Simultaneous Instances of the Tree Property

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Abstract

From large cardinals, we combine methods of Cummings-Foreman and Sinapova to show it is consistent to have a singular strong limit κ with the tree property holding simultaneously at κ^{+n} for each natural number $n \geq 1$. With the same assumptions, we also add collapses to show that it is consistent to have a $\aleph_{\omega^2}$ strong limit with the tree property holding simultaneously at $\aleph_{\omega^2+n}$ for each natural number $n \geq 1$.

1 Introduction

The tree property is a compactness property in set theory motivated by König's result that an infinite finitely branching tree has an infinite branch (equivalently, ω has the tree property). A natural follow-up question is whether or not that this extends to ω_1. This was settled in the negative by Aronszajn who proved that there is an ω_1-Aronszajn tree. What about for infinite cardinals larger than $\aleph_1$?

Interestingly, the consistency of the tree property at $\aleph_2$ turns out to have large cardinal strength. In particular, results of Mitchell and Silver in 1972 tell us that it is consistent that the tree property holds at $\aleph_2$ if and only if it is consistent that there is a weakly compact cardinal. This paper will be in the same vain as the converse. In other words, we will concern ourselves with how certain large cardinal hypotheses entail certain instances of the tree property.

Given the large cardinal consistency strength of the tree property at $\aleph_2$, it becomes an interesting question as to whether or not the only outright ZFC-provable result on the failure tree property at regular cardinals occurs at $\aleph_1$. (The singular cardinal case is not very interesting because it is an easy exercise to show that the tree property always fails at a singular cardinal.) In the case where κ is inaccessible, the tree property at κ is actually equivalent to κ being weakly compact. One could ask an even stronger question: is there a model in which the tree property holds simultaneously every regular cardinal $\kappa > \aleph_1$? This would provide a model of ZFC with a lot of compactness in the universe. Such a result must have large cardinal cardinal strength greater than a weakly compact, because if the tree property holds simultaneously at $\aleph_2$ and $\aleph_3$, then $0^{\#}$ exists [1].

Towards getting the tree property to hold simultaneously, the first results are due to Abraham [1] in 1983, who showed that from sufficient large cardinal hypotheses, it is consistent to have the tree property holding at both $\aleph_2$ and $\aleph_3$. Inspired by Abraham, Cummings and Foreman [3] in 1998 showed that it is consistent (modulo large cardinals) to have the tree property at $\aleph_n$ for each natural number $n > 1$.

We reach a difficulty, though, if try to get the tree property to hold simultaneously above a singular strong limit cardinal. In particular, Specker [15] proved that if $\kappa^{<\kappa} = \kappa$, then the tree property fails at κ^+. It follows that, for the tree property to hold at the double successor of a singular κ, SCH must fail at κ. This presents a host of difficulties.

An outline of this paper is as follows. First we give some preliminaries that are necessary for understanding the remainder of the paper. Then we give the definition of the main forcing notion and prove that it satisfies many of the same properties as the original Cummings-Foreman forcing [3]. Next, we argue that κ^{+n} satisfies the tree property in the resulting model. The case for $n \geq 2$ is modeled after [3], whereas the case for $n = 1$ is modeled after [12]. Afterwards, we give the definition of the forcing that adds collapses, and argue that the conclusion of Theorem 1.2 holds.

1.1 Main Theorems

Theorem 1.1. Let $(\kappa_n : n < \omega)$ and $(\lambda_n : n < \omega)$ be increasing sequences of supercompact cardinals with $\sup_n \kappa_n < \lambda_0$, and $\kappa_0 = \kappa$. There is a forcing extension in which κ is singular strong limit and the tree property holds simultaneously at κ^{+n} for each natural number $n \geq 1$.

Theorem 1.2. Let $(\kappa_n : n < \omega)$ and $(\lambda_n : n < \omega)$ be increasing sequences of supercompact cardinals with $\sup_n \kappa_n < \lambda_0$. There is a forcing extension in which $\aleph_{\omega^2}$ is strong limit and the tree property holds simultaneously at $\aleph_{\omega^2+n}$ for each natural number $n \geq 1$.

2 Background and preliminaries

2.1 Trees, Forcing, and Branch Lemmas

Let us recall the following notions related to the tree property:

Definition 2.1. Let κ be an infinite cardinal.

- For $\alpha < ht(T)$, the α^{th}-**level** of T is defined as $Lev_\alpha(T) = \{x \in T : ht(x) = \alpha\}$.
- A κ-**tree** T is a tree of height κ where each level of T has size strictly less than κ.
- A **branch** b through T is a totally ordered subset of T such that $b \cap Lev_\alpha(T)$ is non-empty for each $\alpha < ht(T)$.
- The **tree property holds at** κ, denoted TP_κ, if every κ-tree T has a branch. The witness to the failure of TP_κ is called a κ-Aronszajn tree.

We also recall some elementary properties of forcing posets:

Definition 2.2. Let κ be an infinite cardinal and $\mathbb{P}$ be a forcing poset.

- $\mathbb{P}$ is κ-closed if every decreasing $(p_i : i < \theta)$ with $\theta < \kappa$ has a lower bound.
- $\mathbb{P}$ is (canonically) κ-directed closed if every directed set $D \subseteq \mathbb{P}$ has a (greatest) lower bound. Recall that D is directed if every two elements in D has a common extension in D.
- $\mathbb{P}$ is $< \kappa$-distributive if, whenever $f \in V[G]$ is a function from some $\lambda < \kappa$ into V, then $f \in V$.

If we have a forcing notion $\mathbb{P}$ which has some chain condition and another $\mathbb{Q}$ which is closed, it is useful to understand the closure and chain condition of these forcings after iterating $\mathbb{P}$ followed by $\mathbb{Q}$ or vice versa. This is the content of Easton's Lemma:

Lemma 2.1 (Easton's Lemma)**.** Let κ be regular. If $\mathbb{P}$ has the κ-cc and $\mathbb{Q}$ is κ-closed, then

1. $\Vdash_{\mathbb{Q}} \mathbb{P}$ has the κ-cc.
2. $\Vdash_{\mathbb{P}} \mathbb{Q}$ is $< \kappa$-distributive.
3. If G is $\mathbb{P}$-generic over V and H is $\mathbb{Q}$-generic over V then G and H are mutually generic (i.e G is $\mathbb{P}$-generic over $V[H]$ and vice versa).

Assume that we have a tree T in our ground model. Of great importance in tree property arguments is knowing which properties about a poset $\mathbb{P}$ allow you to conclude that you do not add a branch through T after forcing with $\mathbb{P}$. The first noteworthy branch lemma is that Knaster forcings do not add branches through κ-trees. Normally the result assumes that the κ-tree is branchless in the ground model (as in [3] and [16]), and Spencer Unger even says in [16] that it "seems like it should be able to be eliminated." Indeed it can be eliminated and we present the proof below. The result when $\kappa = \omega_1$ is proven in Baumgartner's Survey of Iterated Forcing [2].

Lemma 2.2. If $\mathbb{P}$ is κ-Knaster, then forcing with $\mathbb{P}$ does not add a new branch through a κ-tree T.

Proof. Assume otherwise. Without loss of generality, assume that $\dot{b}$ be a $\mathbb{P}$-name for a branch through T such that $\Vdash \dot{b} \notin V$. Consider the set $T^* = \{t \in T \colon \exists p \in \mathbb{P}, p \Vdash t \in \dot{b}\}$. For any $\alpha < \kappa$, we may find a $p_\alpha \in \mathbb{P}$ and a $t_\alpha \in T^*$ such that $p_\alpha \Vdash t_\alpha \in \dot{b} \cap Lev_\alpha(T)$. Since $\dot{b}$ is a name for a new branch, it follows that there is some $A \subseteq \kappa$ of size κ such that $p_i \neq p_j$ for distinct $i, j \in A$. Since $\mathbb{P}$ is κ-Knaster, let $B \subseteq A$ be of size κ such that any two elements of B are compatible. Since the conditions of B are compatible, it follows that $\{t_\alpha \colon \alpha \in B\}$ induces a branch d through T. Since $\Vdash \dot{b} \notin V$, it follows that for any $t \in T^*$ there are incompatible $x, y \in T^*$ such that $t \leq x, y$. Since d is a branch through T, it must be that at least one of these nodes is not in d. So, if we define $T^{**} \subseteq T^*$ as

$$T^{**} = \{t \in T^* \colon t \text{ is } \leq\text{-minimal such that } t \notin d\},$$

we have that T^{**} has size κ and elements of T^{**} are pairwise incompatible. This implies that the set $\{p_i \colon i < \kappa \text{ and } t_i \in T^{**}\}$ is an antichain of size κ, a contradiction. □

Lemma 2.3 (Silver's Branch Lemma)**.** Let κ be a regular cardinal. Assume that T is a κ-tree and $\mathbb{P}$ is τ^+-closed for some regular $\tau < \kappa$ such that $2^\tau \geq \kappa$. Then forcing with $\mathbb{P}$ does not add new branches through T.

We also have the following key branch lemma, see [17].

Lemma 2.4. Suppose that in V, $\mathbb{P}$ is κ-cc, $\mathbb{Q}$ is κ-closed forcing, and κ is not strong limit. Let $G \times H$ be $\mathbb{P} \times \mathbb{Q}$ generic over V. If $T \in V[G]$ is a κ-tree and T has a branch in $V[G][H]$, show that T has a branch in $V[G]$.

2.2 Projections

We summarize some well-known information on projections as they are crucial for analyzing the various forcings later in this paper.

Definition 2.3. We say that $\pi \colon \mathbb{P} \to \mathbb{Q}$ is a **projection**[1] if the following hold:

1. $\pi(1_{\mathbb{P}}) = 1_{\mathbb{Q}}$
2. for any $p, q \in \mathbb{P}$, if $p \leq q$ then $\pi(p) \leq \pi(q)$,
3. for any $q \in \mathbb{Q}$, $p \in \mathbb{P}$, if $q \leq \pi(p)$, then there's an $r \leq p$ such that $\pi(r) \leq q$.

Lemma 2.5. If G is $\mathbb{P}$-generic over V, then the upwards closure $H = (\pi``G) \uparrow$ [2] is $\mathbb{Q}$-generic over V. In particular, $V^{\mathbb{Q}} \subseteq V^{\mathbb{P}}$.

Lemma 2.6. Let H be $\mathbb{Q}$-generic over V. In $V[H]$, define $\mathbb{P}/H = \pi^{-1}[H]$, ordered as a suborder of $\mathbb{P}$. Then, $\mathbb{P}/H$ is nonempty, and if G is $\mathbb{P}/H$-generic over $V[H]$, then it's actually $\mathbb{P}$-generic over V. Further, $H = (\pi``G) \uparrow$.

Lemma 2.7. We may factor forcing with $\mathbb{P}$ as a forcing first by $\mathbb{Q}$ followed by a forcing by $\mathbb{P}/H$. More specifically, given a $\mathbb{P}$-generic G, there's a $\mathbb{Q}$-generic H such that $V[G] = V[H][G]$.

Projections can also give us a sufficient condition on when we may lift an elementary embedding. For this we first state a result by Silver.

Lemma 2.8 (Silver's Lifting Criterion)**.** Let $j \colon V \to M$ be an elementary embedding. Suppose that G is $\mathbb{P}$-generic over V and G^* is $j(\mathbb{P})$-generic over N. If $j"G \subseteq G^*$, then we may lift j to an elementary embedding $j \colon V[G] \to M[G^*]$, where $j(G) = G^*$. Furthermore, if $G^* \in V[G][H]$ for some generic extension of $V[G]$, then our lifted embedding is definable in $V[G][H]$.

Lemma 2.9. Assume that $j \colon V \to M$ is an elementary embedding, $j(\mathbb{P})$ projects to $\mathbb{P}$ via π, and $q \leq_{j(\mathbb{P})} j(\pi(q))$ for any $q \in j(\mathbb{P})$. Then, for any $\mathbb{P}$-generic G over V we may find a $j(\mathbb{P})$-generic H over M letting us lift our embedding to $j \colon V[G] \to M[H]$.

2.3 Prikry Forcing

In the original Cummings-Foreman forcing notion, the left coordinate of the original factor was Cohen forcing. Since our goal is singularize κ and get the tree property at κ^+, however, we going to have to modify the Cummings-Foreman forcing to include a Prikry poset. We will be summarizing the information in Section 3 of Spencer Unger's paper [16] and so the enthusiastic reader should refer to his paper for more details.

Let $(\kappa_n \colon n < \omega)$ be an increasing sequence of supercompact cardinals, $\kappa = \kappa_0$, $\kappa_\omega = \sup_n \kappa_n$, and $\mu = \kappa_\omega^+$. Assume that κ is indestructibly supercompact. Further, let λ_0 be a supercompact cardinal above μ and U^* a normal measure on λ_0. Let $\mathbb{A} = \mathrm{Add}(\kappa, \lambda_0)$. Working in $V^{\mathbb{A}}$, κ is still supercompact, so we may let U be the supercompactness measure

[1]Often a slightly weaker notion of projection is used; namely, that there is a projection from $\mathbb{P}$ to $\mathbb{Q}$ if, given any $\mathbb{P}$-generic G, we can define a $\mathbb{Q}$-generic H such that $V[H] \subseteq V[G]$. This implies that there is a projection (in our sense) from $\mathbb{P}$ to $\mathrm{RO}(\mathbb{Q})$, where $\mathrm{RO}(\mathbb{Q})$ is the complete boolean algebra that $\mathbb{Q}$ densely embeds into.

[2]$(\pi``G) \uparrow$ is the upwards closure of the pointwise image of G under π

on $\mathcal{P}_\kappa(\mu)$ and U_n be the projections of U to $\mathcal{P}_\kappa(\kappa_n)$. For notation, if x and y are sets of ordinals, let κ_y denote the set $\kappa \cap y$ and $x \prec y$ hold when $x \subseteq y$ and and $ot(x) < \kappa_y$. In $V^{\mathbb{A}}$, define the diagonal Prikry forcing $\mathbb{I}$, originally developed by Gitik and Sharon in [6] and by Itay Neeman in [11], as follows:

Definition 2.4. $\mathbb{I}$ has conditions of the form

$$p = (x_0, x_1, \ldots, x_{n-1}, A_n, A_{n+1}, \ldots)$$

where

1. $x_i \in \mathcal{P}_\kappa(\kappa_i)$ for $i < n$,
2. $x_i \prec x_{i+1}$ for $i < n-1$, and
3. $A_i \in U_i$ for $i \geq n$.

The string $(x_0, \ldots, x_{n-1})$ is the stem of p an denoted stem(p). Given another condition

$$q = (y_0, y_1, \ldots, y_{m-1}, B_m, B_{m+1} \ldots),$$

we say that $p \leq q$ if

1. $m \leq n$,
2. $\text{stem}(p) \restriction m = \text{stem}(q)$,
3. $A_i \subseteq B_i$ for $i \geq n$, and
4. $x_i \in B_i$ for $m \leq i < n$.

In other words, extensions of q lengthen the stem of q by choosing elements from the B_i's while also shrinking the B_i's. This forcing adds a generic sequence $(x_n \colon n < \omega) \in \prod_{n<\omega} \mathcal{P}_\kappa(\kappa^n)$ such that $\bigcup_{n<\omega} x_n = \kappa_\omega$. This generic sequence singularizes each κ_n to have cofinality ω and forces $\mu = \kappa^+$.

Lemma 2.10. Importantly, $\mathbb{I}$ sastifies the Prikry property: for any statement φ and any $p \in \mathbb{I}$, there is a direct extension $q \leq^* p$ deciding φ.

Definition 2.5. Given a formula φ and a stem h, write $h \Vdash^* \varphi$ if there is a condition $p \in \mathbb{I}$ with stem h forcing φ.

Let $\dot{U}$ be an $\mathbb{A}$-name for U and for $\alpha < \lambda_0$, let $\mathbb{A}_\alpha = \text{Add}(\kappa, \alpha)$. It is important that we are able to project our Prikry posets onto smaller Prikry posets, so we show that we can do this on a measure 1 set.

Lemma 2.11. There is a $B \subseteq \lambda$ of Mahlo cardinals with $B \in V$ such that

1. if g is $\mathbb{A}$-generic over V, then $\dot{U}_G \cap V[g \restriction \alpha] \in V[g \restriction \alpha]$ and
2. $B \in U^*$.

From this, for each $\alpha \in B$ and each $\mathbb{A}$-generic g over V, we can define supercompactness measures U^α on $\mathcal{P}_\kappa(\lambda_0)$ in $V[g \restriction \alpha]$ and the diagonal Prikry forcing $\mathbb{I}_\alpha$ obtained from U^α as in Definition 2.4. The generic object for $\mathbb{A} * \dot{\mathbb{I}}$ induces generic objects for $\mathbb{A}_\alpha * \dot{\mathbb{I}}_\alpha$ with $\alpha \in B$, so we have the following relationships between the Prikry posets and their regular open algebras.

Lemma 2.12. For all $\alpha \in B$ there is a projection $\pi_\alpha \colon \mathbb{A} * \dot{\mathbb{I}} \to \mathrm{RO}(\mathbb{A}_\alpha * \dot{\mathbb{I}}_\alpha)$

Lemma 2.13. For all $\alpha, \beta \in B$ with $\alpha < \beta$ there is a projection $\pi_{\alpha,\beta} \colon \mathbb{A}_\beta * \dot{\mathbb{I}}_\beta \to \mathrm{RO}(\mathbb{A}_\alpha * \dot{\mathbb{I}}_\alpha)$

Lemma 2.14. $\mathbb{A} * \dot{\mathbb{I}}$ is μ-cc and $\mathbb{A}_\alpha * \dot{\mathbb{I}}_\alpha$ is μ-cc for each $\alpha \in B$.

Lemma 2.15. Forcing with $\mathbb{A} * \dot{\mathbb{I}}$ yields the following cardinal structure:

1. κ is singular strong limit of cofinality ω,
2. $\kappa^+ = (\kappa_\omega^+)^V = \mu$, and
3. $2^\kappa = \lambda_0$.

2.4 The Cummings-Foreman Model

Our main forcing notion modifies the factors of the Cummings-Foreman model in [3], so it is worth summarizing the material found in the original paper. The original definition of the forcing notion is quite general, so it will be helpful for us when proving that the generic extension follows a certain the cardinal structure.

Definition 2.6. Let $V \subseteq W$ be models of set theory. Suppose that τ and κ are cardinals such that $W \models \tau$ is regular and κ is inaccessible. Let $\mathbb{P} = \mathrm{Add}(\tau, \kappa)_V$ and assume that $W \models \mathbb{P}$ is τ^+-cc and $< \tau$-distributive. Also, let $\mathbb{P} \restriction \beta = \mathrm{Add}(\tau, \beta)_V$ for $\beta < \kappa$. Let $F \in W$ be a function from κ to $(V_\kappa)_W$. Define $\mathbb{R} = \mathbb{R}(\tau, \kappa, V, W, F)$ in W by recursion on $\beta \leq \kappa$ and set $\mathbb{R} = \mathbb{R} \restriction \kappa$. Let $\mathbb{R} \restriction 0$ is the trivial forcing. Otherwise, (p, q, f) is a condition in $\mathbb{R} \restriction \beta$ when the following hold:

1. $p \in \mathbb{P} \restriction \beta$,
2. q is a partial function on β and $|\mathrm{dom}(q)| \leq \tau$, and if $\alpha \in \mathrm{dom}(q)$, then
   (a) α is a successor ordinal,
   (b) $q(\alpha) \in W^{\mathbb{P} \restriction \alpha}$, and
   (c) $\Vdash^W_{\mathbb{P} \restriction \alpha} q(\alpha) \in \mathrm{Add}(\tau^+, 1)_{W^{\mathbb{P} \restriction \alpha}}$,
3. f is a partial function on β and $|\mathrm{dom}(f)| \leq \tau$, and if $\alpha \in \mathrm{dom}(f)$, then
   (a) $\Vdash^W_{\mathbb{R} \restriction \alpha} F_0(\alpha)$ is a canonically τ^+-directed closed forcing,
   (b) α is a limit ordinal,
   (c) $f(\alpha) \in W^{\mathbb{R} \restriction \alpha}$, and
   (d) $\Vdash^W_{\mathbb{R} \restriction \alpha} f(\alpha) \in F(\alpha)$.

We also define the ordering $(p_1, q_1, f_2) \leq (p_2, q_2, f_2)$ when the following hold:

1. $p_1 \leq_{\mathbb{P} \restriction \alpha} p_2$,
2. $\mathrm{dom}(q_2) \subseteq \mathrm{dom}(q_1)$ and if $\alpha \in \mathrm{dom}(q_2)$, then $p_1 \restriction \alpha \Vdash^W_{\mathbb{P} \restriction \alpha} q_1(\alpha) \leq q_2(\alpha)$,
3. $\mathrm{dom}(f_2) \subseteq \mathrm{dom}(f_1)$ and if $\alpha \in \mathrm{dom}(f_2)$, then $(p_1, q_1, f_1) \restriction \alpha \Vdash^W_{\mathbb{R} \restriction \alpha} f_1(\alpha) \leq f_2(\alpha)$.

Lemma 2.16. This forcing satisfies the following properties. Each reference is from [3].

1. (Lemma 3.2) $|\mathbb{R}| = \kappa$ and $\mathbb{R}$ is κ-Knaster.
2. (Lemma 3.3) We have that $\mathbb{R}$ projects onto $\mathbb{P}$, $\mathbb{P} \upharpoonright \alpha * \mathrm{Add}(\tau^+, 1)_{W^{\mathbb{P} \upharpoonright \alpha}}$, and $\mathbb{R} \upharpoonright \alpha * F(\alpha)$.
3. (From Section 3.3) Let $\mathbb{U}$ be all conditions in $\mathbb{R}$ of the form $(0, q, f)$ with the ordering induced from $\mathbb{R}$. Then, $\mathbb{U}$ is canonically τ^+-directed closed, κ-cc, and the product forcing $\mathbb{P} \times \mathbb{U}$ projects onto $\mathbb{R}$.
4. (Variant of Lemma 3.6) If $\theta \leq \tau$ and $\mathbb{P}$ is canonically θ-directed closed in W, then $\mathbb{R}$ is canonically θ-directed closed in W.
5. (Lemma 3.11) $\mathbb{U}$ is $\leq \tau$-distributive in $W^{\mathbb{P}}$.
6. (Corollary 3.16) $\mathbb{R}$ is $< \tau$-distributive in W.
7. (Lemma 3.20) Let G be $\mathbb{R}$-generic over W and $\mathbb{S}$ be the quotient forcing of $\mathbb{P} \times \mathbb{U}$ defined in $W[G]$. It follows that $W[G] \models$ "$\mathbb{S}$ is $< \tau^+$-distributive, τ-closed, and κ-cc."
8. (From Section 3.3) Let G be $\mathbb{R}$-generic over W.
   (a) If g is the $\mathbb{P}$-generic induced by G, then $W[G]$ and $W[g]$ have the same τ-sequences of ordinals.
   (b) $W[G] \models \tau^+$ is preserved and $2^\tau = \kappa = \tau^{++}$.
   (c) Every set of ordinals of size τ in $W[G]$ is covered by a set of size τ in W.
9. (Corollary 3.17) $W^{\mathbb{U}} \models \kappa = \tau^{++}$.
10. (From Section 3.5) $\mathbb{R}$ projects onto $\mathbb{R} \upharpoonright \alpha$. In $W^{\mathbb{R} \upharpoonright \alpha}$, there are forcings $\mathbb{P}^*, \mathbb{U}^*$ such that $\mathbb{P}^*$ is τ^+-cc, $\mathbb{U}^*$ is τ^+-closed, and $\mathbb{P}^* \times \mathbb{U}^*$ projects onto $\mathbb{R}/\mathbb{R} \upharpoonright \alpha$.

We also collect some more facts from [3] that will be useful later. Again, the reference is from the original paper.

Lemma 2.17 (Lemma 2.6)**.** Let $\tau < \kappa$, and assume that $V \models$ "τ is regular and κ is inaccessible". Let $\mathbb{P} = \mathrm{Add}(\tau, \eta)$. Let $W \supseteq V$ be a model of ZFC such that

1. κ and τ are cardinals in W,
2. if $X \in W$ is a set of ordinals such that $W \models |X| < \kappa$, then there is a $Y \supseteq X$ such that $Y \in V$ and $V \models |Y| < \kappa$.

Then $\mathbb{P}$ is κ-Knaster in W.

Lemma 2.18 (Lemma 2.13)**.** Let τ be regular and let $\mathbb{A} = \mathrm{Add}(\tau, \eta)$ for some η. Let κ be inaccessible with $\tau < \kappa$. Then

1. If $\mathbb{Q}$ is κ-cc and $\mathbb{Q}$ is a projection of $\mathbb{P} \times \mathbb{U}$, where $\mathbb{P}$ is τ-cc and $\mathbb{U}$ is τ-closed, then $V^{\mathbb{Q}}$ believes that $\mathbb{A}$ is κ-Knaster and $< \tau$-distributive.
2. Suppose that $V^{\mathbb{Q}}$ believes that $\mathbb{Q}^*$ is a projection of $\mathrm{Add}(\tau, \zeta)_V \times \mathbb{U}^*$ and that $\mathbb{U}^*$ is κ-closed. Then $V^{\mathbb{Q} * \dot{\mathbb{Q}}^*} \models \mathbb{A}$ is κ-Knaster.

3 The factor $\mathbb{Q}_0$

For the remainder of this paper, let $(\kappa_n : n < \omega)$ be an increasing sequence of indestructibly supercompact cardinals, $\kappa = \kappa_0$, $\kappa_\omega = \sup_n \kappa_n$, and $\mu = \kappa_\omega^+$. Further, let $(\lambda_n : n < \omega)$ be another increasing sequence of supercompact cardinals with $\mu < \lambda_0$. Let $\lambda = \sup(\lambda_i)$. Fix Laver functions $(F_n : n < \omega)$ for the λ_n's. Set $\mathbb{P}_0 = \mathbb{A} * \dot{\mathbb{I}}$ and $\mathbb{P}_{0,\beta} = \mathbb{A}_\beta * \dot{\mathbb{I}}_\beta$. If $\beta \notin B$, abuse notation and let $\mathbb{P}_{0,\beta} := \mathbb{P}_{0,\gamma}$ where $\gamma > \beta$ is least such that $\gamma \in B$. If $\beta = \lambda_0$, set $\mathbb{P}_{0,\beta} := \mathbb{P}_0$.

We will begin by defining a modification of the forcings $\mathbb{R}$ in [12] and [3], to be denoted $\mathbb{Q}_0$. This definition will be by induction, where we define forcings $\mathbb{Q}_0 \upharpoonright \beta$ for $\beta \leq \lambda_0$. We finally set $\mathbb{Q}_0 := \mathbb{Q}_0 \upharpoonright \lambda_0$.

Definition 3.1. Let $\mathbb{Q}_0 \upharpoonright 0$ be the trivial forcing. Otherwise, $((f, \dot{p}), r, g)$ is a condition in $\mathbb{Q}_0 \upharpoonright \beta$ when the following hold:

1. $(f, \dot{p}) \in \mathbb{P}_{0,\beta}$
2. r is a partial function on β with $\mathrm{dom}(r) \subseteq B$, and $|\mathrm{dom}(r)| < \mu$
3. if $\alpha \in \mathrm{dom}(r)$ then $\Vdash_{\mathbb{P}_{0,\alpha}} r(\alpha) \in \mathrm{Add}(\mu, 1)_{V^{\mathbb{P}_{0,\alpha}}}$
4. g is a partial function on β, $|g| < \mu$,
   and $\mathrm{dom}(g) \subseteq \{\alpha : \Vdash_{\mathbb{Q}_0 \upharpoonright \alpha} F_0(\alpha)$ is a canonically μ-directed closed forcing$\}$
5. if $\alpha \in \mathrm{dom}(g)$, then $\Vdash_{\mathbb{Q}_0 \upharpoonright \alpha} g(\alpha) \in F_0(\alpha)$

The ordering is defined by $(f_1, \dot{p}_1, r_1, g_1) \leq (f_2, \dot{p}_2, r_2, g_2)$ exactly when

1. $(f_1, \dot{p}_1) \leq_{\mathbb{P}_{0,\beta}} (f_2, \dot{p}_2)$
2. $\mathrm{dom}(r_1) \supseteq \mathrm{dom}(r_2)$ and for every $\alpha \in \mathrm{dom}(r_2)$, we have that
   $(f_1, \dot{p}_1) \upharpoonright \alpha \Vdash_{\mathbb{P}_{0,\alpha}} r_1(\alpha) \leq r_2(\alpha)$
3. $\mathrm{dom}(g_1) \supseteq \mathrm{dom}(g_2)$ and for every $\alpha \in \mathrm{dom}(g_2)$, we have that
   $(f_1, \dot{p}_1, r_1, q_1) \upharpoonright \alpha \Vdash_{\mathbb{Q}_0 \upharpoonright \alpha} g_1(\alpha) \leq g_2(\alpha)$.

In the above, $(f_1, \dot{p}_1) \upharpoonright \alpha$ is really $\pi_\alpha(f_1, \dot{p}_1)$, where π_α is the projection from $\mathbb{P}_0$ to $RO(\mathbb{P}_{0,\alpha})$. Similarly, $(f_1, \dot{p}_1, r_1, q_1) \upharpoonright \alpha$ is $(\pi_\alpha(f_1, \dot{p}_1), r_1 \upharpoonright \alpha, q_1 \upharpoonright \alpha)$.

Structural Properties of $\mathbb{Q}_0$

Proposition 3.1. $|\mathbb{Q}_0| = \lambda_0$ and $\mathbb{Q}_0$ has the λ_0-Knaster property, and for all $\beta \in B$, $\mathbb{Q}_0 \upharpoonright \beta$ is β-Knaster.

Proof. $|\mathbb{Q}_0| = \lambda_0$ follows since $|\mathbb{P}_0| = \lambda_0$, $|\mathbb{P}_{0,\beta}| < \lambda_0$ for each $\beta \in B$ and for any α, there are less than λ_0 possibilities for $q(\alpha)$ or $f(\alpha)$. This follows since $\Vdash_{\mathbb{Q}_0 \upharpoonright \alpha} g(\alpha), f(\alpha) \in V_{\lambda_0}$. The Knasterness part of the proposition follows from a Δ-system argument and since Prikry conditions with the same stem are compatible. □

Proposition 3.2. For $\alpha \in B$, $\mathbb{Q}_0$ can be projected to $\mathbb{P}_0$, $\mathbb{Q}_0 \upharpoonright \alpha * F_0(\alpha)$, $\mathbb{P}_{0,\alpha} * \mathrm{Add}(\mu, 1)_{V[\mathbb{P}_{0,\alpha}]}$ and $\mathbb{Q}_0 \upharpoonright \alpha$.

Proof. The projections are the following:

1. $\pi_1\colon ((f,\dot p),r,g)\mapsto (f,\dot p)$
2. $\pi_2\colon ((f,\dot p),r,g)\mapsto ((f,\dot p,r,g)\restriction\alpha, g(\alpha))$
3. $\pi_3\colon ((f,\dot p),r,g)\mapsto ((f,\dot p)\restriction\alpha, r(\alpha))$
4. $\pi_4\colon ((f,\dot p),r,g)\mapsto (f,\dot p,r,g)\restriction\alpha$

We prove that π_2 and π_4 are projections and leave the rest to the imagination of the reader. Recall that $(f,\dot p,r,q)\restriction\alpha$ is $(\pi_\alpha(f,\dot p), r\restriction\alpha, q\restriction\alpha)$. We start with π_4 and then deal with π_2. To check π_4 is order preserving, observe that if $(f_1,\dot p_1,r_1,g_1)\le(f_2,\dot p_2,r_2,g_2)$, then $\pi_\alpha(f_1,\dot p_1)\le\pi_\alpha(f_2,\dot p_2)$ because π_α is a projection. We still have that $\mathrm{dom}(r_1)\restriction\alpha\supseteq\mathrm{dom}(r_2)\restriction\alpha$ and for every $\beta\in\mathrm{dom}(r_2)$, $(f_1,\dot p_1)\restriction\beta\Vdash_{\mathbb{P}_{0,\beta}} r_1(\beta)\le r_2(\beta)$. The similar condition holds for the g_i's and so it follows that $\pi_4(f_1,\dot p_1,r_1,g_1)\le_{\mathbb{Q}_0\restriction\alpha}\pi_4(f_2,\dot p_2,r_2,g_2)$.

Next we check the other condition for a projection. Assume that $(a,\dot b,c,d)\le_{\mathbb{Q}_0\restriction\alpha}\pi_4(f_2,\dot p_2,r_2,g_2)$. It follows that $(a,\dot b)\le\pi_\alpha(f_2,\dot p_2)$, and since π_α is a projection we may find a $(f_1,\dot p_1)\le(f_2,\dot p_2)$ such that $\pi_\alpha(f_1,\dot p_1)\le(a,\dot b)$. We define r_1 by first setting $\mathrm{dom}(r_1)=\mathrm{dom}(c)\cup\mathrm{dom}(r_2)$. Then, we let $r_1(\beta)=c(\beta)$ when $\beta\in\mathrm{dom}(c)$ and $r_1(\beta)=r_2(\beta)$ otherwise.

Finally, define g_1 by setting $\mathrm{dom}(g_1)=\mathrm{dom}(d)\cup\mathrm{dom}(g_2)$ and letting $g_1(\beta)=d(\beta)$ when $\beta\in\mathrm{dom}(d)$ and $g_1(\beta)=g_2(\beta)$ otherwise. It follows by construction that $(f_1,\dot p_1,r_1,g_1)\le(f_2,\dot p_2,r_2,g_2)$ and that $\pi_4(f_1,\dot p_1,r_1,g_1)\le(a,\dot b,c,d)$. So π_4 is a projection.

For π_2, assume that $(f_1,\dot p_1,r_1,g_1)\le(f_2,\dot p_2,r_2,g_2)$. Since π_4 is a projection we know that $(f_1,\dot p_1,r_1,g_1)\restriction\alpha\le(f_2,\dot p_2,r_2,g_2)\restriction\alpha$. Further, by definition of the ordering on $\mathbb{Q}_0$ we have that $(f_1,\dot p_1,r_1,g_1)\restriction\alpha\Vdash q_1(\alpha)\le q_2(\alpha)$. So, $\pi_2(f_1,\dot p_1,r_1,g_1)\le\pi_2(f_2,\dot p_2,r_2,g_2)$ by definition of a two-step iteration.

Next, assume that $((a,\dot b,c,d),e)\le_{\mathbb{Q}_0\restriction\alpha*F_0(\alpha)}((f_2,\dot p_2,r_2,g_2)\restriction\alpha, g_2(\alpha))$. From the first coordinate, we know that $(a,\dot b,c,d)\le(f_2,\dot p_2,r_2,g_2)\restriction\alpha$. Since π_4 is a projection, there's some $(f_1,\dot p_1,r_1,g_1)\le_{\mathbb{Q}_0}(f_2,\dot p_2,r_2,g_2)$ such that $(f_1,\dot p_1,r_1,g_1)\restriction\alpha\le_{\mathbb{Q}_0\restriction\alpha}(a,\dot b,c,d)$. The idea is to modify g_1 to g_1^* by setting $g_1^*=g_1$ below α, by setting $g_1^*(\alpha)=e$, and by setting $g_1^*=g_2$ above α. Since $(a,\dot b,c,d)\Vdash_{\mathbb{Q}_0\restriction\alpha} e\le g_2(\alpha)$, it follows that $(f_1,\dot p_1,r_1,g_1^*)\le_{\mathbb{Q}_0}(f_2,\dot p_2,r_2,g_2)$. But then we also have that $((f_1,\dot p_1,r_1,g_1^*)\restriction\alpha, g_1^*(\alpha))\le_{\mathbb{Q}_0\restriction\alpha*F_0(\alpha)}((a,\dot b,c,d),e)$ as desired. ☐

Definition 3.2. Let $\mathbb{U}$ have conditions of the form $(0,0,q,f)\in\mathbb{Q}_0$ with the ordering inherited from $\mathbb{Q}_0$.

Proposition 3.3. $\mathbb{P}_0$ is μ-cc.

Proposition 3.4. $\mathbb{U}$ is μ-canonically directed closed and λ_0-cc (Knaster).

Proof. The major difference between our new factor and the one originally the given in [3] is the leftmost coordinate. Since our definition of $\mathbb{U}$ fixes the leftmost coordinate, we then notice that we may give exactly the same argument as the original paper (Lemmas 3.9 and 3.8 respectively). We present the directed closed argument from [3] for completeness.

Fix a directed set of conditions $\{(0,0,q_\eta,f_\eta)\colon\eta<\theta\}$ for a cardinal $\theta<\mu$. We define a lower bound $(0,0,r,g)$ as follows. Define $A_1=\bigcup_{\eta<\theta}\mathrm{dom}(q_\eta)$. Notice that $|A_1|<\mu$. Define a function q with domain A_1. For $\alpha\in A_1$ consider $\{q_\eta(\alpha)\colon\eta<\theta\}$. If $\eta,\zeta<\theta$ then for some $\gamma<\theta$ we have that $(0,0,q_\gamma,f_\gamma)\le(0,0,q_\eta,f_\eta),(0,0,q_\zeta,f_\zeta)$, and so $\Vdash q_\gamma(\alpha)\le q_\eta(\alpha),q_\zeta(\alpha)$. Then, there's a $\mathbb{P}_{0,\alpha}$-name for the directed set $\{q_\eta(\alpha)\colon\eta<\theta\}\subseteq\mathrm{Add}(\mu,1)_{V^{\mathbb{P}_{0,\alpha}}}$ and so we may let $r(\alpha)$ be a name forced to be the greatest lower bound to this directed set.

Next, define $A_2 = \bigcup_{\eta<\theta} \mathrm{dom}(f_\eta)$ and notice $|A_2| < \mu$. Observe that we may define a function g by induction on α with domain A_2 such that $(0,0,r,g) \restriction \alpha \Vdash g(\alpha) \leq f_\eta(\alpha)$ for any α and η. The induction step is similar to the previous paragraph, where we say that $g(\alpha)$ is a name for the greatest lower bound of $\{f_\eta(\alpha) : \eta < \theta\}$ if that set is directed, and the trivial condition otherwise.

It's not hard to see that $(0,0,r,g)$ is in fact the greatest lower bound. $\square$

Lemma 3.1. The following results are standard and proofs are similar to that in [3]:

- $\mathbb{P}_0 \times \mathbb{U}$ projects onto $\mathbb{Q}_0$.
- $\mathbb{P}_0 \times \mathbb{U}$ is λ_0-cc and all $< \mu$-sequences of ordinals in $V^{\mathbb{P}_0 \times \mathbb{U}}$ are in $V^{\mathbb{P}_0}$.
- Assume G is $\mathbb{Q}_0$-generic over V and g is the $\mathbb{P}_0$-generic induced from G. If $X \in V[G]$ is a set of ordinals of size $< \mu$, then $X \in V[g]$.
- $\mathbb{Q}_0$ collapses every cardinal between μ and λ_0 to μ.
- $\mathbb{Q}_0$ preserves μ and forces $2^\kappa = \lambda_0 = \mu^+$.
- There's a projection from $\mathbb{Q}_0$ to $\mathbb{Q}_0 \restriction \alpha$ for $\alpha \leq \lambda_0$.

Definition 3.3. Given $\alpha < \lambda_0$ and a V-generic G_α for $\mathbb{Q}_0 \restriction \alpha$, define $\mathbb{R}^* = \mathbb{Q}_0/G_\alpha$ and $\mathbb{U}^* = \{(0,0,q,f) \in \mathbb{Q}_0 : (0,0,q,f) \in \mathbb{R}^*\}$.

Like $\mathbb{Q}_0$, we have that $\mathbb{R}^*$ may be factored as the product of a "Prikry" poset and a μ-closed poset. Proofs are similar to those in [1] and [3].

Proposition 3.5. There's a projection of $\mathbb{P}_0/\mathbb{P}_{0,\alpha} \times \mathbb{U}^*$ onto $\mathbb{R}^*$, given by $((f,\dot{p}),(0,0,r,g)) \mapsto (f,\dot{p},r,g)$

Proposition 3.6. In $V[G_\alpha]$, $\mathbb{U}^*$ is forcing equivalent to a μ-closed forcing.

4 The forcing $\mathbb{R}_\omega$

We define the Cummings-Foreman (CF) variant $\mathbb{R}_\omega$ as follows:

Definition 4.1. We proceed in the same manner as [3]:

1. Let $\mathbb{R}_1 = \mathbb{Q}_0$.
2. Let $\dot{F}_1$ be a $\mathbb{Q}_0$-name for a function on λ_1 such that $\Vdash_{\mathbb{Q}_0} \dot{F}_1(\alpha) = F_1(\alpha)$ when $F_1(\alpha)$ is a $\mathbb{Q}_0$-name and $\Vdash_{\mathbb{Q}_0} \dot{F}_1(\alpha) = 0$ otherwise. Then define $\dot{\mathbb{Q}}_1$ to be the canonical name for $\mathbb{R}(\mu, \lambda_1, V, V[\mathbb{Q}_0], F_1^*)$ where F_1^* is the interpretation of $\dot{F}_1$ in $V[\mathbb{Q}_0]$. Let $\mathbb{R}_2 = \mathbb{Q}_0 * \dot{\mathbb{Q}}_1$.
3. Similarly, for $n \geq 2$, let $\mathbb{R}_n = \mathbb{Q}_0 * \ldots * \dot{\mathbb{Q}}_{n-1}$ and let $\dot{F}_n$ be a $\mathbb{R}_n$-name for a function on λ_n such that $\Vdash_{\mathbb{R}_n} \dot{F}_n(\alpha) = F_n(\alpha)$ when $F_n(\alpha)$ is a $\mathbb{R}_n$-name and $\Vdash_{\mathbb{R}_n} \dot{F}_n(\alpha) = 0$ otherwise. Then define $\dot{\mathbb{Q}}_n$ to be the canonical name for $\mathbb{R}(\lambda_{n-2}, \lambda_n, V[\mathbb{R}_{n-1}], V[\mathbb{R}_n], F_n^*)$ where F_n^* is the interpretation of $\dot{F}_n$ in $V[\mathbb{R}_n]$.
4. Finally, let $\mathbb{R}_\omega$ be the inverse limit of $(\mathbb{R}_n : n < \omega)$.

For Definition 4.1 to actually make sense, we have to show that after forcing with $\mathbb{Q}_0$ we satisfy the hypotheses of Definition 2.6 to make $\mathbb{R}(\mu, \lambda_1, V, V[\mathbb{Q}_0], F_1^*)$ valid.

Lemma 4.1. Let G be $\mathbb{Q}_0$-generic over V and g be the $\mathbb{P}_0$-generic induced by G. In $V[G]$ we have that

1. μ is regular and λ_1 is inaccessible,
2. $\mathrm{Add}(\mu, \lambda_1)_V$ is μ^+-cc and $< \mu$-distributive.

Proof. Since $|\mathbb{Q}_0| = \lambda_0 < \lambda_1$, it follows that λ_1 is still inaccessible in $V[G]$. To see that μ is regular in $V[G]$, assume otherwise and fix an unbounded sequence $f : \tau \to \mu$ with $\tau < \mu$ such that $f \in V[G]$. Since $\mathbb{U}$ is μ-closed, it follows that $f \in V[g]$. This contradicts Lemma 2.15 because μ is regular in $V[g]$.

The second part of this follows from Lemma 2.18(1), where $\tau = \mu$, $\kappa = \lambda_0$, $\mathbb{Q} = \mathbb{Q}_0$, and $\mathbb{U}$ as in the previous section. In particular we have that $\mathrm{Add}(\mu, \lambda_1)_V$ is λ_0-Knaster and $< \mu$-distributive in $V[G]$. Since $V[G] \models \lambda_0 = 2^\kappa = \mu^+$, the result follows. $\square$

It follows that the definition of $\mathbb{R}_2$ makes sense. Since the terms $\dot{\mathbb{Q}}_n$ for $n \geq 1$ are simply names for the factors in the original Cummings-Foreman paper, it follows in turn that we satisfy the following properties:

Lemma 4.2 (Lemma 4.3 in [3])**.** Let $n \geq 1$. Let $\mathbb{R}_n = \mathbb{Q}_0 * \dots \dot{\mathbb{Q}}_{n-1}$, let $\mathbb{P}_1 = \mathrm{Add}(\mu, \lambda_1)_V$, and let $\mathbb{P}_n = \mathrm{Add}(\lambda_{n-2}, \lambda_n)_{V[\mathbb{R}_{n-1}]}$ for $n \geq 2$. Further, let $\mathbb{U}_1 = \mathbb{U}(\mu, \lambda_1, V, V[\mathbb{Q}_0], F_1^*)$ and $\dot{\mathbb{U}}_n = \mathbb{U}_n(\lambda_{n-2}, \lambda_n, V[\mathbb{R}_{n-1}], V[\mathbb{R}_n], F_n^*)$ for $n \geq 2$, where $\mathbb{U}_n$ is the poset $\mathbb{U}$ corresponding to $\mathbb{Q}_n$ defined in preliminary subsection 2.4 on the Cummings-Foreman Model. We abuse notation and will occasionally denote $\kappa = \lambda_{-2}$ and $\mu = \lambda_{-1}$.

1. In $V[\mathbb{R}_n]$, we have $2^{\lambda_{i-2}} = \lambda_i$ for $i < n$ and the λ_i's are still inaccessible for $i \geq n$.
2. $V[\mathbb{R}_n] \models \mathbb{Q}_n$ is $< \lambda_{n-2}$-distributive, λ_n-Knaster, and $|\mathbb{Q}_n| = \lambda_n$. If $n \geq 2$, then $\mathbb{Q}_n$ is also λ_{n-3}-closed in $V[\mathbb{R}_n]$.
3. All $< \lambda_{n-2}$-sequences of ordinals from $V[\mathbb{R}_n * \dot{\mathbb{Q}}_n]$ are in $V[\mathbb{R}_{n-1} * \dot{\mathbb{P}}_{n-1}]$.
4. All cardinals up to λ_{n-2} are preserved in $V[\mathbb{R}_n * \dot{\mathbb{Q}}_n]$.
5. $V[\mathbb{R}_n] \models$ "$\mathbb{Q}_n$ is a projection of $\mathbb{P}_n \times \mathbb{U}_n$," and we also have that $V[\mathbb{R}_n * \dot{\mathbb{P}}_n] \subseteq V[\mathbb{R}_n * \dot{\mathbb{Q}}_n] \subseteq V[\mathbb{R}_n * (\dot{\mathbb{P}}_n \times \dot{\mathbb{U}}_n)]$.
6. All λ_{n-2}-sequences of ordinals from $V[\mathbb{R}_n * \dot{\mathbb{Q}}_n]$ are in $V[\mathbb{R}_n * \dot{\mathbb{P}}_n]$.
7. λ_{n-1} is preserved in $V[\mathbb{R}_n * \dot{\mathbb{Q}}_n]$. In $V[\mathbb{R}_n * \dot{\mathbb{Q}}_n]$, we have $2^{\lambda_{i-2}} = \lambda_i$ for $i \leq n$.
8. $\mathrm{Add}(\lambda_{n-2}, \eta)_{V[\mathbb{R}_{n-1}]}$ is λ_{n-1}-Knaster in $V[\mathbb{R}_n * \dot{\mathbb{Q}}_n]$ for any ordinal η.
9. $V[\mathbb{R}_n * \dot{\mathbb{Q}}_n] \models$ "$\mathrm{Add}(\lambda_{n-1}, \eta)_{V[\mathbb{R}_n]}$ is $< \lambda_{n-1}$-distributive and λ_n-Knaster" for any ordinal η.

To show that the cardinal arithmetic works out after forcing with $\mathbb{R}_\omega$, we use the following lemma.

Lemma 4.3 (Lemma 4.4 in [3])**.** Let G_ω be $\mathbb{R}_\omega$-generic and $X \in V[G_\omega]$ is a λ_n-sequence of ordinals. Then $X \in V[G_0][\cdots][G_n][G_{n+1}][g_{n+2}]$, where $G_0 * \cdots * G_n * G_{n+1} * g_{n+2}$ is the initial segment of G_ω which is V-generic for $\mathbb{Q}_0 * \cdots * \dot{\mathbb{Q}}_{n+1} * \dot{\mathbb{P}}_{n+2}$. In the case where $X \in V[G_\omega]$ is a μ-sequence of ordinals, it follows that $X \in V[G_0][g_1]$.

Proof. For clarity, we show the case when $n = 0$. Since $\mathbb{R}_\omega/\mathbb{R}_4$ is λ_1-closed, it follows that $X \in V[G_0][G_1][G_2][G_3]$. Since $\mathbb{Q}_3$ is $< \lambda_1$-distributive, it follows $X \in V[G_0][G_1][G_2]$. We therefore have that $X \in V[G_0][G_1][g_2]$ since all λ_0-sequences of ordinals in $V[G_0][G_1][G_2]$ are in $V[G_0][G_1][g_2]$. □

From the previous results we have the following:

Lemma 4.4. After forcing with $\mathbb{R}_\omega$ we have the following cardinal structure:

1. $\mathrm{cf}(\kappa) = \omega$,
2. $\mu = \kappa^+$,
3. $\lambda_n = \kappa^{+n+2}$ for all n,
4. $2^\kappa = \lambda_0$,
5. $2^\mu = \lambda_1$,
6. $2^{\lambda_n} = \lambda_{n+2}$ for all n.

5 Tree property at κ^{++}

In this section we modify the Cummings-Foreman argument that the tree property holds in $V^{\mathbb{R}_\omega}$ to account for the presence of $\mathbb{Q}_0$. If T is a λ_0-tree, it follows by Lemma 4.3 that $T \in V[G_0][G_1][g_2]$. It is therefore enough to show that there is no λ_0-Aronszajn tree in $V[G_0][G_1][g_2]$.

5.1 First we lift...

Recall that $\lambda = \sup \lambda_n$. Fix an elementary embedding $j\colon V \to M$ with $\mathrm{crit}(j) = \lambda_0 > \lambda$, ${}^{\lambda}M \subseteq M$, and where $j(F_0)(\lambda_0)$ is the canonical $\mathbb{Q}_0$ name for $\mathbb{P}_2 \times \mathbb{U}_1$. Observe that $j(F_0)(\lambda_0)$ is a $\mathbb{Q}_0$-name for a λ_0 directed closed forcing in $V[\mathbb{Q}_0]$.

We start with a set of claims which will let us do some heavy lifting to turn $j\colon V \to M$ into $j\colon V[G_0][G_1][g_2] \to M[H_0][H_1][h_2]$. This argument is essentially the Six Stages listed in Section 4 of [3], but is fleshed out for completeness, for clarity, and to emphasize the parts where we are using the new factor $\mathbb{Q}_0$.

Claim 5.1. There's a $V[G_0][g_2]$-generic filter $g_1 \times u_1$ for $\mathbb{P}_1 \times \mathbb{U}_1$ such that $g_1 \times u_1 \times g_2$ is generic for $\mathbb{P}_1 \times \mathbb{U}_1 \times \mathbb{P}_2$ over $V[G_0]$.

Proof. Since $G_0 * G_1 * g_2$ is V-generic for $\mathbb{Q}_0 * \mathbb{Q}_1 * \mathbb{P}_2$, it follows that G_1 and g_2 are mutually generic over $V[G_0]$. So, in $V[G_0][g_2][G_1]$, we may consider the quotient forcing $\mathbb{S}$ of $\mathbb{P}_1 \times \mathbb{U}_1$ and $\mathbb{Q}_1$. If $g_1 \times u_1$ is generic for $\mathbb{S}$ over $V[G_0][g_2][G_1]$, then it's also generic for $\mathbb{P}_1 \times \mathbb{U}_1$ over $V[G_0][g_2]$. The product lemma implies that g_2 and $g_1 \times u_1$ are mutually generic over $V[G_0]$, and so $g_1 \times u_1 \times g_2$ is generic for $\mathbb{P}_1 \times \mathbb{U}_1 \times \mathbb{P}_2$ over $V[G_0]$. □

Claim 5.2. There's a H_0 such that H_0 is $V[g_1]$-generic for $j(\mathbb{Q}_0)$ with $H_0 \restriction \lambda_0 + 1 = G_0 * (g_2 \times u_1)$ and H_0 collapses λ_0 and λ_1 to cardinality μ. Further, we may lift to j to $j\colon V[G_0] \to M[H_0]$.

Proof. In M, observe that $j(\mathbb{Q}_0) \restriction \lambda_0 = \mathbb{Q}_0$ and that elementarity implies $j(\mathbb{Q}_0)$ projects onto $\mathbb{Q}_0 * j(F_0)(\lambda_0) = \mathbb{Q}_0 * (\mathbb{U}_1 \times \mathbb{P}_2)$. Now, our choice of j implies $\mathbb{P}_2$ and $\mathbb{U}_1$ are really the forcings defined in $V[G_0]$ (also using V as a parameter for $\mathbb{U}_1$). However, the chain condition of $\mathbb{Q}_0$ and the closure of M imply $\mathbb{P}_2$ and $\mathbb{U}_1$ are the same in $V[G_0]$ as in $M[G_0]$ (using M instead of V in the definition for $\mathbb{U}_1$).

Anyways, the projection given by elementarity is in $V[G_0][g_1 \times u_1 \times g_2]$, so let H_0 be $V[G_0][g_1 \times u_1 \times g_2]$-generic for $j(\mathbb{Q}_0)/(G_0 * (g_2 \times u_1))$. Facts about projections tell us that H_0 is $V[g_1]$-generic for $j(\mathbb{Q}_0)$ and is generated from $G_0 * (g_2 \times u_1)$. Also, we get that $H_0 \restriction \lambda_0 + 1 = G_0 * (g_2 \times u_1)$.

Further, we have that $H_0 \restriction \lambda_0 = G_0$. Since the elements in $\mathbb{Q}_0$ have size less than the critical point of j, it follows that we may lift to $j\colon V[G_0] \to M[H_0]$. $j(\mathbb{Q}_0)$ collapses all ordinals between $j(\mu) = \mu$ and $j(\lambda_0)$ to μ, and since $\lambda_0, \lambda_1 < j(\lambda_0)$, we have that H_0 collapses λ_0 and λ_1 to μ. □

Claim 5.3. There's a generic object h_1 for $j(\mathbb{P}_1)$ over $V[G_0][H_0]$ so that $j"g_1 \subseteq h_1$. This allows us to lift to $j\colon V[G_0][g_1] \to M[H_0][h_1]$.

Proof. Observe that elementarity and sufficient closure of M implies that $j(\mathbb{P}_1) = \mathrm{Add}(\mu, j(\lambda_1))_V$. Observe also that there's a projection of $j(\mathbb{P}_1)$ onto $\mathbb{P}_1$. So, we may take an h_1 generic for $j(\mathbb{P}_1)/g_1$ over $V[G_0][H_0][g_1]$. This will satisfy $j"g_1 \subseteq h_1$ and let us lift the embedding j. □

Claim 5.4. There's a generic object x_1 for $j(\mathbb{U}_1)$ over $V[G_0][H_0][h_1]$ such that $j"u_1 \subseteq x_1$. Further, x_1 and h_1 are mutually generic over $M[H_0]$ by Easton's Lemma, so $h_1 \times x_1$ generates a filter H_1 generic for $j(\mathbb{Q}_1)$ over $M[H_0]$. We also have that $j"G_1 \subseteq H_1$, allowing us to lift to $j\colon V[G_0][G_1] \to M[H_0][H_1]$.

Proof. Observe that in $M[G_0][g_2 \times u_1]$, $|u_1| \leq |\mathbb{U}_1| = \lambda_1 < \lambda$. Since M is closed under λ sequences and the $\mathbb{Q}_0$, $\mathbb{P}_2$, and $\mathbb{U}_1$ have chain conditions smaller than λ (simply by cardinality considerations), it follows in $V[G_0][g_2 \times u_1]$ that ${}^{\lambda}M[G_0][g_2 \times u_1] \subseteq M[G_0][g_2 \times u_1]$. This implies that in $j"u_1 \in M[H_0]$ by choice of H_0.

Now, elementarity implies that, in $M[H_0]$, $j(\mathbb{U}_1)$ is $j(\mu^+)$-directed closed, where $j(\mu^+) = (\mu^+)^{M[H_0]}$ (note that μ is less than the critical point of j). We observed in Claim 5.2 that λ_1 is collapsed to μ in $M[H_0]$, and so it follows that there's a lower bound t in $j(\mathbb{U}_1)$ for $j"u_1$.

This let's us fix a $V[G_0][H_0][h_0]$-generic filter x_1 for $j(\mathbb{U}_1)$ with $j"u_1 \subseteq x_1$. The choice of x_1 and the product lemma implies that $x_1 \times h_1$ is generic over $M[H_0]$. By elementarity, we have that $j(\mathbb{P}_1) \times j(\mathbb{U}_1)$ projects onto $j(\mathbb{Q}_1)$, and so it follows $x_1 \times h_1$ induces a filter H_1 on $j(\mathbb{Q}_1)$ over $M[H_0]$. The argument for why $j"G_1 \subseteq H_1$ is exactly the same as "4.2.5 Stage Five" in [3], and so we may lift to $j\colon V[G_0][G_1] \to M[H_0][H_1]$. □

Claim 5.5. There's a $V[G_0][H_0][h_1][x_1]$-generic h_2 for $j(\mathbb{P}_2)$ such that $j"g_2 \subseteq h_2$. This lets us lift to $j\colon V[G_0][G_1][g_2] \to M[H_0][H_1][h_2]$.

Proof. This is "4.2.6 Stage Six" in [3]. □

5.2 ... then we go fishing.

Let T be a λ_0-tree in $V[G_0][G_1][g_2]$. The standard argument shows that T has a branch b in $M[H_0][H_1][h_2]$. Since M is closed under λ sequences and the $\mathbb{Q}_0$, $\mathbb{Q}_1$, and $\mathbb{P}_2$ have chain conditions smaller than λ (by cardinality considerations), we have in $V[G_0][G_1][g_2]$

that ${}^{\lambda}M[G_0][G_1][g_2] \subseteq M[G_0][G_1][g_2]$. It follows then that $T \in M[G_0][G_1][g_2]$. We start pulling back the branch with a lot of claims and some facts.

Fact 5.1 (Structural Fact)**.** $M[H_0][h_1] = M[G_0][g_2 \times u_1][H_0^*][g_1 \times h_1^*]$, where H_0^* is a generic object for $j(\mathbb{Q}_0)/(G_0 * (g_2 \times u_1))$ and h_1^* is a generic object for $j(\mathbb{P}_1)/\mathbb{P}_1$.

Fact 5.2 (Structural Fact)**.** $M[G_0][g_2 \times u_1][h_1] = M[G_0][g_1 \times u_1 \times g_2][h_1^*]$

Fact 5.3 (Structural Fact)**.** $M[H_0][h_1] \subseteq M[G_0][h_1 \times u_1 \times g_2][h_0^* \times u_0^*]$, where h_0^* is a generic object for $j(\mathbb{P}_0)/g_0$ and u_0^* is a generic object for $\mathbb{U}^*$

Proof of Structural Facts. By our choice of generics H_0 and h_1 we may decompose them in this way, and h_1 and H_0 are mutually generic over $M[G_0]$ by Claim 5.3. Also, $j(\mathbb{P}_0)/\mathbb{P}_0 \times \mathbb{U}^*$ projects onto $j(\mathbb{Q}_0)/(G_0 * (g_2 \times u_1))$. □

Fact 5.4. In $M[G_0][G_1][g_2]$, the quotient forcing $\mathbb{S} = (\mathbb{P}_1 \times \mathbb{U}_1)/\mathbb{Q}_1$ is μ-closed.

Proof. By Lemma 3.20 in [3], we have that $\mathbb{S}$ is μ-closed in $M[G_0][G_1]$. Next, $\mathbb{P}_2$ is $< \lambda_0$-distributive in $M[G_0][G_1]$ by Lemma 4.3(9) in [3], and so by an Easton's lemma variant we have that $\mathbb{S}$ is still μ-closed in $M[G_0][G_1][g_2]$. □

Fact 5.5. All κ-sequences of ordinals from $M[G_0][h_1 \times u_1 \times g_2]$ are in $M[G_0]$.

Proof. Found on the bottom of page 18 and top of page 19 in [3]. □

Fact 5.6. $j(\mathbb{P}_1)/\mathbb{P}_1$ is λ_0-Knaster and $< \mu$-distributive in $M[G_0][g_1 \times u_1 \times g_2]$.

Proof. Found on the bottom of page 18 in [3]. □

Claim 5.6. $b \in M[H_0][h_1]$

Proof. We know that $b \in M[H_0][H_1][h_2]$. Lemma 4.3(9) in [3] implies that $\mathbb{P}_2$ is $< \lambda_0$-distributive in $V[G_0][G_1]$ and so $j(\mathbb{P}_2)$ is $< j(\lambda_0)$-distributive in $M[H_0][H_1]$. Since b has length λ_0 it follows that $b \in M[H_0][H_1]$. By Claim 5.4 we have that $b \in M[H_0][h_1 \times x_1]$. By Lemma 3.11(2) in [3], we know $\mathbb{U}_1$ is $\leq \mu$-distributive in $V[G_0][g_1]$. Elementarity implies that $j(\mathbb{U}_1)$ is $\leq \mu$-distributive in $M[H_0][h_1]$. By elementarity we have that λ_0 is collapsed to μ in $M[H_0][h_1]$ and so it follows that $b \in M[H_0][h_1]$. □

Claim 5.7. $b \in M[G_0][h_1 \times u_1 \times g_2][u_0^*]$

Proof. For the remainder of this claim, let $\mathbb{T} = j(\mathbb{P}_0)/g_0$. By Fact 5.3 and Claim 5.6, we know $b \in M[G_0][h_1 \times u_1 \times g_2][u_0^* \times h_0^*]$. Forcing with $\mathbb{U}^*$ in $M[G_0][h_1 \times u_1 \times g_2]$ collapses λ_0 to μ, and so by Lemma 2.4 in [16], it's enough to argue that $M[G_0][h_1 \times u_1 \times g_2][u_0^*] \models \mathbb{T}^2$ is μ-cc. We will work backwards and reduce this hypothesis to a case solved by Unger in [16]. First observe that $\mathbb{U}^*$ is μ-closed in $M[G_0][h_1 \times u_1 \times g_2]$ because all κ-sequences of elements of $M[G_0][h_1 \times u_1 \times g_2]$ are in $M[G_0]$. It follows by Easton's lemma that it's enough to show $M[G_0][h_1 \times u_1 \times g_2] \models \mathbb{T}^2$ is $\mu - cc$.

Next, because $\mathbb{U}_1 \times \mathbb{P}_2$ is λ_0-directed closed in $M[G_0]$ and $j(\mathbb{P}_1)$ is λ_0-Knaster in $M[G_0]$ (Lemma 4.2 in [3]), it follows by Easton's Lemma that $u_1 \times g_2$ is generic for a μ-distributive forcing over $M[G_0][h_1]$. It's therefore enough to show that $M[G_0][h_1] \models \mathbb{T}^2$ is $\mu - cc$ because an antichain in $M[G_0][h_1 \times u_1 \times g_2]$ of $\mathbb{T}^2$ with size μ will be in $M[G_0][h_1]$ by distributivity.

Observe that h_1 and G_0 are mutually generic, and so we have that $M[G_0][h_1] = M[h_1][G_0]$. It's enough then to show that $M[h_1][g_0 \times u_0] \models \mathbb{T}^2$ is μ-cc because $\mathbb{P}_0 \times \mathbb{U}$ projects onto $\mathbb{Q}_0$.

Towards this end, observe that Lemma 5.3 in [16] implies that $\Vdash^M_{\mathbb{P}_0} \mathbb{T}^2$ is μ-cc. This implies that $M \models \mathbb{P}_0 * \mathbb{T}^2$ is μ-cc. $j(\mathbb{P}_1)$ is μ-closed in M, and so $M[h_1] \models \mathbb{P}_0 * \mathbb{T}^2$ is μ-cc. Finally, $\mathbb{U}$ is μ-closed in $M[h_1]$, so Easton's lemma implies that $M[h_1][u_0] \models \mathbb{P}_0 * \mathbb{T}^2$ is μ-cc. So, $M[h_1][u_0 \times g_0] \models \mathbb{T}^2$ is μ-cc, completing the claim.

□

Claim 5.8. $b \in M[G_0][h_1 \times u_1 \times g_2]$

Proof. Notice that forcing with $\mathbb{U}^*$ is μ-closed in $M[G_0][h_1 \times u_1 \times g_2]$ by Fact 5.5 and that $2^\kappa = \lambda_0$ in $M[G_0][h_1 \times u_1 \times g_2]$. It follows by Silver's branch lemma that $b \in M[G_0][h_1 \times u_1 \times g_2]$. □

Claim 5.9. $b \in M[G_0][g_1 \times u_1 \times g_2]$

Proof. $j(\mathbb{P}_1)/g_1$ is λ_0-Knaster in $M[G_0][g_1 \times u_1 \times g_2]$ by Fact 5.6, and T is a tree of height λ_0 in this model. It follows that forcing with T won't add any new branches, and so $b \in M[G_0][g_1 \times u_1 \times g_2]$. □

Claim 5.10. $b \in M[G_0][G_1][g_2]$

Proof. By Fact 5.4 and since $2^\kappa = \lambda_0$ in $M[G_0][G_1][g_2]$, we have by Silver's branch lemma that forcing with $\mathbb{S}$ to get $M[G_0][g_1 \times u_1 \times g_2]$ doesn't add any branches. So $b \in M[G_0][G_1][g_2]$.

□

6 Tree property at κ^{+n} for $n \geq 3$

The argument when $n \geq 3$ is almost exactly the same as the argument in the Cummings-Foreman paper [3]. Following their notation for this section, we let $V_n \coloneqq V[G_0][G_1]\cdots[G_n]$ and $M_n \coloneqq M[G_0][G_1]\cdots[G_n]$. The main difference between our forcing and the forcing in Cummings-Foreman paper is the first factor $\mathbb{Q}_0$, and the main difference between our argument from the previous section and their argument was Claim 5.7. When $n \geq 3$, though, we may immediately lift our embedding to $j\colon V_{n-3} \to M_{n-3}$ because $\mathrm{crit}(j) = \lambda_n$ and $|\mathbb{R}_n| = \lambda_{n-1}$. This avoids the need for Claim 5.7 because we no longer have pull a branch from $M[\cdots H_0 \cdots]$ back to $M[\cdots G_0 \cdots]$, and so we no longer have to deal with the generic objects h_0^* and u_0^*.

7 Generalization of Sinapova's Forcing

In this section and the following section we use the framework developed in [12] to argue that the tree property holds at κ^+. Much of the argument is the same, so we aim to describe the relevant forcing, prove some relevant structural properties about that forcing, and summarize how the argument in [12] is done in this particular circumstance. Recall that $\mathbb{A} = \mathrm{Add}(\kappa, \lambda_0)$ and $\mathbb{P}_0 = \mathbb{A} * \dot{\mathbb{I}}$. Consider the μ-closed forcing $\mathbb{Q} = \mathbb{U} \times \mathbb{P}_1$, where $\mathbb{P}_1 = \mathrm{Add}(\mu, \lambda_1)_V$ and $\mathbb{U}$ are the conditions of the form $(0, 0, q, f) \in \mathbb{Q}_0$ with the induced suborder from $\mathbb{Q}_0$. One of the main differences between [12] and our situation is that the poset $\mathbb{Q}$ in [12] doesn't include the poset $\mathbb{P}_1$.

Definition 7.1. Let $\dot{p}$ be a name for a condition in $\dot{\mathbb{I}}$. Define $\mathbb{R}_{\dot{p}}$ to have underlying set $\mathbb{Q}_0 \times \mathbb{P}_1$ with the following (modified) ordering:

Declare that $(f_1, \dot{p}_1, r_1, g_1, a_1) \leq_{\dot{p}} (f_2, \dot{p}_2, r_2, g_2, a_2)$ exactly when

1. $a_1 \leq_{\mathbb{P}_1} a_2$
2. $(f_1, \dot{p}_1) \leq_{\mathbb{P}_0} (f_2, \dot{p}_2)$
3. $\mathrm{dom}(r_1) \supseteq \mathrm{dom}(r_2)$ and for every $\alpha \in \mathrm{dom}(r_2)$, we have that $(f_1, \dot{p}) \restriction \alpha \Vdash_{\mathbb{P}_{0,\alpha}} r_1(\alpha) \leq r_2(\alpha)$
4. $\mathrm{dom}(g_1) \supseteq \mathrm{dom}(g_2)$ and for every $\alpha \in \mathrm{dom}(g_2)$, we have that $(f_1, \dot{p}, r_1, g_1) \restriction \alpha \Vdash_{\mathbb{Q}_0 \restriction \alpha} g_1(\alpha) \leq g_2(\alpha)$.

Lemma 7.1. $\mathbb{P}_0 \times \mathbb{Q}$ projects to $\mathbb{R}_{\dot{p}}$, witnessed by the identity.

Proof. It is straightforward to see that the identity map is order preserving. To show the other requirement, suppose that $(f_1, \dot{p}_1, r_1, g_1, a_1) \leq_{\dot{p}} (f_2, \dot{p}_2, r_2, g_2, a_2)$. We define $\overline{r}$ by setting $\mathrm{dom}(\overline{r}) = \mathrm{dom}(r_1)$ and define $\overline{r}(\alpha)$ by mixing names so that

$$\Vdash_{\mathbb{P}_{0,\alpha}} \overline{r}(\alpha) \leq r_2(\alpha) \text{ and } (f_1, \dot{p}) \restriction \alpha \Vdash_{\mathbb{P}_{0,\alpha}} \overline{r}(\alpha) = r_1(\alpha).$$

More specifically, we may let

$$\overline{r}(\alpha) = \{\langle \sigma, b \rangle \colon b \leq \langle f_1, \dot{p} \rangle \restriction \alpha \text{ and } b \Vdash_{\mathbb{P}_{0,\alpha}} \sigma \in r_1(\alpha)\} \cup \{\langle \sigma, b \rangle \colon b \perp \langle f_1, \dot{p} \rangle \restriction \alpha \text{ and } b \Vdash_{\mathbb{P}_{0,\alpha}} \sigma \in r_2(\alpha)\}.$$

Next, by induction, we define $\overline{g}(\alpha)$ by induction so that

$$(0, 0, \overline{r}, \overline{g}) \restriction \alpha \Vdash_{\mathbb{Q}_0 \restriction \alpha} \overline{g}(\alpha) \leq g_2(\alpha) \text{ and } (f_1, \dot{p}, \overline{r}, \overline{g}) \restriction \alpha \Vdash_{\mathbb{Q}_0 \restriction \alpha} \overline{g}(\alpha) = g_1(\alpha).$$

The construction during the induction step is the similar to the construction of $\overline{r}(\alpha)$ above. But then, by definition, we have as desired that

1. $(f_1, \dot{p}_1, \overline{r}, \overline{g}, a_1) \leq_{\mathbb{P}_0 \times \mathbb{Q}} (f_2, \dot{p}_2, r_2, g_2, a_2)$ and
2. $(f_1, \dot{p}_1, \overline{r}, \overline{g}, a_1) \leq_{\dot{p}} (f_1, \dot{p}_1, r_1, g_1, a_1)$.

$\square$

Lemma 7.2. Let $s^* = (0, \dot{p}, 0, 0, 0) \in \mathbb{Q}_0 \times \mathbb{P}_1$. Then $\mathbb{R}_{\dot{p}}/s^* = \{s \in \mathbb{R}_{\dot{p}} |\ s \leq_{\dot{p}} s^*\}$ projects to $(\mathbb{Q}_0 \times \mathbb{P}_1)/s^* = \{s \in \mathbb{Q}_0 \times \mathbb{P}_1 |\ s \leq_{\mathbb{Q}_0 \times \mathbb{P}_1} s^*\}$ witnessed by the identity.

Proof. The proof is similar to the previous lemma. Since the last three coordinates of s^* are trivial, notice that $s \leq_{\dot{p}} s^*$ iff $s \leq_{\mathbb{Q}_0 \times \mathbb{P}_1} s^*$. The identity is order preserving, so it's enough to check the nontrivial condition for projections:

Suppose $(f_1, \dot{p}_1, r_1, g_1, a_1) \leq_{\mathbb{Q}_0 \times \mathbb{P}_1} (f_2, \dot{p}_2, r_2, g_2, a_2)$. Define $\overline{r}(\alpha)$ and $\overline{g}(\alpha)$ similar to the previous lemma so that the following hold:

- $\Vdash_{\mathbb{P}_{0,\alpha}} \overline{r}(\alpha) \leq r_2(\alpha)$,
- $(f_1, \dot{p}_1) \restriction \alpha \Vdash_{\mathbb{P}_{0,\alpha}} \overline{r}(\alpha) = r_1(\alpha)$,
- $(0, 0, \overline{r}, \overline{g}) \restriction \alpha \Vdash_{\mathbb{Q}_0 \restriction \alpha} \overline{g}(\alpha) \leq g_2(\alpha)$, and
- $(f_1, \dot{p}_1, \overline{r}, \overline{g}) \restriction \alpha \Vdash_{\mathbb{Q}_0 \restriction \alpha} \overline{g}(\alpha) = g_1(\alpha)$.

It follows as desired that

1. $(f_1, \dot p_1, \overline{r}, \overline{g}, a_1) \leq_{\dot p} (f_2, \dot p_2, r_2, g_2, a_2)$ and
2. $(f_1, \dot p_1, \overline{r}, \overline{g}, a_1) \leq_{\mathbb{Q}_0 \times \mathbb{P}_1} (f_1, \dot p_1, r_1, g_1, a_1)$.

□

Definition 7.2. Let $\mathcal{A}$ be $\mathbb{A}$-generic over V. Let $p = \dot p_{\mathcal{A}}$. Define $\mathbb{Q}_p$ to have underlying set $\mathbb{Q}$ with the ordering $(0, 0, q_1, f_1, p_1) \leq_{\mathbb{Q}_p} (0, 0, q_2, f_2, p_2)$ exactly when

1. $p_1 \leq_{\mathbb{P}_1} p_2$
2. $\mathrm{dom}(q_1) \supseteq \mathrm{dom}(q_2)$ and $\mathrm{dom}(f_1) \supseteq \mathrm{dom}(f_2)$
3. there's an $a \in \mathcal{A}$ such that for every $\alpha \in \mathrm{dom}(q_2)$ and for every $\alpha \in \mathrm{dom}(f_2)$, we have that
   (a) $(a, \dot p) \restriction \alpha \Vdash_{\mathbb{P}_{0,\alpha}} q_1(\alpha) \leq q_2(\alpha)$
   (b) $(a, \dot p, q_1, f_1) \restriction \alpha \Vdash_{\mathbb{Q}_0 \restriction \alpha} f_1(\alpha) \leq f_2(\alpha)$

Lemma 7.3. $\mathbb{Q}_p$ is κ-closed

Proof. Assume that $\{(0, 0, q_i, f_i, p_i) : i < \theta\}$ is decreasing for $\theta < \kappa$. For each $i < \theta$ there's an $a_i \in \mathcal{A}$ such that (3) holds in the above definition. Since $\mathbb{A}$ is κ-directed closed, it follows that $a = \bigcup_{i<\theta} a_i \in \mathcal{A}$. Since $\mathbb{P}_1$ is μ-closed, we may let $\overline{p}$ be a lower bound of the p_i's. Next, observe that for each α and each $i < j < \theta$, we have $(a, \dot p) \restriction \alpha \Vdash_{\mathbb{P}_{0,\alpha}} q_j(\alpha) \leq q_i(\alpha)$. Since the $q_i(\alpha)$'s are forced to be in a μ-closed forcing, it follows that there's a name $\overline{q}(\alpha)$ such that $(a, \dot p) \restriction \alpha \Vdash_{\mathbb{P}_{0,\alpha}} \overline{q}(\alpha) \leq q_i(\alpha)$ for each $i < \theta$. Finally, define $\overline{f}(\alpha)$ by induction on α so that $(a, \dot p, \overline{q}, \overline{f}) \restriction \alpha \Vdash_{\mathbb{Q}_0 \restriction \alpha} \overline{f}(\alpha) \leq f_i(\alpha)$. This is possible since the $f_i(\alpha)$'s are forced to be in a μ-closed forcing as well. Then, $(0, 0, \overline{q}, \overline{f}, \overline{p})$ is a lower bound for the initial decreasing sequence, as desired. □

Lemma 7.4. $\mathbb{R}_p$ is isomorphic to $\mathbb{A} * (\dot{\mathbb{I}} \times \dot{\mathbb{Q}}_p)$

Proof. In $V[\mathcal{A}]$, we argue that $\pi : \mathbb{R}_p/\mathcal{A} \to \mathbb{I} \times \mathbb{Q}_p$ defined by $(f_1, \dot p_1, r_1, g_1, a_1) \mapsto (\dot p_1^{\mathcal{A}}, r_1, g_1, a_1)$ is a dense embedding. Since π is onto, it is enough to show that π is order preserving and that $s \perp_{\mathbb{R}_p/\mathcal{A}} s'$ implies $\pi(s) \perp_{\mathbb{I} \times \mathbb{Q}_p} \pi(s')$.

Assume that $(f_1, \dot p_1, r_1, g_1, a_1) \leq_{\mathbb{R}_p/\mathcal{A}} (f_2, \dot p_2, r_2, g_2, a_2)$. Since $(f_1, \dot p_1) \leq_{\mathbb{P}_0} (f_2, \dot p_2)$ and $f_1 \in \mathcal{A}$, it follows that $\dot p_1^{\mathcal{A}} \leq_{\mathbb{I}} \dot p_2^{\mathcal{A}}$. It follows in turn that $(0, 0, r_1, g_1, a_1) \leq_{\mathbb{Q}_p} (0, 0, r_2, g_2, a_2)$.

Next, assume that $\pi(f_1, \dot p_1, r_1, g_1, a_1)$ and $\pi(f_2, \dot p_2, r_2, g_2, a_2)$ were compatible in $\mathbb{I} \times \mathbb{Q}_p$. Since π is onto, we may let $\pi(f, \dot q, r, g, a)$ witness this. Let $\overline{a}_1 \in \mathcal{A}$ witness that $(0, 0, r, g, a) \leq_{\mathbb{Q}_p} (0, 0, r_1, g_1, a_1)$ and $\overline{a}_2 \in \mathcal{A}$ witness that $(0, 0, r, g, a) \leq_{\mathbb{Q}_p} (0, 0, r_2, g_2, a_2)$. Let $\overline{a} \in \mathcal{A}$ be such that $\overline{a} \leq \overline{a}_1, \overline{a}_2$. By further extending $\overline{a}$, we may assume that $\overline{a} \Vdash_{\mathbb{A}} \dot q \leq \dot p_1, \dot p_2$. Finally, Let $\overline{f} \in \mathcal{A}$ extend $\overline{a}$, f_1, and f_2. It follows that $(\overline{f}, \dot q, r, g, a)$ witnesses that $(f_1, \dot p_1, r_1, g_1, a_1)$ and $(f_2, \dot p_2, r_2, g_2, a_2)$ are compatible in $\mathbb{R}_p/\mathcal{A}$. Therefore, π is a dense embedding. □

7.1 h-splittings and $\dagger_h$

One of the main technical tasks in [12] is proving Proposition 3.4, which is crucial in defining the branch in the forcing extension by the Mitchell poset. In this section we give the relevant definitions and summarize the results necessary to prove this result.

Let $G \times g_1$ be $\mathbb{Q}_0 \times \mathbb{P}_1$-generic, $\mathcal{A}$ be $\mathbb{A}$-generic induced from $G \times g_1$, and $\mathcal{I}$ be the $\mathbb{I}$ generic induced over $V[\mathcal{A}]$ induced by $G \times g_1$. Further, let G^* be $(\mathbb{P}_0 \times \mathbb{Q})/(G \times g_1)$-generic. For every $q \in \mathcal{I}$, let G_q be $\mathbb{R}_{\dot{q}}/(G \times g_1)$-generic induced by G^*. Also, let $\mathcal{Q}$ be $\mathbb{Q}$ generic induced by G^* and for $p \in \mathbb{I}$, let $\mathcal{Q}_p$ be $\mathbb{Q}_p$-generic over $V[\mathcal{A}]$ induced by G^*.

Let $\tau \in V[\mathcal{A}]$ be an $\mathbb{R}/\mathcal{A}$-name for the tree, forced by the empty condition. Let $\dot{T} \in V[\mathcal{A}][\mathcal{Q}]$ be an $\mathbb{I}$ name for the tree, obtained from τ. This means that $q \Vdash_{\mathbb{I}} u <_{\dot{T}} v$ iff there's $a \in \mathcal{A}$ and $r \in \mathcal{Q}_p$ such that $(a, \dot{q}, r) \Vdash_{\mathbb{R}/\mathcal{A}} u <_\tau v$.

Let $\dot{b} \in V[\mathcal{A}]$ be a $(\mathbb{P}_0 \times \mathbb{Q})/\mathcal{A}$-name for the branch given by Itay [11], where we assume that

$$1 \Vdash^{V[\mathcal{A}]}_{\mathcal{I} \times \mathbb{Q}} \dot{b} \text{ is a cofinal branch through } \tau.$$

The following are Definitions 3.3 and 3.4 in [12].

Definition 7.3. Let h be a stem. Say there's an **h-splitting** at a node u if there is a $p \in \mathbb{I}$ with $\text{stem}(p) = h$ and $r \in \mathcal{Q}$ such that $(p, r) \Vdash^{V[\mathcal{A}]}_{\mathbb{I} \times \mathbb{Q}} u \in \dot{b}$ and nodes u_1, u_2 of higher levels and conditions r_1, r_2 such that for $k = 1$ or 2,

1. $r_k \leq_{\mathbb{Q}} r$, $r_k \in \mathcal{Q}_p$,
2. $(p, r_k) \Vdash^{V[\mathcal{A}]}_{\mathbb{I} \times \mathbb{Q}} u_k \in \dot{b}$, and
3. $p \Vdash^{V[\mathcal{A}][\mathcal{Q}]}_{\mathbb{I}} u_1 \perp_{\dot{T}} u_2$

Definition 7.4. Let h be a stem. Say that $\dagger_h$ **holds**, if in $V[\mathcal{A}][\mathcal{Q}]$ there are unbounded $J \subseteq \mu, \xi < \kappa$, and $(p_\alpha | \alpha \in J)$, where each $p_\alpha \in \mathbb{I}$ is a condition with stem h, and setting $u_\alpha = (\alpha, \xi)$ we have:

1. for all $\alpha < \beta$ from J, $p_\alpha \wedge p_\beta \Vdash_{\mathbb{I}} u_\alpha <_{\dot{T}} u_\beta$;
2. for all $\alpha \in J$, $p_\alpha \Vdash_{\mathbb{I}} u_\alpha \in \dot{b}$.

The key property involving $\dagger_h$ is Propostion 3.4 in [12]:

Proposition 7.1. Let h be a stem such that $\dagger_h$ holds; then $\alpha_h < \mu$.

The proof of this key property amounts to proving the splitting lemma stated below. The proof of this splitting lemma is exactly the same as in [12].

Lemma 7.5 (Splitting Lemma)**.** Let h be a stem such that $\dagger_h$ holds. Let $r \leq_{\mathbb{Q}} \overline{r}$ be such that $r \in \mathcal{Q}_q$ for some q with stem h. Then there are nodes $(v_i | i < \kappa_\omega)$ and conditions $((p_i, r_i) | i < \kappa_\omega)$ in $\mathbb{I} \times \mathbb{Q}$ such that:

1. for every i, $\text{stem}(p_i) = h$, $p_i \leq q$, $r_i \leq_{\mathbb{Q}} r$, $r_i \in Q_{p_i}$
2. for every i, $(p_i, r_i) \Vdash_{\mathbb{I} \times \mathbb{Q}} v_i \in \dot{b}$ and
3. for every $i < j$, $p_i \wedge p_j \Vdash_{\mathbb{I}} v_i \perp_{\dot{T}} v_j$.

8 Tree property at κ^+

In this section we finish Theorem 1.1 by proving the following:

Theorem 8.1. In $V[G_\omega]$, the tree property holds at κ^+.

Proof. Assume that T is a μ-tree in $V[G_\omega]$. Working in $V^{\mathbb{R}_3}$, the forcing $\mathbb{R}_\omega/\mathbb{R}_3$ is λ_0-closed and so $T \in V[G_0][G_1][G_2]$. $\mathbb{Q}_2$ is $< \lambda_0$-distributive in $V[G_0][G_1]$, and so $T \in V[G_0][G_1]$. It follows that $T \in V[G_0][g_1]$, where g_1 is $\mathbb{P}_1$-generic over $V[G_1]$. So, it is enough to show that there's no μ-Aronszajn tree in $V[G_0][g_1]$. Assume for the sake of contradiction that T is a μ-Aronszajn tree in $V[G_0][g_1]$.

By construction of $\mathbb{Q}_0$, we have that $\mathbb{Q}_0$ is the projection of $\mathbb{P}_0 \times \mathbb{U}$, where $\mathbb{U}$ is μ-directed closed in V. Since $\mathbb{P}_1$ is μ-directed closed in V, we also have that the product $\mathbb{Q}$ is μ-closed. It follows by indestructibility of each κ_n that in $V^{\mathbb{Q}}$, $\mathbb{P}_0$ is Neeman's forcing from [11]. Then, if we let $\mathcal{A} \times \mathcal{Q}$ be $\mathbb{P}_0 \times \mathbb{Q}$-generic, we have by [11] that μ has the tree property in $V[\mathcal{A} \times \mathcal{Q}]$.

We follow the final section of [12] to define a branch for T in $V[G_0][g_1]$. In particular, by Proposition 7.1 we may define $\alpha = \sup\{\alpha_h \mid \dagger_h \text{ holds}\} < \mu$. Further, let $u \in T_\alpha$ and $s^* \in G^*$ be such that $s^* \Vdash_{(\mathbb{P}_0 \times \mathbb{Q})/\mathcal{A}} u \in \dot{b}$. In $V[G_0][g_1]$, define

$$d = \{v \mid u <_T v \text{ and there's a } s \in G_0 \times g_1 \text{ such that } s \leq_{\mathbb{P}_0 \times \mathbb{Q}} s^* \text{ and } s \Vdash_{(\mathbb{P}_0 \times \mathbb{Q})/\mathcal{A}} v \in \dot{b}\}.$$

The same argument as [12] shows that d induces a branch through T in $V[G_0][g_1]$. This contradicts that T is an Aronszajn tree and yields the result.

□

9 Adding collapses

The method for proving Theorem 1.1 in the previous sections involved modifying the Cummings-Foreman forcing with a Prikry part in the first factor. Let $\overline{V} \models ZFC$ be the ground model. To add collapses, we will use a slightly different method by first preparing the ground model and then forcing with (roughly) the original Cummings-Foreman forcing $\mathbb{R}_\omega$. After forcing with this we will force with the Prikry poset. In particular, we force with an iteration $\mathbb{C}$ of Levy collapses in $\overline{V}$, so that for each n:

1. $\kappa_n = \kappa^{+n}$ and κ_n is generically supercompact, and
2. after forcing with $\mathbb{R}_\omega$, we have normal measures U_n on $P_\kappa(\kappa_n)$, and "guiding generics" K_n for $\text{Coll}(\kappa^{+\omega+3}, < j_{U_n}(\kappa))$ in the ultrapower of $\mathbb{R}_\omega$ by U_n.

Note, the final model in this section is $\overline{V}[\mathbb{C} * \mathbb{R}_\omega * (\text{Prikry})]$. For the remainder of this section, $V = \overline{V}[\mathbb{C}]$.

9.1 The forcing notion

We define the forcing $\mathbb{R}_\omega$, but first we must modify (again) the first factor. Notice that the difference between our new first factor and the forcing $\mathbb{Q}_0$ defined earlier in this paper is that the first coordinate is now just the ordinary Cohen Forcing.

Definition 9.1. Working in V, let $\mathbb{P} = \text{Add}(\kappa, \lambda_1)$ and let $\mathbb{P} \restriction \beta = \text{Add}(\kappa, \beta)$ for $\beta < \lambda_1$. Define $\mathbb{R}_1$ by recursion on $\beta \leq \lambda_1$ and set $\mathbb{R}_1 = \mathbb{R}_1 \restriction \lambda_1$. Let $\mathbb{R}_1 \restriction 0$ is the trivial forcing. Otherwise, (p, q, f) is a condition in $\mathbb{R}_1 \restriction \beta$ when the following hold:

1. $p \in \mathbb{P} \restriction \beta$,
2. q is a partial function on β and $|\mathrm{dom}(q)| < \mu$, and if $\alpha \in \mathrm{dom}(q)$, then
   (a) α is a successor ordinal,
   (b) $q(\alpha) \in V^{\mathbb{P}\restriction\alpha}$, and
   (c) $\Vdash_{\mathbb{P}\restriction\alpha} q(\alpha) \in \mathrm{Add}(\mu, 1)_{V^{\mathbb{P}\restriction\alpha}}$,
3. f is a partial function on β and $|\mathrm{dom}(f)| < \mu$, and if $\alpha \in \mathrm{dom}(f)$, then
   (a) $\Vdash_{\mathbb{R}_1\restriction\alpha} F_0(\alpha)$ is a canonically μ-directed closed forcing,
   (b) α is a limit ordinal,
   (c) $f(\alpha) \in V^{\mathbb{R}\restriction\alpha}$, and
   (d) $\Vdash_{\mathbb{R}_1\restriction\alpha} f(\alpha) \in F(\alpha)$.

We also define the ordering $(p_1, q_1, f_2) \leq (p_2, q_2, f_2)$ when the following hold:

1. $p_1 \leq_{\mathbb{P}\restriction\alpha} p_2$,
2. $\mathrm{dom}(q_2) \subseteq \mathrm{dom}(q_1)$ and if $\alpha \in \mathrm{dom}(q_2)$, then $p_1 \restriction \alpha \Vdash_{\mathbb{P}\restriction\alpha} q_1(\alpha) \leq q_2(\alpha)$,
3. $\mathrm{dom}(f_2) \subseteq \mathrm{dom}(f_1)$ and if $\alpha \in \mathrm{dom}(f_2)$, then $(p_1, q_1, f_1) \restriction \alpha \Vdash_{\mathbb{R}_1\restriction\alpha} f_1(\alpha) \leq f_2(\alpha)$.

Definition 9.2. We proceed in the same manner as [3].

1. Let $\mathbb{R}_1 = \mathbb{Q}_0$.
2. Let $\dot{F}_1$ be a $\mathbb{Q}_0$-name for a function on λ_1 such that $\Vdash_{\mathbb{Q}_0} \dot{F}_1(\alpha) = F_1(\alpha)$ when $F_1(\alpha)$ is a $\mathbb{Q}_0$-name and $\Vdash_{\mathbb{Q}_0} \dot{F}_1(\alpha) = 0$ otherwise. Then define $\dot{\mathbb{Q}}_1$ to be the canonical name for $\mathbb{R}(\mu, \lambda_1, V, V[\mathbb{Q}_0], F_1^*)$ where F_1^* is the interpretation of $\dot{F}_1$ in $V[\mathbb{Q}_0]$. Let $\mathbb{R}_2 = \mathbb{Q}_0 * \dot{\mathbb{Q}}_1$.
3. Similarly, for $n \geq 2$, let $\mathbb{R}_n = \mathbb{Q}_0 * \ldots * \dot{\mathbb{Q}}_{n-1}$ and let $\dot{F}_n$ be a $\mathbb{R}_n$-name for a function on λ_n such that $\Vdash_{\mathbb{R}_n} \dot{F}_n(\alpha) = F_n(\alpha)$ when $F_n(\alpha)$ is a $\mathbb{R}_n$-name and $\Vdash_{\mathbb{R}_n} \dot{F}_n(\alpha) = 0$ otherwise. Then define $\dot{\mathbb{Q}}_n$ to be the canonical name for $\mathbb{R}(\lambda_{n-2}, \lambda_n, V[\mathbb{R}_{n-1}], V[\mathbb{R}_n], F_n^*)$ where F_n^* is the interpretation of $\dot{F}_n$ in $V[\mathbb{R}_n]$.
4. Finally, let $\mathbb{R}_\omega$ be the inverse limit of $(\mathbb{R}_n : n < \omega)$.

By arguments similar to previous sections of this paper and in [3], we have the following:

Lemma 9.1. After forcing with $\mathbb{R}_\omega$, the following is true:

1. $\lambda_1 = \lambda_0^+ = 2^\kappa$,
2. $2^\mu = \lambda_1$,
3. for each n, κ_n is preserved,
4. for each n, $\lambda_{n+2} = \lambda_{n+1}^+ = 2^{\lambda_n}$, and
5. for each n, λ_n has the tree property.

Lemma 9.2. Let C be $\mathbb{C}$-generic and G be $\mathbb{R}_\omega$-generic. There is a λ_0-supercompactness embedding $j : \overline{V}[C * G] \to M$ with $\mathrm{crit}(j) = \kappa$ so that, for each $\alpha < j(\kappa)$, there is a function $f : \kappa \to \kappa$ such that $j(f)(\kappa) = \alpha$.

Proof. $C*G$ is generic for a κ-directed closed forcing, so κ is still indescructibly supercompact in $\overline{V}[C*G]$. Let $\sigma\colon \overline{V}[C][G] \to M[C^*][G^*]$ be some λ_0-supercompactness embedding. Let σ_V be the restriction of σ to V. The plan is to create a new generic object G^{**} by making a small number of changes to G^* and then argue that we can lift σ_V to the desired elementary embedding $j[C*G] \to M[C^*][G^{**}]$.

In $\overline{V}[C*G]$, fix an enumeration $(u_i\colon i<\lambda_1\setminus\lambda_0)$ of $\sigma_{\overline{V}}(\kappa)$. Define the set G^{**} to be the set of all conditions $r\in\sigma(\mathbb{R}_\omega)$ such that if $r(0)=(p,q,f)$, then there is a $r'\in G^*$ such that

- $r(n)=r'(n)$ for each $n>0$,
- $r'(0)=(p_0,q,f)$ where
    1. $\mathrm{dom}(p_0)=\mathrm{dom}(p)$, and
    2. $p_0\restriction\mathrm{dom}(p_0)=p\restriction\mathrm{dom}(p)\setminus(\sigma_{\overline{V}}``\lambda_1\times\{\kappa\})$
- for all $i<\lambda_1$, if $(\sigma_{\overline{V}}(i),\kappa)\in\mathrm{dom}(p)$ then $p(\sigma_{\overline{V}}(i),\kappa)=u_i$.

Claim 9.1. G^{**} is generic over $M[C^*]$.

Proof. Intuitively, all that is happening is that we take each element of G^* and modify only the Cohen part of the first coordinates. So, showing that G^{**} is generic amounts to arguing that the modified Cohen part of the first coordinate is still generic. This follows because the changes that we are making are sufficiently small. A similar argument is found either Lemma 2.1 in [11] or in Section 4.1 of [4]. ☐

Finally, observe that if $r\in G$ and $r(0)=(p,q,f)$, then $\mathrm{dom}(\sigma_{\overline{V}}(p))\subseteq\sigma_{\overline{V}}``\lambda_1\times\kappa$. It follows that $\sigma_{\overline{V}}``G\subseteq G^{**}$, so we may lift $\sigma_{\overline{V}}$ to the embedding $j\colon\overline{V}[C*G]\to M$. By construction, if f_α is the α-th subset added by the Cohen part of G, then $j(f_\alpha)(\kappa)=u_\alpha$. This yields the final part of the lemma. ☐

Fix j obtained from the previous lemma. Working in $V[G]$ we define the following for $n<\omega$:

1. U_n is the supercompactness embedding on $\mathcal{P}_\kappa(\kappa^{+n})$ derived from j,
2. $j_n\colon V[G]\to\mathrm{Ult}(V[G],U_n)\cong M_n$, and
3. k_n is the factor map defined by $j_n(f)(j_n``\kappa^{+n}))\mapsto j(f)(j``\kappa)$ so that $j=k_n\circ j_n$.

The arguments in [6], which are also sketched in [13], give the following:

1. The critical point of k_n is greater than $j(\kappa)$.

   Proof. The additional property of j given by Lemma 9.2 implies that $j(\kappa)+1\subseteq\mathrm{ran}(k_n)$. ☐

2. In $V[G]$, there is a generic K for $\mathrm{Coll}(\kappa^{+\omega+3},<j(\kappa))_{M[j(G)]}$ over $M[j(G)]$.

   Proof. This poset has $\kappa^{+\omega+3}$ antichains in $M[j(G)]$ and is $\kappa^{+\omega+3}$ closed. ☐

3. There is a generic K_n for $\mathrm{Coll}(\kappa^{+\omega+3},<j_n(\kappa))_{M_n}$ over M_n.

Proof. Let $K_n = k_n^{-1}[K] \cap \mathrm{Coll}(\kappa^{+\omega+3}, < j_n(\kappa))_{M_n}$. Observe that antichain A in $\mathrm{Coll}(\kappa^{+\omega+3}, < j_n(\kappa))_{M_n}$ have size less than $j_n(\kappa)$. This implies that $k_n(A) = k_n``A$. □

Definition 9.3. Now, we define the diagonal Gitik-Sharon Prikry forcing Pr with interleaved collapses in the same way as [13]. For $i < \omega$, let $X_i = \{x \in \mathcal{P}_\kappa(\kappa^{+i}) \colon \kappa_x \text{ is inaccessible}, ot(x) = \kappa_x^{+i}\}$. Conditions of Pr have the form $r = (d, x_0, c_0, \dots, x_{n-1}, c_{n-1}, A_n, C_n, \dots)$ where

1. for $i < n$, $x_i \in X_i$, and for $i \geq n$, $A_i \in U_i$, and $A_i \subseteq X_i$.
2. for $i < n-1$, $x_i \prec x_{i+1}$ and $c_i \in \mathrm{Coll}(\kappa_{x_i}^{+\omega+3}, < \kappa_{x_{i+1}})$ and $c_{n-1} \in \mathrm{Coll}(\kappa_{x_i}^{+\omega+3}, < \kappa)$.
3. if $n > 0$, then $d \in \mathrm{Coll}(\omega, \kappa_{x_0}^{+\omega})$, otherwise $d \in \mathrm{Coll}(\omega, \kappa)$.
4. for $i \geq n$, $[C_n]_{U_n} \in K_n$.

The ordering of Pr is the usual one.

Theorem 9.1. The usual arguments give the following:

1. Forcing with Pr makes $(\kappa^{+n})^{V[G]}$ have cofinality ω for each n.
2. Pr has the $\kappa^{+\omega+1}$-chain condition.
3. Pr has the Prikry property.
4. If P is Pr-generic, then in $V[G * P]$, we have the following cardinal structure:
   (a) $\kappa = \aleph_{\omega^2}$
   (b) $(\kappa^{+\omega+1})^{V[G*P]} = \aleph_{\omega^2+1}$
   (c) $(\kappa^{+\omega+n})^{V[G*P]} = \aleph_{\omega^2+n} = \lambda_{n-2}$ for each $n \geq 2$,
   (d) $2^{\aleph_{\omega^2}} = \aleph_{\omega^2+3}$.

Lemma 9.3. $\mathbb{R}_\omega$ is projected onto by a forcing $\mathbb{P} \times \mathbb{Q}$ which is $(\kappa^+\text{-Knaster}) \times (\mu\text{-closed})$.

Proof. Observe that $\mathbb{R}_\omega$ is the projection of the product forcing $\mathbb{Q}_0 \times T(\dot{\mathbb{Q}}_1 * \dot{\mathbb{Q}}_2 * \cdots)$, where $T(-)$ is the term forcing. The forcing $\mathbb{Q}_0$ is projected onto by a forcing $\mathbb{P} \times \mathbb{C}$ which is $(\kappa^+\text{-Knaster}) \times (\mu\text{-closed})$. Since the forcing $\mathbb{C} \times T(\dot{\mathbb{Q}}_1 * \dot{\mathbb{Q}}_2 * \cdots)$ is μ-closed, we obtain our result. □

Lemma 9.4. For each n, let $\mathbb{R}_n = \mathbb{Q}_0 * \mathbb{Q}_1 * \cdots \mathbb{Q}_{n-1}$. Then, in $V^{\mathbb{R}_n}$, $\mathbb{R}_\omega / \mathbb{R}_n$ is the projection of a forcing $\mathbb{P} \times \mathbb{Q}$ which is $(\lambda_{n-1}\text{-Knaster}) \times (\lambda_{n-1}\text{-closed})$.

Proof. Observe that $\mathbb{R}_\omega$ may be factored as $\mathbb{R}_n * \dot{\mathbb{Q}}_n * (\dot{\mathbb{Q}}_{n+1} * \cdots)$. Working in $V^{\mathbb{R}_n}$, we have that $\mathbb{R}_\omega / \mathbb{R}_n$ which is the projection of the product forcing $\mathbb{Q}_n \times T(\dot{\mathbb{Q}}_{n+1} * \cdots)$, where $T(-)$ is the term forcing. The forcing $\mathbb{Q}_n$ is the projection of a forcing $\mathbb{P} \times \mathbb{C}$ which is $(\lambda_{n-1}\text{-Knaster}) \times (\lambda_{n-1}\text{-closed})$. Since the forcing $\mathbb{C} \times T(\dot{\mathbb{Q}}_{n+1} * \cdots)$ is λ_{n-1}-closed, we obtain our result. [3] □

Our final goal is to prove that the tree property holds at μ and at each λ_n. We start with the tree property at μ, as it is more concrete.

[3]Here we use the notation that $\mu = \lambda_{-1}$.

Theorem 9.2. In $V[G][P]$, we have that the tree property holds at μ.

Proof. The argument follows Section 3 in [13]. The main points are the following: we take a Pr-name $\dot{T} \in V[G] \subseteq V[\text{Knaster} \times \text{Closed}]$ for a μ-tree and as in [11] we find a branch for T in $V[\text{Knaster} \times \text{Closed}][P]$. Then we use the Splitting Branch Lemmas in Section 3 of [13] to pull this branch back to $V[G][P]$. The main apparent difference between [13] and this paper is that here we are using $\mathbb{R}_\omega$ instead of the Mitchell forcing. The argument still goes through because in both cases they are the projection of forcings which are Knaster $\times$ Closed. $\square$

To get the tree property at each λ_n, we use the following abstract formulation of an argument from [5].

Theorem 9.3. Let $V' \models ZFC$. Assume $\kappa = \kappa_0$ is supercompact in V', $(\kappa_n : n < \omega)$ is an increasing sequence of regular cardinals with $\mu = (\sup \kappa_n)^+ < \lambda$, where $\lambda = \lambda_n$ for some n. Let $\mathbb{R}$ be a forcing, and G' be $\mathbb{R}$-generic over V'. Let $j : V' \to N$ be an elementary embedding with $\text{crit}(j) = \lambda$, and assume $j \subseteq j : V'[G'] \to N^*$ in $V'[G'][K \times A]$ where K is λ_{n-1}-Knaster and A is λ_{n-1}-closed. In $V'[G']$ suppose that Pr is the diagonal Gitik-Sharon Prikry forcing with collapses, with respect to κ and the κ_n's. Then, $V'[G'][Pr] \models TP_\lambda$.

Proof. This is the same argument as Theorem 3.1 in [5], as these are the structural properties needed in order to carry out the argument in that paper. $\square$

Theorem 9.4. Let $G * \dot{P}$ be generic for $\mathbb{R}_\omega * \dot{Pr}$ over V. Then, in $V[G][P]$, the tree property holds at λ_n for each $n \geq 0$.

Proof. We will use the preceding two theorems. It's enough to argue that we satisfy the hypothesis of Theorem 9.3, which amounts to arguing that we can lift elementary embeddings in certain ways. Assume that $n \geq 0$ and assume $j : V \to M$ is an elementary embedding with $\text{crit}(j) = \lambda_n$. Let G_ω be $\mathbb{R}_\omega$-generic, G_n be $\mathbb{R}_n$-generic, and P be Pr-generic in $V[G]$. Since $|\mathbb{R}_n| = \lambda_{n-1}$, we may lift this embedding to $j : V[G_n] \to M[G_n]$. Elementarity implies that $j(\mathbb{R}_\omega)/\mathbb{R}_n$ is a projection of a forcing which is (λ_{n-1}-Knaster)$\times$(λ_{n-1}-closed). Since $j(\mathbb{R}_\omega)/\mathbb{R}_n$ projects onto $\mathbb{R}_\omega/\mathbb{R}_n$, we may lift the embedding $j : V[G_n] \to M[G_n]$ to $j : V[G] \to M[j(G)]$ in $V[G][A \times K]$ where K is the λ_{n-1}-closed and A is λ_{n-1}-Knaster. This is exactly what we wanted. $\square$

We close with some open problems. This paper makes progress towards the broader goal of getting the tree property everywhere. A natural (and ambitious) extension of this paper is to combine the above with the tree property at all regulars below the singular strong limit κ. Going in the other direction, is it possible to have a singular strong limit κ with the tree property holding simultaneously at an ω_1 sequence above κ? Is it also natural to ask if we can add collapses to get these two results with $\kappa = \aleph_{\omega^2}$. To summarize:

1. Is it consistent modulo large cardinals to have a singular strong limit κ where the tree property holds at all regular cardinals below κ and at κ^{+n} for each $n \geq 1$? Is this possible when $\kappa = \aleph_{\omega^2}$?

2. Is it consistent modulo large cardinals to have a singular strong limit κ with the tree property holds simultaneously at an ω_1 sequence above κ? Is this possible when $\kappa = \aleph_{\omega^2}$?

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