width of this region from its area. Now, repeat the same process for B, C, and D.

Segment III: **The Examination and Its Outcomes**

9. The teacher set to pursue this approach with the entire class: He asked for a value for A and one of the dimensions of Region A. The following is an outline of the classroom exchange that took place. One of the students offered to take $A = 100$ and the length of A to be 5 (units for these values were included in the class discussion but for simplicity they are omitted in this presentation). Accordingly, the areas of the other three regions were determined from the above system of equations to be: $B = 300$, $C = 700$, $D = 1500$. Following this, the dimensions of each region was accordingly determined. The teacher recorded the results on the blackboard as shown in Figure 2: [The parenthetical letters in this and in the figures that follow represent the order in which the dimensions of the regions were determined by the students. For example, as shown in Figure 2, the students began with *A=100* and, accordingly, determined from the set of the three equations, *B=300, C=700,* and *D=1500.* Following this they (a) offered the length of A to be 5, which they used to determine (b) the width of A to be *20.* This led (c) the length of B to be *15* and, in turn (d) the width of C to be 140.]

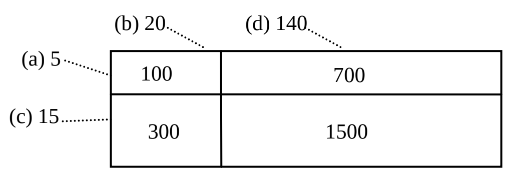

Figure 2: A Trial with Integer Dimensions.

The students immediately realized that 5 cannot be the length of Region A since $15 \times 140 \neq 1500$, so they set to try a different value. They chose 10, but it, too, was found to be invalid since $30 \times 70 \neq 1500$ (Figure 3).

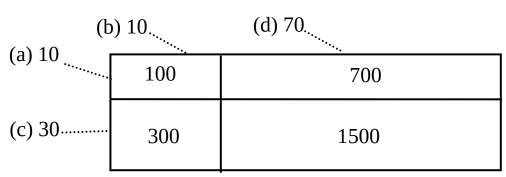

Figure 3: Another Trial with Integer Dimensions

10. This trial and error process of varying values for the length of A and determining the dimensions of the four regions from the corresponding areas continued for some time. The variation of values, however, remained within whole numbers.

11. The teacher indicated this fact to the students, which prompted them to offer fractional values and later irrational values. The following two figures (Figures 4 and 5) depict these exchanges (with the value 2/3 and √2, respectively).

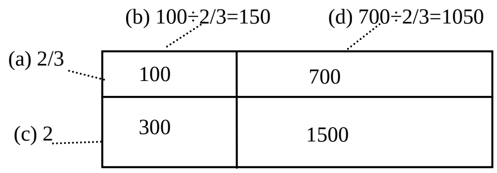

Figure 4: A Trial with Rational Dimensions.

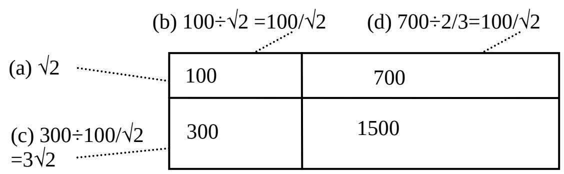

Figure 5: A Trial with Irrational Dimensions.

12. After these repeated attempts, some students expressed doubt as to whether dimensions for the four regions that preserve the given configuration can be found. The teacher responded by recapitulating the solution process carried out thus far, and concluded with the following question, which he wrote on the board:
    Can a figure representing the problem conditions be constructed for $A = 100$?
    The students were asked to work on this question in their small working groups.
13. A few minutes later, one of the groups suggested searching for the length of A by substituting a variable t for it. The teacher followed up on this suggestion. An outline of the exchange that ensued follows (Figure 6).

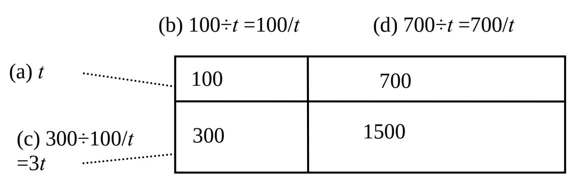

Figure 6: Dimensions Represented by Algebraic Expressions Involving a Variable.

We are looking for positive t for which:

$$(3t)\cdot\frac{700}{t} = 1500$$

Clearly, no such t exists.

Hence, the figure is not constructible for $A = 100$.

14. The students responded to this result by saying something to the effect: It didn't work for $A = 100$, so let's try a different value.
15. The last response ("let's try a different value") was against the teacher's expectation, because, on the one hand, the students' earlier conclusion was that for *any* (positive)

value of A, the total area is determinable (by the function: $\textit{Total Area} = 4A + 2200$) and, on the other hand, their last conclusion is that for the value $A = 100$ the figure is not constructible, and, hence, the total area is not determinable.[2] In addition, the students did seem to notice that the area of Region D is constant in all the cases they examined.

16. Instead of raising these issues with the students, the teacher decided to let the class pursue their approach in their small working groups after, again, recapitulating in general terms what has been achieved this far by the class, concluding with the following statement, which he wrote on the board:

    The figure cannot be constructed for $A = 100$. We will be looking for a value of A different from 100 for which the figure is constructible.

17. The working groups varied in their approaches. Some set out by taking A as a variable to be determined; others chose a particular value for A different from 100 and, as before, substituted different values for its dimension and, accordingly, computed the dimensions of the four regions. However, after some time all the groups were pursuing the first approach. At this point, the teacher resumed a public discussion. An outline of the main elements of the exchange that issued follows (Figure 7).

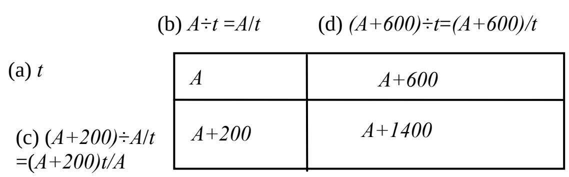

Figure 7: Dimensions Represented by Algebraic Expressions Involving a Parameter

We are looking for positive number A and t for which:

$$\frac{(A+600)}{t} \cdot \frac{(A+200)t}{A} = A + 1400$$

Solving:

$$\frac{(A+600)}{\cancel{t}} \cdot \frac{(A+200)\cancel{t}}{A} = A + 1400$$

$$A^2 + 800A + 120000 = A^2 + 1400A$$

$$\cancel{A^2} + 800A + 120000 = \cancel{A^2} + 1400A$$

$$120000 = 600A$$

$$A = 200$$

Conclusion: The figure is constructible only for $A = 200$.

Segment IV: Lesson(s) Learned

18. The teacher turned to the class with the question:

    Why was the system of equations (in Segment 1) insufficient to solve the problem?

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[2] This behavior may suggest a weak understanding of the concept of function as an input-output process. However, since this issue was not pursued in the lesson, it will not be discussed in the lesson's analysis.

It took some discussion for the students to understand this question. The discussion revolved around the meaning of equation, system of equations, solution set, and solution process. An outcome of this discussion relevant to the question at hand was that the original system of equations yielded an infinite number of solutions, but only one solution exists ($A = 200$), and, hence, there must have been a condition in the problem statement not represented by the system. The question then was: What is that condition?

19. After some further class discussion, one of the working groups suggested that the constraint of the geometric configuration of the land is not represented by this system of equations. That is, the system only represents the relationship between the areas of the four regions, not the constraint that each two neighboring regions share a common side. The regions could be scattered as in Figure 8, in which case there are infinitely many values for the area of the land.

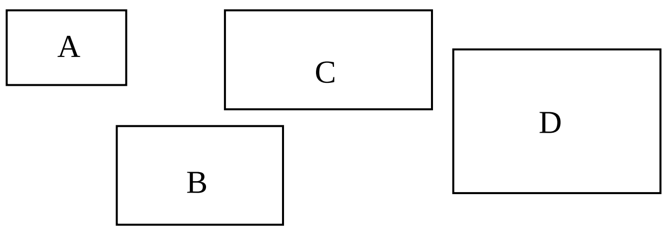

Figure 8: Scattered Regions

20. The lesson ended with the following homework problems:
    1. Your initial system needed an additional equation to represent the geometric condition about the equal dimensions of adjacent regions. Find such an equation and show that your new system has a unique solution.
    2. We found that the area of the land is 200 m^2. What is the perimeter of the rectangular land?
    3. Is the perimeter of the land unique? If not, is there a smallest value or a largest value for the perimeter? If there is a smallest value or a largest value, is it unique?
    4. When you become a teacher, will you offer this problem to your classes? If you were to teach this problem, what would you hope for the students to learn from it? Be specific.
    5. Will you offer this problem in classes that have not been exposed to systems of equations?

In Section 4 we return to analyze this lesson in terms of the *DNR* framework. The questions addressed in this analysis will include: What are the instructional objectives of the lesson? What are the instructional principles that guided the teacher's moves? What is the nature of the mathematics that the students seem to have learned as a result of these moves?

3. *DNR* Structure[3]

Lester (2005) defines a *research framework* as

> … a basic structure of the ideas (i.e., abstractions and relationships) that serve as the basis for a phenomenon that is to be investigated. These abstractions and the (assumed) interrelationships among them represent the relevant features of the phenomenon as

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[3] This section is an abridged and modified version of several sections in the papers Harel (2008a, 2008b, 2008c).

> determined by the research perspective that has been adopted. The abstractions and interrelationships are then used as the basis and justification for all aspects of the research. (p. 458)

Following Eisenhart (1991), Lester also points out that

> … conceptual frameworks are built from an array of current and possibly far-ranging sources. The framework used may be based on different theories and various aspects of practitioner knowledge, depending on what the researcher can argue will be relevant and important to address about a research problem. (p. 460)

In the macro level, the phenomena *DNR* aims at are two fundamental questions: (a) what is the mathematics that should be taught in school and (b) how should that mathematics be taught? The basic structure of the *DNR* ideas that serve as the basis the formulation and investigation of these questions and their instantiations is a system consisting of three categories of constructs:

1. *Premises*—explicit assumptions underlying the *DNR* concepts and claims.
2. *Concepts*—referred to as *DNR determinants.*
3. *Instructional principles*—claims about the potential effect of teaching actions on student learning.

In the rest of this section, these three constructs are discussed in this order.

3.1 Premises

DNR is based on a set of eight premises, seven of which are taken from or based on known theories. The premises are loosely organized in four categories:

1. Mathematics

- ***Mathematics:*** Knowledge of mathematics consists of all *ways of understanding* and *ways of thinking* that have been institutionalized throughout history (Harel, 2008a)

2. Learning

- ***Epistemophilia****:* Humans—all humans—possess the capacity to develop a desire to be puzzled and to learn to carry out mental acts to solve the puzzles they create. Individual differences in this capacity, though present, do not reflect innate capacities that cannot be modified through adequate experience. (Aristotle, see Lawson-Tancred, 1998)
- ***Knowing:*** Knowing is a developmental process that proceeds through a continual tension between assimilation and accommodation, directed toward a (temporary) equilibrium (Piaget, 1985).
- ***Knowing-Knowledge Linkage****:* Any piece of knowledge humans know is an outcome of their resolution of a problematic situation (Piaget, 1985, Brousseau, 1997).
- ***Context Dependency:*** Learning is context dependent.

3. Teaching

- ***Teaching:*** Learning mathematics is not spontaneous. There will always be a difference between what one can do under expert guidance or in collaboration with more capable peers and what he or she can do without guidance (Vygotsky's, 1978)

4. Ontology

- ***Subjectivity:*** Any observations humans claim to have made is due to what their mental structure attributes to their environment (Piaget's constructivism theory, see, for example, von Glasersfeld, 1983; information processing theories, see, for example, Chiesi, Spilich, & Voss, 1979; Davis, 1984)
- ***Interdependency****:* Humans' actions are induced and governed by their views of the world, and, conversely, their views of the world are formed by their actions.

These premises—with the exception of the Mathematics Premise, which is discussed in length in Harel, 2008a—are taken from or based on known theories, as the corresponding references for each premise indicate. As a conceptual framework for the ***learning*** and ***teaching*** of ***mathematics,*** *DNR* needs lenses through which to see the realities of the different actors involved in these human activities—mathematicians, students, teachers, school administrators. In addition, *DNR* needs a stance on the nature of the targeted knowledge to be taught—mathematics—and of the learning and teaching of this knowledge.

Starting from the end of the premises list, the two Ontology Premises—Subjectivity and Interdependency—orient our interpretations of the actions and views of students and teachers. The Epistemophilia Premise is about humans' propensity to know, as is suggested by the term "epistemophilia:" love of episteme. Not only do humans desire to solve puzzles in order to construct and impact their physical and intellectual environment, but also they seek to be puzzled.[4] The Epistemophilia Premise also claims that *all* humans are capable of learning if they are given the opportunity to be puzzled, create puzzles, and solve puzzles. While it assumes that the propensity to learn is innate, it rejects the view that individual differences reflect innate basic capacities that cannot be modified by adequate experience.

The Knowing Premise is about the mechanism of knowing: that the means—the only means—of knowing is a process of assimilation and accommodation. A failure to assimilate results in a disequilibrium, which, in turn, leads the mental system to seek equilibrium, that is, to reach a balance between the structure of the mind and the environment.

The Context Dependency Premise is about contextualization of learning. The premise does not claim that learning is entirely dependent on context—that knowledge acquired in one context is not transferrable to another context, as some scholars (Lave, 1988) seem to suggest. Instead, the Context Dependency Premise holds that ways of thinking belonging to a particular domain are best learned in the context and content of that domain. Context dependency exists even within sub-disciplines of mathematics, in that each mathematical content area is characterized by a unique set of ways of thinking (and ways of understanding).

The Teaching Premise asserts that expert guidance is indispensible in facilitating learning of mathematical knowledge. This premise is particularly needed in a framework oriented within a constructivist perspective, like *DNR,* because one might minimize the role of expert guidance in learning by (incorrectly) inferring from such a perspective that individuals are responsible for their own learning or that learning can proceed naturally and without much intervention (see, for example, Lerman, 2000). The Teaching Premise rejects this claim, and, after Vygotsky, insists that expert guidance in acquiring scientific knowledge—mathematics, in our case—is indispensable to facilitate learning.

Finally, the Mathematics Premise comprises its own category; it concerns the nature of the mathematics knowledge—the targeted domain of knowledge to be taught—by stipulating that *ways of understanding* and *ways of thinking* are the constituent elements of this discipline, and therefore instructional objectives must be formulated in terms of both these elements, not only in terms of the former, as currently is largely the case, as we will now explain.

[4] The term "puzzle" should be interpreted broadly: it refers to problems intrinsic to an individual or community, not only to recreational problems, as the term is commonly used. Such problems are not restricted to a particular category of knowledge, though here we are solely interested in the domain of mathematics.

3.2 Concepts

This section focuses mainly on two central concepts of *DNR: way of understanding* and *way of thinking.* As was explained in Harel (2008a), these are fundamental concepts in *DNR,* in that they define the mathematics that should be taught in school.

Judging from current textbooks and years of classroom observations, teachers at all grade levels, including college instructors, tend to view mathematics in terms of “subject matter,” such as definitions, theorems, proofs, problems and their solutions, and so on, not in terms of the “conceptual tools” that are necessary to construct such mathematical objects. Undoubtedly, knowledge of and focus on subject matter is indispensable for quality teaching; however, it is not sufficient. Teachers should also concentrate on conceptual tools such as problem-solving approaches, believes about mathematics, and proof schemes.

What exactly are these two categories of knowledge? To define them, it would be helpful to first explain their origin in my earlier work on proof. In Harel and Sowder (1998; 2007), proving is defined as the *mental act* a person (or community) employs to remove doubts about the truth of an assertion. The proving act is instantiated by one of two acts, *ascertaining* and *persuading*, or by a combination thereof. *Ascertaining* is the act an individual employs to remove her or his own doubts about the truth of an assertion, whereas *persuading* is the act an individual employs to remove others’ doubts about the truth of an assertion. A *proof* is the particular argument one produces to ascertain for oneself or to convince others that an assertion is true, whereas a *proof scheme* is a collective cognitive characteristic of the proofs one produces. For example, when asked why 2 is an upper bound for the sequence, $\sqrt{2}$, $\sqrt{2+\sqrt{2}}$, $\sqrt{2+\sqrt{2+\sqrt{2}}}$, $\ldots$, some undergraduate students produced the proof: “$\sqrt{2}=1.41$, $\sqrt{2+\sqrt{2}}=1.84$, $\sqrt{2+\sqrt{2+\sqrt{2}}}=1.96$ [five more items of the sequence were evaluated] we see that [the results] are always less than 2, … Hence, all items of the sequence are less than 2.” Other students produced the proof: “Clearly, $\sqrt{2}$ is less than 2. The second item is less than 2 because it is the square root of a number that is smaller than 4, this number being the sum of 2 and a number that is smaller than 2. The same relationship exists between any two consecutive terms in the sequence.” These two proofs are products resulting from carrying out the proving act, either in the form of ascertainment or persuasion. They may suggest certain persistent characteristics of these students’ act of proving. For example, on the basis of additional observations of proofs produced by these two groups of students, we may characterize the proving act of the first group as empirical and that of the second group as deductive, if the respective proofs they produce are similar in nature to the ones presented here. Thus, we have here a triad of concepts: *proving act, proof,* and *proof scheme*. A *proof* is a cognitive product of the proving act, and *proof scheme* is a cognitive characteristic of that act. Such a characteristic is a common property among one’s proofs. Based on students’ work and historical development, Harel and Sowder (1998) offered a taxonomy of proof schemes consisting of three classes: External Conviction, Empirical, and Deductive.

As I engaged deeply in the investigation of students’ conceptions of proof, I came to realize that while the triad, *proving, proof,* and *proof scheme*, is useful, even critical, to understanding the processes of learning and teaching mathematical proof, it is insufficient to document and communicate clinical and classroom observations. This is so because proving itself is never carried out in isolation from other mental acts, such as “interpreting,” “connecting,” “modeling,” “generalizing,” “symbolizing,” etc. As with the act of proving, we often wish to talk about the products and characteristics of such acts. Thus, the following definitions:

> A person's statements and actions may signify cognitive products of a mental act carried out by the person. Such a product is the person's *way of understanding* associated with that mental act. Repeated observations of one's ways of understanding may reveal that they share a common cognitive characteristic. Such a characteristic is referred to as a *way of thinking* associated with that mental act.

It is clear from these definitions that a proof is a way of understanding, whereas a proof scheme is a way of thinking. Likewise, in relation to the mental act of interpreting, for example, a particular interpretation one gives to a term, a statement, or a string of symbols is a way of understanding, whereas a cognitive characteristic of one's interpretations is a way of thinking associated with the interpreting act. For example, one's ways of understanding the string $y = -3x + 5$ may be: (a) an equation—a constraint on the quantities, x and y; (b) a number-valued function—for an input x, there corresponds the output $y = -3x + 5$; (c) a truth-valued function—for an input (x, y), there corresponds the output "True" or "False"; and (d) "a thing where what you do on the left you do on the right." The first three ways of understanding suggest a mature way of thinking: that "symbols in mathematics represent quantities and quantitative relationships." On the other hand, the fourth way of understanding, which was provided by a college freshman, is likely to suggest *a non-referential symbolic* way of thinking—a way of thinking where mathematical symbols are free of coherent quantitative or relational meaning. Other examples of ways of understanding and ways of thinking will emerge as the paper unfolds.

Mathematicians, the practitioners of the discipline of mathematics, practice mathematics by carrying out mental acts with particular characteristics—ways of thinking—to produce particular constructs—ways of understanding. Accordingly, in *DNR*, mathematics is defined as a discipline consisting of these two sets of knowledge. Specifically:

> Mathematics is a union of two sets: The first set is a collection, or structure, of structures consisting of particular axioms, definitions, theorems, proofs, problems, and solutions. This subset consists of all the *institutionalized*[5] ways of understanding in mathematics throughout history. The second set consists of all the ways of thinking that are characteristics of the mental acts whose products comprise the first set.

The main pedagogical implication of this definition is that mathematics curricula at all grade levels, including curricula for teachers, should be thought of in terms of the constituent elements of mathematics—ways of understanding and ways of thinking—not only in terms of the former, as currently is largely the case.

There is also an important implication for research in mathematics education concerning ways of thinking. Humans' reasoning involves numerous mental acts such as interpreting, conjecturing, inferring, proving, explaining, structuring, generalizing, applying, predicting, classifying, searching, and problem solving. Humans perform such mental acts, and they perform them in every domain of life, not just in science and mathematics. Although all the aforementioned examples of mental acts are important in the learning and creation of mathematics, they are not unique to mathematics—people interpret, conjecture, justify, abstract, solve problems, etc. in every area of their everyday and professional life. Professionals from

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[5] *Institutionalized* ways of understanding are those the mathematics community at large accepts as correct and useful in solving mathematical and scientific problems. A subject matter of particular field may be viewed as a structure of institutionalized ways of understanding.

different disciplines are likely to differ in the extent they carry out certain mental acts; for example, a painter is likely to abstract more often than a carpenter, a chemist to model more often than a pure mathematician, and the latter to conjecture and justify more often than a pianist. But a more interesting and critical difference among these professionals is the ways of thinking associated with mental acts they perform. A biologist, chemist, physicist, and mathematician all carry out problem-solving acts in every step in their professional activities and attempt to justify any assertions they make. The four, however, are likely to differ in the problem-solving approaches and in the nature of their justifications. Hence, an important goal of research in mathematics education is to identify these ways of thinking and recognize, when possible, their development in learners and in the history of mathematics, and, accordingly, develop and implement mathematics curricula that target them.

3.4 Instructional Principles

This section discusses *DNR'*s three foundational instructional principles, *duality, necessity,* and *repeated reasoning,* in this order.

The *Duality Principle*. This principle asserts:

1. Students develop ways of thinking through the production of ways of understanding, and, conversely,
2. The ways of understanding they produce are impacted by the ways of thinking they possess.

Students do not come to school as blank slates, ready to acquire knowledge independently of what they already know. Rather, what students know now constitutes a basis for what they will know in the future. This is true for all ways of understanding and ways of thinking associated with any mental act; the mental act of proving is no exception. In everyday life and in science, the means of justification available to humans are largely limited to empirical evidence. Since early childhood, when we seek to justify or account for a particular phenomenon, we are likely to base our judgment on similar or related phenomena in our past (Anderson, 1980). Given that the number of such phenomena in our past is finite, our judgments are typically empirical. Through such repeated experience, which begins in early childhood, our hypothesis evaluation becomes dominantly empirical; that is, the proofs that we produce to ascertain for ourselves or to persuade others become characteristically inductive or perceptual. If, during early grades, our judgment of truth in mathematics continues to rely on empirical considerations, the empirical proof scheme will likely dominate our reasoning in later grades and more advanced classes, as research findings clearly show (Harel & Sowder, 2007). While unavoidable, the extent of the dominance of the empirical proof scheme on people is not uniform. Children who are raised in an environment where sense making is encouraged and debate and argumentation are an integral part of their social interaction with adults are likely to have a smoother transition to deductive reasoning than those who are not raised in such an environment.

A simple, yet key, observation here is this: the arguments children produce to prove assertions and account for phenomena in everyday life impact the kind and robustness of the proof schemes they form. Proofs, as was explained earlier, are ways of understanding associated with the mental act of proving, and proof schemes are ways of thinking associated with the same act. Hence, a generalization of this observation is: for any mental act, the ways of understanding one produces impact the quality of the ways of thinking one forms.

Of equal importance is the converse of this statement; namely: for any mental act, the ways of thinking one has formed impact the quality of the ways of understanding one produces.

The latter statement is supported by observations of students' mathematical behaviors, for example, when proving. As was indicated earlier, the empirical proof scheme does not disappear upon entering school, nor does it fade away effortlessly when students take mathematics classes. Rather, it continues to impact the proofs students produce.

This analysis points to a reciprocal developmental relationship between ways of understanding and ways of thinking, which is expressed in the *Duality Principle*. The principle is implied from the Interdependency Premise. To see this, one only needs to recognize that a person's ways of thinking are part of her or his view of the world, and that a person's ways of understanding are manifestations of her or his actions. Specifically, the statement, ways of understanding students produce are impacted by the ways of thinking they possess, is an instantiation of the premise's assertion that humans' actions are induced and governed by their views of the world, whereas the statement, students develop ways of thinking through the production of ways of understanding, is an instantiation of the premise's assertion that humans' views of the world are formed by their actions. Furthermore, the Context Dependency Premise adds a qualification to this statement: ways of thinking belonging to a particular discipline best develop from or are impacted by ways of understanding belonging to the same discipline.

The *Necessity Principle*. This principle asserts:

> For students to learn the mathematics we intend to teach them, they must have a need for it, where 'need' here refers to intellectual need.

There is a lack of attention to students' intellectual need in mathematics curricula at all grade levels. Consider the following two examples: After learning how to multiply polynomials, high-school students typically learn techniques for factoring (certain) polynomials. Following this, they learn how to apply these techniques to simplify rational expressions. From the students' perspective, these activities are intellectually purposeless. Students learn to transform one form of expression into another form of expression without understanding the mathematical purpose such transformations serve and the circumstances under which one form of expression is more advantageous than another. A case in point is the way the quadratic formula is taught. Some algebra textbooks present the quadratic formula before the method of completing the square. Seldom do students see an intellectual purpose for the latter method—to solve quadratic equations and to derive a formula for their solutions—rendering completing the square problems alien to most students (see Harel, 2008a for a discussion on a related way of thinking: *algebraic invariance*). Likewise, linear algebra textbooks typically introduce the pivotal concepts of "eigenvalue," "eigenvector," and "matrix diagonalization" with statements such as the following: "The concepts of "eigenvalue" and "eigenvector" are needed to deal with the problem of factoring an $n \times n$ matrix A into a product of the form XDX^{-1}, where D is diagonal. The latter factorization would provide important information about A, such as its rank and determinant." Such introductory statements aim at pointing out to the student an important problem. While the problem is intellectually intrinsic to its poser (a university instructor), it is likely to be alien to the students because a regular undergraduate student in an elementary linear algebra course is unlikely to realize from such statements the nature of the problem indicated, its mathematical importance, and the role the concepts to be taught ("eigenvalue," "eigenvector," and "diagonalization") play in determining its solution. What these two examples demonstrate is that the intellectual need element in (the *DNR* definition of) learning is largely ignored in teaching. The Necessity Principle attends to the indispensability of intellectual need in learning:

The *Repeated Reasoning Principle*. This principle asserts:

Students must practice reasoning in order to internalize desirable ways of understanding and ways of thinking.

Even if ways of understanding and ways of thinking are intellectually necessitated for students, teachers must still ensure that their students internalize, retain, and organize this knowledge. Repeated experience, or practice, is a critical factor in achieving this goal, as the following studies show: Cooper (1991) demonstrated the role of practice in organizing knowledge; and DeGroot (1965) concluded that increasing experience has the effect that knowledge becomes more readily accessible: "[knowledge] which, at earlier stages, had to be abstracted, or even inferred, [is] apt to be immediately perceived at later stages." (p. 33-34). Repeated experience results in fluency, or effortless processing, which places fewer demands on conscious attention. "Since the amount of information a person can attend to at any one time is limited (Miller, 1956), ease of processing some aspects of a task gives a person more capacity to attend to other aspects of the task (LaBerge and Samuels, 1974; Schneider and Shiffrin, 1977; Anderson, 1982; Lesgold et al., 1988)" (quote from Bransford, Brown, & Cocking, p. 32). The emphasis of *DNR-based instruction* is on repeated reasoning that reinforces desirable ways of understanding and ways of thinking. Repeated reasoning, not mere drill and practice of routine problems, is essential to the process of internalization, where one is able to apply knowledge autonomously and spontaneously. The sequence of problems given to students must continually call for thinking through the situations and solutions, and problems must respond to the students' changing intellectual needs. This is the basis for the *repeated reasoning principle*.

4 Analysis of the Lesson

In this section we return to the lesson presented in Section 2 and discuss how the design and implementation of this lesson was guided by the *DNR* framework. It should be pointed out at the outset that the accounts given here are based on the teacher's own retrospective notes taken immediately after the lesson, combined with an external observer's notes taken during the lesson. Thus, claims made in these accounts about students' conceptualizations, actions, and reactions should not be held in the same standards of evidence required from a formal teaching experiment. Rather, these accounts should be viewed in the spirit of Steffe & Thompson's (2000) notion of *exploratory teaching experiment*, in that they are the teacher's "on the fly" construction of temporal models for the students' ways of understanding and ways of thinking needed to inform his teaching actions during the lesson.

The lesson described in Section 2 is one in a series of lessons with a recurring theme that both geometry and algebra are systems for drawing logical conclusions from given data. To exploit the power of these systems by drawing the strongest conclusions, it is necessary to "tell geometry" or "tell algebra" all the given conditions: the conditions must be stated in a form which these systems can process, and all must be used nontrivially in the reasoning. The lesson reported here aimed at promoting the way of thinking "*when representing a problem algebraically,* ***all*** *of the problem constraints must be represented*." Our experience suggests that students usually lack this way of thinking. This is part of a general phenomenon where students either do not see a need or do not know how to translate verbal statements into algebraic representations and fail, as a result, to make logical derivations. For example, we observed students working on linear algebra problems fail to represent *all* the problem information algebraically. Statements critical to the problem solution (e.g., "v is in the span of u_1 and u_2," "u_1 and u_2 are linearly independent," "v is in the eigenspace of A") often are not translated in algebraic terms by these students even when they seem to understand their meaning.

We refer to the problem-solving approach of representing a given problem algebraically and applying known procedures to the algebraic representation (such as "elimination of variables" to solve systems of equations) in order to obtain a solution to the problem as the *algebraic representation approach*. Clearly, representing *all* the problem conditions algebraically is an essential ingredient of this approach. Problem-solving approaches are (one kind of) ways of thinking (see Harel, 2008a).

As is evident from Segment I, the students in this lesson applied the algebraic representation approach, but only partially, in that they did not represent *all* the problem constraints algebraically. The Rectangular Land Problem was designed to intellectually compel the students to appreciate the need to "tell algebra" all the conditions stated—sometimes not so explicitly—in the problem. This was done by bringing the students, in a later segment of the lesson, into a conflict with their own conclusion that there are infinitely many values for the total area of the land.[6]

On the basis of the conclusion reached in Segment I, the teacher embarked on the next phase in the lesson: to necessitate an examination of this conclusion, where he began by asking the class to provide *two* of these solutions (Fragment 6). The reason for asking for *two* solutions was to ensure that the students see that at least one of their solutions is incorrect, and will experience, as a result, an intellectual perturbation that compels them to reflect on and examine their own solution, whereby utilizing the *Necessity Principle.*

At first, the students viewed the teacher's task as unproblematic; they chose two arbitrary numbers for *A* and obtained two corresponding values for the total area by using the formula $\textit{Total Area} = 4A + 2200$ they had derived earlier from their system of equations. It took some negotiation with the students for the them to understand that they must also show that their answer is viable; namely, that their values for *A, B, C,* and *D* correspond to regions A, B, C, and D that fit into the given geometric configuration (Fragment 7). We note that in none of the lessons on the Rectangular Land Problem we conducted did this understanding lead the students at this stage of the lesson to realize that their initial system of equations needs to be amended by an equation representing the geometric condition given in the figure.

Following this, the students tried to obtain the dimensions of the four regions, a task they now deemed necessary, though not difficult (Fragment 8). This is the content of Segment III, the longest in the lesson, which contains the process that led the students to examine their earlier conclusion. In this process, they concluded that there is a unique solution to the problem, not infinitely many solutions as they had previously thought. In this segment, the students first attempted to determine the dimensions of Region A by substituting different numbers, focusing exclusively on whole numbers. The teacher accepted the students' attempts but also prompted them to vary the domains of these numbers: from whole numbers to rational numbers and to irrational numbers. This number-domain extension was assumed by the teacher to be natural to the students since, based on their academic stage, they must have known that in principle the value sought can be non-integer. The repeated failure to find the missing value may have led the students to doubt the existence of such value—doubts the teacher formulated in terms of a question: Can a figure representing the problem conditions be constructed for $A = 100$? (Fragment 12). Further, the repeated trials of values from different domains seemed to have triggered the students to represent the missing value by a variable *t,* and, in turn, to answer the question by algebraic means; namely, by showing, algebraically, that no such value exists, they

[6] Cognitive conflict is not the only means for intellectual necessity (see Harel, 2008b)

determined that the figure cannot be constructed for $A = 100$ (Fragment 13). Still further, this experience seemed to serve as a conceptual basis for the students' next step, where they reapplied the same technique but this time they set both the area of Region A, *A,* and its length, *t,* as unknowns. This provided the opportunity for the teacher (not included in the lesson's description) to distinguish between the status of *A* and *t:* while the former is a *parameter,* the latter is a *variable*.

At least two ways of thinking were utilized in this segment: the first has to do with beliefs about mathematics, that mathematics involves trial and error and proposing and refining conjectures until one arrives at a correct result; and the second has to do with proof schemes, that algebraic means are a powerful tool to prove—to remove doubts about conjectures. These ways of thinking—which are strongly related in the context of our lesson—were not preached to or imposed on the students; rather, in accordance with the *Necessity Principle*, they were necessitated through problematic situations that were meaningful to the students. Although the way of thinking about the power and use of algebra in proving was not foreign to these students, it is evident from the lesson accounts that it was not spontaneous for them either. With respect to the mental act of proving, the students' actions that were available to the teacher during the lesson were the repeated attempts by the students to construct a desired figure by (haphazard) substitutions of different whole-number values for the length of Region A. In the *DNR'*s terminology, these were the students' current ways of understanding associated with the proving act, which the teacher assumed were governed by the students' empirical proof scheme (see Harel & Sowder, 2007). In accordance with the *Duality Principle,* the teacher built on these ways of understanding by prompting the students to expand the variation of values from other domains of numbers known to them and, subsequent, necessitate the manipulation of algebraic expression involving these values (Fragment 11). His goal and hope was that that this change in ways of understanding (i.e., particular solution attempts) will trigger the application of a different way of thinking—the deductive proof scheme.

The fourth, and last, segment of the lesson was to help the students account for the conflicting conclusions they reached about the number of solutions to the problems. This is a crucial stage, for the lesson's design was use the resolution of this conflict to advance the way of thinking that, when representing a problem algebraically, *all* the problem constraints must be represented. At this point, the teacher felt that it was clear to the students that their initial conclusion that there are infinitely many solutions to the problem was wrong, but they did not understand why the system of equations they initially constructed did not result in the correct answer. At no point during the lesson did the students realize that absent from their initial system is a representation of the geometric constraints entailed from the given figure. A mathematically mature person would likely have inferred from the discrepancy between the numbers of solutions—one versus many—that the system with many solutions is missing at least one equation that is independent of the other equations in the system. Conceptual prerequisites for this realization include several way of understanding: that a system of equations is a set of quantitative constraints, that a solution set of the system is determined by the independent equations in the system, and that, therefore, to reduce the size of the solution set one must add additional independent equations to the system. The teacher operated on the assumption that the students did not fully possess these ways of understanding. Although they represented (some) of the problem constraints by equations, whereby they constructed their initial system, they also constructed a 4^{th} equation as a (linear) combination of—and therefore dependent on—the previous three equations. In addition, many of their manipulations of the system's equations were rather haphazard, unaware of the fact that a method for solving a system of equation is a process of transforming the given system into a (simpler) system with the same solution set. A few students approached the solution process more systematically, by using row reduction for example.

The discrepancy between the two outcomes that the students faced—infinitely many solutions versus a single solution—offered the teacher the opportunity to compel the students to revisit their meanings for equation, system of equations, solution set, dependent and independent equations, and operations on a system to obtain a solution. The refined ways of understanding of these concepts that resulted from the classroom discussion (Fragment 17) led the students to review their earlier action and, in turn, to a realization that there was nothing in their initial system representing the condition that adjacent regions share one side in common. They even proceeded to draw the figure in Segment IV to illustrate this observation and explained that "scattered" regions would indeed entail infinitely many solutions.

In sum, this analysis demonstrates how the entire lesson—its conceptualization, design, and implementation—was oriented and driven by the *DNR* conceptual framework. In particular, we see the application of the *Duality Principle* and the *Necessity Principle,* along with adherence to the *DNR* premises. Absent from this discussion is the *Repeated Reasoning Principle*. It is unrealistic to expect that the students will internalize the lesson's targeted ways of thinking and other ways of thinking that the lesson afforded them in a single 90-minute session. To internalize these ideas, the students must be given the opportunity to repeatedly reason about problem situations where similar ways of thinking and ways of understanding are likely to emerge. Indeed, our program for these students included a sequence of problems whose goals included the targeted ways of thinking discussed here.

We conclude this section by noting that, in this and in other lessons about the Rectangular Land Problems, there were other (correct) solutions. In this lesson, they were expressed in the homework assignments 1 and 5 (Fragment 19). In Problem 1, for example, some students added the condition *A/C=B/D.* Question 5 led some students to approach the problem quantitatively without resorting to algebraic equations. One of the solutions based such an approach is demonstrated in Figure 9. To explain the solution, view the figure as a matrix and use the problem givens. It is easy to see that the cells a_{11} represents Region A, the union of cells a_{12} and a_{13} represent Region B, the union of cells a_{21} and a_{31} represent Region C, and the union of the remaining cells represent Region D. Now, by considering the dimensions of these remaining cells, conclude that *A=200.*

A	A	600
A	A	600
200	200	600

Figure 9: A Quantitative Approach to the Rectangular Land Problem.

5 Final Comments

Formulating instructional objectives in terms of ways of thinking is of paramount importance in *DNR,* as is entailed from *DNR's* definition of mathematics. The goal of the Rectangular Land Problem was to enhance the *algebraic representation approach* among the targeted population of students; in the lesson reported here, the students were preservice secondary teachers. However, ways of thinking, according to the *Duality Principle,* can develop only through ways of understanding, which, by the *Necessity Principle,* must be intellectually necessitated through problematic situations. On the other hand, intellectual need is not a uniform construct. One must take into account students' current knowledge, especially—again, by the *Duality Principle*—their ways of thinking. Furthermore, a single problem is not sufficient for students to fully internalize a way of thinking. It is necessary, by the *Repeated Reasoning Principle,* to repeatedly provide the students with situations that necessitate the application of a targeted way of thinking.

This was done by bringing the students into a conflict with their own conclusion that there are infinitely many values for the total area of the land. This conclusion was not a consensus in each lesson in which the Rectangular Land Problem was presented. In some lessons, there were some students who approached the problem differently, by assigning variables to the dimensions of the regions A, B, C, and D, and so, by default, represented algebraically the constraints entailed from the given geometric configuration.[7] This approach led them, in turn, to a unique solution to the problem. Surprisingly perhaps, the presence of these multiple approaches and their corresponding outcomes never led the students to account for the discrepancy by attending to the geometric constraints given in the problem. This suggests that the students' consideration of the rectangles' dimensions was "accidental" rather than with the intention to represent the geometric constraints. Nevertheless, the presence of multiple solutions did not alter the teacher's goal of bringing the students to realize that their system of equations must include the condition entailed from the particular configuration of the geometric figure. On the contrary: the presence of conflicting solutions strengthened the intellectual need to reexamine and compare between the solutions so as to account for the conflict.

In *DNR*, teaching actions are sequenced so that one action is built on the outcomes of its predecessors for the purpose of furthering an instructional objective. Many, though not all, of these actions are aimed at *intellectually necessitating* for the students the ways of understanding and ways of thinking targeted. How does one determine students' intellectual need? *DNR* provides a framework for addressing this question, but detailed methodologies, together with suitable pedagogical strategies, for dealing with this question are yet to be devised. The framework consists of a classification of intellectual needs into five interrelated categories. Briefly, these categories are:

- The *need for certainty* is the need to prove, to remove doubts. One's certainty is achieved when one determines—by whatever means he or she deems appropriate—that an assertion is true. Truth alone, however, may not be the only need of an individual, and he or she may also strive to explain *why* the assertion is true.
- The *need for causality* is the need to explain—to determine a cause of a phenomenon, to understand what makes a phenomenon the way it is.
- The *need for computation* includes the need to quantify and to calculate values of quantities and relations among them. It also includes the need to optimize calculations.
- The *need for communication* includes the need to persuade others than an assertion is true and the need to agree on common notation.

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[7] This solution approach occurred more often—not surprisingly—when the problem statement included another part: "… *The farmer's Son, Dan, asked: What are the dimensions of our land? … What conclusion will Dan conclude from his father's answer?*"

- The *need for connection and structure* includes the need to organize knowledge learned into a structure, to identify similarities and analogies, and to determine unifying principles.

In general, intellectual need is a subjective construct—what constitute intellectual need for one person may not be so for another. However, in the classroom the teacher must make an effort to create a collective intellectual need. A necessary condition for this to happen is that classroom debates are *public* rather than *pseudo public.* To explain, consider the first segment of the lesson. This lesson concluded with the teacher stating publically the conclusion reached by the class. The teacher's effort focused on ensuring that *all* students share a common understanding of the conclusion asserted—that are infinitely many solutions to the problem. It was on the basis of this shared understanding that the teacher carried out the next step in his lesson plan, whose goal was to bring the students into disequilibrium and, in turn, to a realization that not all the problem constraints have been represented in the system of equations (Fragment 5). The teaching practice of ensuring that the entire class reaches a common understanding—though not necessarily agreement—of the assertion(s) made or the problem(s) at hand is critical in *DNR*-based instruction. Without it, any ensuing classroom discussion is likely to be a *pseudo public* rather than a *genuine public* discussion. In a pseudo public debate, the classroom discussion proceeds without all students having formed a common and coherent way of understanding the issue under consideration. A pseudo public debate manifests itself in the teacher communicating individually with different groups of students and often with a single student while the rest of the class is not part of the exchange.

References

Anderson, J. R. (1980). Cognitive psychology and its implications. San Francisco: Freeman.

Anderson, J. R. (1982). Acquisition of cognitive skill. Psychological Review, 89, 369–406.

Balacheff, N. (1988). Aspects of proof in pupil's practice of school mathematics. In D. Pimm (Ed.), *Mathematics, teachers and children* (pp. 216 – 235).

Boero, P., Garuti, R., Lemut, E., & Mariotti, M. A. (1996). Challenging the traditional school approach to theorems: a hypothesis about the cognitive unity of theorems. In Proceedings of the PME-XX (pp. 113–120).

Bransford, J. R., Brown, A. L., & Cocking, R. R. (Eds.). (1999). How people learn: Brain, experience and school. Washington, DC: National Academy Press.

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield (Eds. and Trans.). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Chazan, D. (1993). High school geometry students' justification for their views of empirical evidence and mathematical proof. *Educational Studies in Mathematics, 24,* pp. 359 – 387.

Chiesi, H., Spilich, G., & Voss, J. (1979). Acquisition of domainrelated information in relation to high and low domain information. Journal of Verbal Learning and Verbal Behavior, 18, 257–273.

Cooper, R. (1991). The role of mathematical transformations and practice in mathematical development. In L. Steffe (Ed.), Epistemological foundations of mathematical experience. New York: Springer-Verlag.

Davis, R. B. (1984). Learning mathematics: The cognitive science approach to mathematics education. New Jersey: Ablex Publishing.

DeGroot, A. D. (1965). Thought and choice in chess. The Hage: Mouton.

Eisenhart, M. A. (1991). *Conceptual frameworks for research circa 1991: Ideas from a cultural anthropologist; implications for mathematics education researchers*. Proceedings of the 13th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (vol. 1, pp. 202 – 219). Blacksburg, VA.

Fischbein, E., & Kedem, I. (1982). Proof and certitude in the development of mathematical thinking. In A. Vermandel (Ed.), Proceedings of the Sixth International Conference for the Psychology of Mathematics Education (128–131). Antwerp, Belgium: Universitaire Instelling.

Hanna, G., & Jahnke, H.N. (1996). Proof and proving. In a. Bishop, K. Clements, C. Keiterl, J. Kilpatrick, & C. Laborde (Eds.). *International handbook of mathematics education* (Part 2, pp. 877 – 908). Dordrecht, the Netherlands: Kluwer.

Harel, G. (1985). Teaching linear algebra in high school. Unpublished doctoral dissertation. Ben-Gurion University of the Negev, Beersheba, Israel.

Harel, G. (1989). Learning and teaching linear algebra: Difficulties and an alternative approach to visualizing concepts and processes. *Focus on Learning Problems in Mathematics*, *11*, 139-148.

Harel, G. (1990). Using geometric models and vector arithmetic to teach high-school students basic notions in linear algebra, *International Journal for Mathematics Education in Science and Technology*, *21*, 387-392.

Harel, G. (2001). The Development of Mathematical Induction as a Proof Scheme: A Model for DNR-Based Instruction. In S. Campbell & R. Zaskis (Eds.). Learning and Teaching

Number Theory. In C. Maher (Ed.). *Journal of Mathematical Behavior*. New Jersey, Ablex Publishing Corporation, 185-212.

Harel, G. (2008a). What is Mathematics? A Pedagogical Answer to a Philosophical Question. In R. B. Gold & R. Simons (Eds.), *Current Issues in the Philosophy of Mathematics from the Perspective of Mathematicians,* Mathematical American Association.

Harel, G. (2008b). DNR Perspective on Mathematics Curriculum and Instruction: Focus on Proving, Part I. *Zentralblatt fuer Didaktik der Mathematik,* 40, pp. 487–500.

Harel, G. (2008c). DNR Perspective on Mathematics Curriculum and Instruction, Part II. *Zentralblatt fuer Didaktik der Mathematik.* 40, pp. 893 – 907.

Harel, G., Behr, M. (1995). Teachers' solutions for multiplicative problems, *Hiroshima Journal for Research in Mathematics Education*, pp. 31-51.

Harel, G., Behr, M., Post, T., & Lesh, R. (1992). The blocks task; comparative analyses with other proportion tasks, and qualitative reasoning skills among 7th grade children in solving the task, *Cognition and Instruction*, *9*, pp. 45-96.

Harel, G., & Sowder, L. (1998). Students' proof schemes. *Research on Collegiate Mathematics Education*, Vol. III. In E. Dubinsky, A. Schoenfeld, & J. Kaput (Eds.), AMS, 234-283.

Harel, G., & Sowder, L (in press). Toward a comprehensive perspective on proof, In F. Lester (Ed.), *Handbook of Research on Teaching and Learning Mathematics* (2nd edition). Greenwich, CT: Information Age Publishing.

LaBerge, D., & Samuels, S. J. (1974). Toward a theory of automatic information processing in reading. Cognitive Psychology, 6, pp. 293–323.

Lave, J. (1988). Cognition in practice: Mind, mathematics and culture in everyday life. New York: Cambridge University Press.

Lerman, S. (2000). The social turn in mathematics education research. In J. Boaler (Ed.), Multiple Perspectives on Mathematics Teaching and Learning (19–44). Westport, CT: Ablex.

Lesgold, A. M., Rubison, H., Peltovich, P., Glaser, R., Klopfer, D., & Wang, Y. (1988). Expertise in a complex skill: Diagnosing X-ray pictures. In M. T. H. Chi, R. Glaser & M. Farr (Eds.), The nature of expertise (pp. 311–342). Hillsdale: Erlbaum.

Lester, F. K. (2005). On the theoretical, conceptual, and philosophical foundations for research in mathematics education. *International Reviews on Mathematical Education*, vol.37,no.6, pp. 457-467.

Lawson-Tancred, H. T. (1998). Aristotle: The Metaphysics. Harmondsworth: Penguin.

Martin, G., & Harel, G. (1989). Proof frame of preservice elementary teachers, *Journal for Research in Mathematics Education*, *20*, 41-5.

Miller, G. A. (1956). The magical number seven, plus or minus two: some limits on our capacity to process information. Psychological Review, 63, pp. 81–87.

Piaget, J. (1985). The equilibration of cognitive structures: the central problem of intellectual development. Chicago: University of Chicago Press.

Post, T., Harel, G., Behr, M. & Lesh, R. (1991). Intermediate teachers' knowledge of rational number concepts. In E. Fennema , T. P. Carpenter, and S. J. Lamon (Eds.) *Integrating Research on Teaching and Learning Mathematics*. Albany , New York : SUNY Press, pp. 177-198.

Schneider, W., & Shiffrin, R. M. (1977). Controlled and automatic human information processing: Detection, search and attention. Psychological Review, 84, pp. 1–66.

Schoenfeld, A. (2000). Purposes and methods of research in mathematics education, *Notices*, American Mathematical Society.

Sowder, L., & Harel, G., (2003). Case Studies of Mathematics Majors' Proof Understanding, Production, and Appreciation. *Canadian Journal of Science, Mathematics and Technology Education*. *3*, pp. 251-267.

Steffe, L., & Thompson, P. (2000). Teaching experiment methodology: Underlying principles and essential elements. In Kelly, A., & Lesh, R. (Eds.). *Handbook of Research Design in Mathematics and Science Education.* Lawrence Erlbaum Associates, pp. *267-306*.

Stokes, E. (1997). *Pasteur's quadrant: Basic science and technological innovation.* Washington, DC: Brookings Institution Press.

Thompson, P. (1998). Thompson, personal communication.

von Glasersfeld, E. (1983). Learning as constructive activity. Proceedings of the 5th Annual Meeting of the North American Group of Psychology in Mathematics Education, pp.41–101.

Vygotsky, L. S. (1978). Mind in society: The development of higher psychological processes (M. Cole, V. John-Steiner, S. Scribner, & E. Souberman, Eds. & Trans.): Cambridge: Harvard University Press.(Original work published 1934).