## HOMEWORK 1

This problem set is due Friday September 4. You may work on the problem set in groups; however, the final write-up must be yours and reflect your own understanding. In all these exercises assume that $k$ is an algebraically closed field and $R$ is a commutative ring with unit.

Problem 0.1. (1) Show that the union of the coordinate axes in $\mathbb{A}_{k}^{3}$ is a closed algebraic set. Determine generators for its ideal.
(2) Consider the curve in $\mathbb{A}_{k}^{3}$ given in parametric form $C=\left\{\left(t, t^{2}, t^{3}\right) \in \mathbb{A}^{3} \mid t \in k\right\}$. Determine generators for the ideal of $C$.
(3) Consider the set $\{(0,0),(1,1),(0,1),(1,0)\}$ of four points in $\mathbb{A}_{\mathbb{C}}^{2}$. Find generators for its ideal.
(4) Consider the set $\{(0,0),(1,1),(2,2),(1,0)\}$ of four points in $\mathbb{A}_{\mathbb{C}}^{2}$. Find generators for its ideal. How does your answer differ from the previous part? What is special about these four points?

Problem 0.2. Consider the set $V=\left\{\left(t^{3}, t^{4}, t^{5}\right) \mid t \in k\right\}$ in $\mathbb{A}_{k}^{3}$. Show that $V$ is an affine variety. Find generators of its ideal. How many generators do you need? Can $V$ be described as the zero locus of two polynomials?

Problem 0.3. Let $f$ be a polynomial in $k\left[x_{1}, \ldots, x_{n}\right]$. Show that $\mathbb{A}^{n}-V(f)$ can be realized as the affine variety $V\left(x_{n+1} f-1\right)$ in $\mathbb{A}^{n+1}$. Conclude that the general linear group $G L(n, k)$ (invertible $n \times n$ matrices with entries in $k$ under usual matrix multiplication) can be realized as an affine variety in $\mathbb{A}_{k}^{n^{2}+1}$.

Problem 0.4. Let $S=\mathbb{A}_{k}^{2}-\{(0,0)\}$ be the complement of the origin in $\mathbb{A}_{k}^{2}$. Find $I(S)$, the set of polynomials vanishing on $S$. What is $V(I(S))$ ? Can $S$ be an affine variety?
Problem 0.5. Let $X \subset \mathbb{A}^{n}$ and $Y \subset \mathbb{A}^{m}$ be two affine varieties. Prove that $X \times Y \subset \mathbb{A}^{n+m}$ is an affine variety.

Problem 0.6. Show that any two ordered sets of $n+2$ points in general position in $\mathbb{P}^{n}$ are projectively equivalent. Show that two sets of four points in $\mathbb{P}^{1}$ are projectively equivalent if and only if their crossratios are equal. Harder: Characterize when $n+3$ points in general linear position in $\mathbb{P}^{n}$ are projectively equivalent.

Problem 0.7. Let $\Gamma$ be a set of points in $\mathbb{P}^{n}$ of cardinality d. Show that $\Gamma$ can be expressed as the zero locus of polynomials of degree at most d. Show that if all the points in $\Gamma$ do not lie on a line, then in fact $\Gamma$ can be expressed as the zero locus of polynomials of degree $d-1$ or less.

