CURVE COUNTING À LA GÖTTSCHE

PROBLEM SESSION, 25 AUG 2011 — STEVEN L. KLEIMAN, MIT

Let n_δ be the number of δ-nodal curves lying in a suitably ample complete linear system and passing through appropriately many points on a smooth projective complex algebraic surface. A major problem is to understand the behavior of n_δ, specifically to finish off Lothar Göttsche's mostly proved 1997 conjectures [16] and then to go on to treat the new refinements by Göttsche and Vivek Shende [17].

The general subject area has been very active for over fifteen years, and is now busier and more exciting than ever. Among many other people involved have been Joe Harris himself, some of his students, and some of theirs. The area is rather broad, embracing ideas from physics, symplectic differential geometry, complex analytic geometry, algebraic geometry, tropical geometry, and combinatorics.

Problem number one is to find the two power series

$$B_1(q),\ B_2(q) \in \mathbb{Z}[[q]]$$

appearing in Göttsche's remarkable formula for the generating function of the n_δ. The formula expresses the function, so the n_δ, in terms of the four basic numerical invariants of the system and the surface. In fact, n_δ is a polynomial in the four. See (1) and (4) and (5) below.

Göttsche [16, Rmk. 2.5(2)] computed the coefficients of $B_1(q)$ and $B_2(q)$ up to degree 28 on the basis of the recursive formula for the n_δ of the plane due to Lucia Caporaso and Harris [8, Thm. 1.1]. (A different recursion had been given earlier by Ziv Ran [28, Thm. 3C.1].) Göttsche checked the result against much of what was known, including Ravi Vakil's enumeration [33] for the Hirzebruch surfaces.

The problem is to find a closed form for each $B_i(q)$ or else a functional equation.

Second, given δ, how ample is suitable, so that n_δ has the predicted value? After all, for any system, the polynomial yields a number, but it isn't always n_δ. For example, consider plane curves of degree d. If $d = 1$, then n_3 is the number of 3-nodal lines, namely 0, but the polynomial yields 75. Considering the geometry, Göttsche [16, Cnj. 4.1, Rmk. 4.4] conjectured the polynomial works if $\delta \le 2d-2$.

The latter conjecture was proved for $\delta \le 8$ by Ragni Piene and the author [22, Thm. 3.1] using algebraic methods, then for $\delta \le 14$ by Florian Block [4, Prp. 1.4]. He built on ideas of Sergey Fomin and Grigory Mikhalkin [14, Thm. 5.1], who used tropical methods to set up the enumeration from scratch and to validate its predictions for $\delta \le d/2$. In principle, the problem is purely combinatorial: to show formally the Caporaso–Harris recursion yields a polynomial in d for $\delta \le 2d-2$.

On any surface, Martijn Kool, Shende, and Richard Thomas [23, Prp. 2.1] proved δ-very ample works. Piene and the author [21, Thm. 1.1] proved $M^{\otimes m} \otimes N$ works if M is very ample, $m \ge 3\delta$, and N is spanned, provided $\delta \le 8$. Both results were inspired by Göttsche's [16, Prp. 5.2]; in turn, Göttsche had been inspired by Harris and Rahul Pandharipande's preprint [18], which treats $\delta \le 3$ in the plane.

Often, the curves are taken to lie in a generic linear subsystem, but it suffices to impose point conditions as above, owing to Piene and the author's [21, Lem. (4.7)].

The problem is to determine just when the polynomial yields n_δ.

Third, what about nonlinear systems? After all, Gromov–Witten theory fixes not the linear equivalence class, but the homology class, and this class determines the four basic invariants (1). Jim Bryan and Naichung Conan Leung [5, Thm. 1.1] handled primitive complete nonlinear systems on generic Abelian surfaces for all δ. They used symplectic methods. Piene and the author [22, § 5] obtained similar results algebraically, but for $\delta \leq 8$.

Israel Vainsencher [32, § 6.2] treated a remarkable system. His parameter space was the Grassmannian of $\mathbb{P}^2$ in $\mathbb{P}^4$. His surface was $\mathbb{P}^2$, but moving in $\mathbb{P}^4$. His curves arose by intersecting the moving $\mathbb{P}^2$ with a fixed general quintic 3-fold X. Thus he found X contains 17,601,000 irreducible 6-nodal quintic plane curves. Piene and the author [22, Thm. 4.3] validated the number. Pandharipande [10, (7.54)] noted each curve has six double covers previously unconsidered in mirror symmetry.

Given any suitably general algebraic system of curves on surfaces, Piene and the author [22, Thm. 2.5 and Rmk. 2.7] found on the parameter space the class of the curves with δ nodes for $\delta \leq 8$ and conjectured the formula generalizes to any δ.

The problem is to generalize the formula (4) for n_δ to algebraic systems.

Fourth, what about higher singularities? This question is related to the previous one, about algebraic systems. For example, given a system, consider those curves with a triple point and δ double points. Their number can be viewed as the number of curves with δ double points in the following system: take the subsystem of curves with a triple point, and resolve the locus of triple points. This example was treated for $0 \leq \delta \leq 3$ by Vainsencher and by Piene and the author [21, Thm. 1.2]. Other work has been done; see Dmitry Kerner's papers [19], [20] and their references.

The problem is to enumerate the curves of fixed global equisingularity type lying in a given system — that is, to find on the parameter space the class of these curves.

Fifth, what about positive characteristic? Sometimes an enumeration is more tractable modulo a prime. Thus Göttsche [15, Thm. 0.1] found the Betti numbers of the Hilbert schemes of points on a smooth surface. (In [12, p. 175–178], he and Barbara Fantechi discuss other proofs and refinements of the result.) This result (among others) led to the celebrated formula of Shing-Tung Yau and Eric Zaslow [34, p. 5] enumerating rational curves on a K3 surface. They developed ideas of Cumrun Vafa et al.: see [30, p. 438] for a similar-looking formula; see [31, p. 44] for the use of Göttsche's result; see [1, p. 437] for the use of varying Jacobians. In turn, Arnaud Beauville [3] and Fantechi, Göttsche, and Duco van Straten [13] developed the ideas in [34] further, but left open the question of whether the curves are nodal.

The Yau–Zaslow formula inspired Göttsche to develop his conjectures. For K3 surfaces and Abelian surfaces, $B_1(q)$ and $B_2(q)$ disappear, leaving explicit formulas in any geometric genus. These formulas were proved for primitive classes on generic such surfaces by Bryan and Leung; see [6] for a lovely survey.

The problem is to see when Göttsche's conjectures hold in positive characteristic.

To define the $B_i(q)$, denote the surface by S, its canonical bundle by K, and the line bundle of the linear system by L. The four basic invariants are these numbers:

$$x := L^2, \quad y := L \cdot K, \quad z := K^2, \quad t := c_2(S). \tag{1}$$

For $\delta \leq 6$, Vainsencher [32, § 5] worked out formulas for the n_δ, getting humongous polynomials in x, y, z, t. Afterwards, it was natural to conjecture this statement:

(2) *The number n_δ is given by a universal polynomial of degree δ in $\mathbb{Q}[x, y, z, t]$.*

For plane curves of degree d, we have $(x, y, z, t) = (d^2, -3d, 9, 3)$. So Philippe

Di Francesco and Claude Itzykson [11, p. 85] conjectured n_δ is for $\delta \le \binom{d-1}{2}$ given by a polynomial in d of a certain shape. Youngook Choi [9, p. 12] established their conjecture for $\delta \le d$ on the basis of Ran's work [28]. Göttsche [16, § 4] refined the conjecture. Given (2) in the form of (4), Nikolay Qviller [26, § 4] established most of Göttsche's refinements concerning the shape.

In full generality, (2) was given a symplectic proof and an algebraic proof by Ai-ko Liu [24, 25]. It was given new algebraic proofs by Yu-Jong Tzeng [29, Thm. 1.1] and Kool, Shende, and Thomas [23, Thm. 4.1]. The latter have caused quite a stir!

Göttsche [16, Cnj. 2.1] did conjecture (2) in full generality, but his development of (2) is far more important. First, he proved (2) is equivalent to this statement:

$$\sum n_\delta u^\delta = A_1^x A_2^y A_3^z A_4^t \quad \text{for some } A_i \in \mathbb{Q}[[u]]. \tag{3}$$

The A_i are the exponentials of their logarithms. Hence (3) is equivalent to this:

$$n_\delta = P_\delta(a_1, \dots, a_\delta)/\delta! \quad \text{where } \textstyle\sum_{\delta\ge 0} P_\delta u^\delta/\delta! = \exp\bigl(\sum_{\kappa\ge 1} a_\kappa u^\kappa/\kappa!\bigr) \tag{4}$$

for some *linear* forms $a_\kappa(x, y, z, t)$. The polynomials P_δ were studied extensively in 1934 by Eric Temple Bell [2]. Piene and the author [21, p. 210] determined a_κ for $\kappa \le 8$, and found the coefficients to be integers. Recently, Qviller [26, Thm. 2.4] (see [27, § 6] too) proved the coefficients are always integers.

The $B_i(q)$ appear in the next formula, called the *Göttsche–Yau–Zaslow Formula*:

$$\sum n_\delta u(q)^\delta = B_1(q)^z B_2(q)^y B_3(q)^\chi B_4(q)^{-\nu/2} \tag{5}$$

where $u(q)$, $B_3(q)$, $B_4(q) \in \mathbb{Z}[[q]]$ are explicit quasi-modular forms and where

$$\chi := \chi(L) = (x - y)/2 + \nu \quad \text{and} \quad \nu := \chi(\mathcal{O}_S) = (z + t)/12.$$

Göttsche [16, Cnj. 2.4] conjectured (5). He [16, Rmks. 2.5(1), 3.1] noted (5) implies (2) and generalizes the Yau–Zaslow Formula. Tzeng [29, Thm. 1.2] derived (5) from (3) via Bryan and Leung's work on K3 surfaces [7, Thm. 1.1] and Piene and the author's [21, Lem. 5.3]; it asserts there are enough suitably ample primitive classes.

Finally, inspired by Kool, Shende, and Thomas's [23], Göttsche and Shende [17] conjectured (5) can be refined as follows. There should be polynomials $n_\delta(v) \in \mathbb{Z}[v]$ and power series with polynomial coefficients $B_i(v, q) \in \mathbb{Z}[v][[q]]$ such that

$$\sum n_\delta(v)\, u(v, q)^\delta = B_1(v, q)^z B_2(v, q)^y B_3(v, q)^\chi B_4(v, q)^{-\nu/2}.$$

Again, $u(v, q)$ and $B_3(v, q)$ and $B_4(v, q)$ are known. Putting $v = 1$ recovers (5). Further, if S is real (that is, invariant under complex conjugation), then $n_\delta(-1)$ is the Welschinger invariant — the number of real δ-nodal curves lying in a suitably ample real complete linear system and passing through a real set of appropriately many points, each curve counted with an appropriate sign.

The refined problem number one is to find $B_1(v, q)$ and $B_2(v, q)$.

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References

[1] Berhadsky, Michael, Vafa, Cumrun, and Sadov, Vladimir, *D-branes and topological field theories,* Nuclear Phys. B **463** (1996), no. 2-3, 420–434.
[2] Bell, Eric Temple, *Exponential polynomials,* Ann. Math. **35** (1934), 258–277.
[3] Beauville, Arnaud, *Counting rational curves on K3 surfaces,* Duke Math. J. **97** (1999), no. 1, 99–108.
[4] Block, Florian, "Computing Node Polynomials for Plane Curves," arXiv:1006.0218v3.

[5] Bryan, Jim, and Leung, Naichung Conan, *Generating functions for the number of curves on Abelian surfaces,* Duke Math. J. **99** (1999), no. 2, 311–328.

[6] Bryan, Jim, and Leung, Naichung Conan, *Counting curves on irrational surfaces,* in "Differential geometry inspired by string theory," pp. 313–339, Surv. Differ. Geom., **5**, Int. Press, Boston, MA, 1999; = `http://www.math.ubc.ca/~jbryan/papers/survey.pdf`.

[7] Bryan, Jim, and Leung, Naichung Conan, *The enumerative geometry of K3 surfaces and modular forms,* J. Amer. Math. Soc. **13** (2000), no. 2, 371–410.

[8] Caporaso, Lucia, and Harris, Joe, *Counting plane curves of any genus,* Invent. Math. **131** (1998), no. 2, 345–392.

[9] Choi, Youngook, "Severi Degrees in Cogenus 4," arXiv: alg-geom/9601013.

[10] Cox, David, and Katz, Sheldon, "Mirror symmetry and algebraic geometry," Math. Surveys and Monographs, Vol. **68**, AMS 1999.

[11] Di Francesco, Philippe, and Itzykson, Claude, *Quantum intersection rings,* in "The moduli space of curves (Texel Island, 1994)," pp. 81–148, Progr. Math., **129**, Birkhäuser Boston, Boston, MA, 1995.

[12] Fantechi, Barbara, and Göttsche, Lothar, *Local properties and Hilbert schemes of points* in "Fundamental algebraic geometry," pp. 139–178, Math. Surveys Monogr., **123**, Amer. Math. Soc., Providence, RI, 2005.

[13] Fantechi, B., Gttsche, L., and van Straten, Duco, *Euler number of the compactified Jacobian and multiplicity of rational curves,* J. Algebraic Geom. **8** (1999), no. 1, 115–133.

[14] Fomin, Sergey, and Mikhalkin, Grigory, *Labeled Floor diagrams,* J. Eur. Math. Soc. (JEMS) **12** (2010), no. 6, 1453–1496.

[15] Göttsche, Lothar, *The Betti numbers of the Hilbert scheme of points on a smooth projective surface,* Math. Ann. **286** (1990), no. 1-3, 193–207.

[16] Göttsche, Lothar, *A conjectural generating function for numbers of curves on surfaces,* Comm. Math. Phys. **196** (1998), no. 3, 523–533.

[17] Göttsche, Lothar, "Refined curve counting," ELGA Lecture, Cordoba Argentina, 9 Aug 2011.

[18] Harris, Joe, and Pandharipande, Rahul, "Severi degrees in cogenus 3," arXiv: alg-geom/9504003v1.

[19] Kerner, Dmitry, "Enumeration of singular algebraic curves," Israel J. Math. **155** (2006), 1–56; revised as arXiv:math/0407358v4.

[20] Kerner, Dmitry, "On the enumeration of complex plane curves with two singular points," Int. Math. Res. Not. IMRN 2010, no. 23, 4498–4543.

[21] Kleiman, Steven, and Piene, Ragni, "Enumerating singular curves on surfaces," in "Algebraic geometry — Hirzebruch 70," Cont. Math. **241** (1999), 209–238; corrections and revision in arXiv:math/0111299v1.

[22] Kleiman, Steven, and Piene, Ragni, *Node polynomials for families: methods and applications,* Math. Nachr. **271** (2004), 1–22.

[23] Kool, Martijn, Shende, Vivek, Thomas, Richard, "A short proof of the Göttsche conjecture," arXiv:1010.3211v2.

[24] Liu, Ai-ko, *Family blowup formula, admissible graphs and the enumeration of singular curves. I.,* J. Differential Geom. **56** (2000), no. 3, 381–579.

[25] Liu, Ai-ko, "The algebraic proof of the universality theorem," arXiv:math/0402045v1.

[26] Qviller, Nikolay "The Di Francesco-Itzykson-Göttsche Conjectures for Node Polynomials of $\mathbb{P}^2$," arXiv:1102.2092v2.

[27] Qviller, Nikolay "Structure of Node Polynomials for Curves on Surfaces," arXiv:1010.2377v3.

[28] Ran, Ziv, *Enumerative geometry of singular plane curves,* Invent. Math. **97** (1989), 447–465.

[29] Tzeng, Yu-Jong, "A Proof of the Göttsche–Yau–Zaslow Formula," arXiv:1009.5371v3.

[30] Vafa, Cumrun, *Instantons on D-branes,* Nuclear Phys. B **463** (1996), no. 2-3, 435–442.

[31] Vafa, Cumrun, and Witten, Edward, *A strong coupling test of S-duality,* Nuclear Phys. B **431** (1994), no. 1-2, 3–77.

[32] Vainsencher, Israel, *Enumeration of n-fold tangent hyperplanes to a surface,* J. Alg. Geom. **4** (1995), 503–526.

[33] Vakil, Ravi, *Counting curves on rational surfaces,* Manuscripta Math. **102** (2000), no. 1, 53–84.

[34] Yau, Shing-Tung, and Zaslow, Eric, *BPS states, string duality, and nodal curves on K3,* Nuclear Phys. B **471** (1996), no. 3, 503–512.