Abelian varieties with singular theta divisors – some problems

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1 Abelian varieties with singular points on the theta divisor

Let $\mathcal{A}_g$ be the moduli space of principally polarized abelian varieties of dimension g (over the compex numbers). An important divisor in $\mathcal{A}_g$ is the theta-null divisor $\Theta_{\mathrm{null}}^{(g)}$ consisting of ppav whose theta divisor has a singular point (at an even 2-torsion point). Its class, as well as its restriction to $\mathcal{M}_g$, are well understood. Here we suggest studying more generally the stratification given by singularities of the theta divisor at 2-torsion points.

We first denote by $I_3^{(g)}$ the subset of ppav (A, Θ) where Θ is a symmetric theta divisor with a point of order at least 3 at an odd 2-torsion point.

Conjecture 1.1 (Grushevsky, Salvati-Manni) *$I_3^{(g)}$ is empty for $g \leq 2$ and of pure codimension g for $g \geq 3$.*

This kind of conjecture can be viewed as a generalization to ppav of a question raised by Joe Harris on the moduli of curves with a theta characteristic of a given number of sections. Indeed, for $g \leq 3$ this is classical and follows from the theory of theta divisors. More generally, this conjecture has been proven by Casalaina-Martin and Friedman for $g \leq 5$ [CMF] with an independent proof given by Grushevsky and Hulek in [GH1].

More precisely, for small values of g one has

$$I_3^{(3)} = \mathrm{Sym}^3(\mathcal{A}_1),\ I_3^{(4)} = \mathcal{A}_1 \times \Theta_{\mathrm{null}}^{(3)},\ I_3^{(5)} = \overline{IJ} \times \Theta_{\mathrm{null}}^{(4)}.$$

where $\Theta_{\mathrm{null}}^{(g)}$ is the theta null divisor in $\mathcal{A}_g$ and IJ is the locus of ppav's which are the intermediate Jacobians of cubic threefolds.

The fact that g is the expected codimension of $I_3^{(g)}$ can be seen by the following desrciption. For a principally polarized abelian variety A we set

$$f_m(\tau) = \mathrm{grad}_z \theta(\tau, z)|_{z=m}$$

where $m \in A[2]$ is an odd 2-torsion point. On the level cover $\mathcal{A}_g(4,8)$ associated to the theta group we find that $f_m \in H^0(\mathcal{A}_g(4,8), \mathbb{E} \otimes (\det \mathbb{E})^{1/2})$

where $\mathbb{E}$ is the rank g Hodge bundle and $(\det \mathbb{E})^{1/2}$ is a suitable root of the Hodge line bundle. One then has $I_3^{(g)} = p(\{f_m = 0\})$ where $p : \mathcal{A}_g(4,8) \to \mathcal{A}_g$ is the natural projection. Since f_m is a section of a rank g bundle, the expected codimension is indeed g.

Problem 1.2 *Prove or disprove conjecture (1.1).*

This problem can be generalized by considering the loci $I_k^{(g)}$ where the theta divisor has a prescribed point of order k at an odd/even 2-torsion point (depending on the parity of k).

2 Going to the boundary

One way to approach the above problem is to study degenerations of abelian varieties. For our purposes it is suitable to work with the so-called perfect cone compactification $\mathcal{A}_g^{\mathrm{perf}}$ of $\mathcal{A}_g$, which is the toroidal compactification given by the perfect cone or first Voronoi decomposition. It has the property that the boundary divisor is irreducible. It was shown in [GH2] that the sections $f_m \in H^0(\mathcal{A}_g(4,8), \mathbb{E} \otimes (\det \mathbb{E})^{1/2})$ extend on $\mathcal{A}_g(4,8)^{\mathrm{perf}}$ to sections $\overline{f}_m \in H^0(\mathcal{A}_g(4,8)^{\mathrm{perf}}, \mathbb{E} \otimes (\det \mathbb{E})^{1/2} \otimes \mathcal{O}(-\sum \varepsilon_\alpha D_\alpha))$, where D_α runs through the boundary components of $\mathcal{A}_g(4,8)^{\mathrm{perf}}$ and ε_α is a suitable collection of integers $\varepsilon_\alpha \in \{0,1\}$ depending on m. We set

$$\overline{G_{3,m}^{(g)}} = \{\overline{f}_m = 0\}, \quad \overline{G_3^{(g)}} = p(\overline{G_{3,m}^{(g)}}).$$

Notice that the latter is independent of m. Since the sections $\overline{f}_m$ are extensions of the sections f_m we have

$$\overline{I_3^{(g)}} \subset \overline{G_3^{(g)}}.$$

Conjecture 2.1 (Grushevsky, Hulek) *There is an equality* $\overline{I_3^{(g)}} = \overline{G_3^{(g)}}$ *(and this locus is of pure codimension g).*

This conjecture was shown to hold for all $g \leq 5$ in [GH2].

Problem 2.2 *Prove or disprove conjecture (2.1).*

3 Restriction to $\mathcal{M}_g$

The restriction of the loci $I_{k+1}^{(g)}$ to $\mathcal{M}_g$ are the loci $\mathcal{M}_g^k$ of curves of genus g having a theta characteristic with at least $k+1$ sections and the same parity as $k+1$. These loci were studied, in particular, by Harris [H] and also by Teixidor i Bigas [TB].

Problem 3.1 (Harris) *Determine the dimension of the loci* $\mathcal{M}_g^k$.

Harris gave a lower bound for the dimension of these loci and Teixidor i Bigas provided an upper bound thus showing that the codimension of $\mathcal{M}_g^2 = I_3^{(g)} \cap \mathcal{M}_g$ in $\mathcal{M}_g$ is 3. This also shows that this intersection is highly not transversal.

We denote by $t : \mathcal{M}_g \to \mathcal{A}_g$ the Torelli map. It was very recently shown by Alexeev and Brunyate [AB] that the Torelli map extends to a morphism $\bar{t} : \overline{\mathcal{M}}_g \to \mathcal{A}_g^{\mathrm{perf}}$. (The same result has long been known for the second Voronoi compactification $\mathcal{A}_g^{\mathrm{Vor}}$, where it was shown by Namikawa.)

Problem 3.2 *Describe the restriction of* $\overline{G_3^{(g)}}$ *to* $\overline{\mathcal{M}}_g$, *in particular the number and dimension of the components of these pullbacks and compute the class of these cycles.*

The class of $\overline{G_3^{(g)}}$ in $\mathcal{A}_g^{\mathrm{perf}}$ was computed in [GH2] (assuming that the cacle has the expected dimension).

References

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