

HOMEWORK 5

This problem set is due Monday October 17. You may work on the problem set in groups; however, the final write-up must be yours and reflect your own understanding. In all these exercises assume that k is an algebraically closed field and R is a commutative ring with unit.

Problem 0.1. Compute the multiplication table for the cohomology of $G(2, 5)$.

Problem 0.2. Prove Pieri's formula

$$\sigma_1 \cdot \sigma_{\lambda_1, \dots, \lambda_k} = \sum_{\lambda_i \leq \mu_i \leq \lambda_{i-1}, \sum \mu_i = 1 + \sum \lambda_i} \sigma_{\mu_1, \dots, \mu_k}$$

where $\sigma_{\lambda_1, \dots, \lambda_k}$ and $\sigma_{\mu_1, \dots, \mu_k}$ are Schubert cycles in $G(k, n)$.

Problem 0.3. We say that a plane curve $F = 0$ has a cusp at p if the Taylor expansion of F at p has the form

$$L^2 + \text{h.o.t.}$$

where L is a line containing p and h.o.t. denotes higher order terms. Show that for $d > 2$ plane curves of degree d that have a cusp form a projective subvariety of codimension two in $\mathbb{P}^{d(d+3)/2}$, the space of plane curves of degree d . (Hint: Linearize the problem by considering plane curves that have a cusp at p with tangent direction L .)

Problem 0.4. Let $X \subset \mathbb{P}^n$ be a projective variety. The secant variety to X is the closure of the union of lines spanned by distinct points on X

$$\text{Sec}(X) = \overline{\cup_{p, q \in X, p \neq q} \overline{pq}}.$$

Prove that $\text{Sec}(X)$ is a projective variety of dimension less than or equal to $\min(2 \dim(X) + 1, n)$. We say that the secant variety is defective if $\dim(\text{Sec}(X)) < \min(2 \dim(X) + 1, n)$. Prove that $\text{Sec}(X)$ is defective if and only if every point $x \in \text{Sec}(X)$ lies on infinitely many secant lines to X . Show that the secant variety of the Veronese image $\nu_2(\mathbb{P}^2)$ in \mathbb{P}^5 is defective. Hard Challenge: Show that a surface S in \mathbb{P}^5 which is not contained in any hyperplane has a defective secant variety if and only if S is the Veronese image $\nu_2(\mathbb{P}^2)$.

Problem 0.5. More generally, let $X \subset \mathbb{P}^n$ be a projective variety. The r -secant variety $\text{Sec}_r(X)$ to X is the closure of the union of the \mathbb{P}^{r-1} 's spanned by r distinct points p_1, \dots, p_r in X in general linear position. Prove that $\text{Sec}_r(X)$ is a projective variety of dimension less than or equal to $\min(r \dim(X) + r - 1, n)$. We say that $\text{Sec}_r(X)$ is defective if the dimension of $\text{Sec}_r(X)$ is strictly less than $\min(r \dim(X) + r - 1, n)$. Show that $\text{Sec}_r(X)$ is defective if and only if every point on $\text{Sec}_r(X)$ is contained in infinitely many secant \mathbb{P}^{r-1} 's to X . Show that the fourth Veronese image $\nu_4(\mathbb{P}^2) \subset \mathbb{P}^{14}$ has a defective 5-secant variety $\text{Sec}_5(\nu_4(\mathbb{P}^2))$. Hard Challenge: Show that among the secant varieties to the Veronese images of \mathbb{P}^2 , $\text{Sec}_2(\nu_2(\mathbb{P}^2))$ and $\text{Sec}_5(\nu_4(\mathbb{P}^2))$ are the only defective secant varieties.