HOMEWORK 7

This problem set is due Friday November 6. You may work on the problem set in groups; however, the final write-up must be yours and reflect your own understanding. In this problem set assume that all the varieties are defined over the complex numbers.

Problem 0.1. *Show that the projective tangent space to the Veronese variety $v_d(\mathbb{P}^n)$ in $\mathbb{P}^{\binom{n+d}{d}-1}$ at a point L^d may be described as the set of polynomials of degree d divisible by L^{d-1}. Conclude that the Veronese varieties are smooth.*

Problem 0.2. *Let X be a smooth curve in $\mathbb{P}^2$. A line l is called a flex line to X if it has contact of order three at a point $x \in X$. Find the flex lines to the curve $x^3 + y^3 + z^3 = 0$ (hint: there are nine of them). How many of these flex lines are defined over the real numbers? Challenge: Show that a smooth, irreducible curve over the complex numbers has finitely many flex lines.*

Problem 0.3. *Let $X = v_2(\mathbb{P}^2) \subset \mathbb{P}^5$ be the Veronese surface in $\mathbb{P}^5$ given by*

$$(x_0, x_1, x_2) \mapsto (x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2, x_1x_2).$$

Consider the projection of X from the point $(0, 0, 0, 0, 0, 1)$ to $\mathbb{P}^5$ given by

$$(x_0, x_1, x_2) \mapsto (x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2).$$

Is the image of this projection normal? (Remark: Zariski's Main Theorem asserts that if $f : X \to Y$ is a regular and birational map between projective varieties and Y is normal, then the fibers $f^{-1}(y)$ of f are connected. What is the relevance to this exercise?)

Problem 0.4. *Prove that if $n \geq 3$, then the quadric cone $x_1^2 + \cdots + x_n^2$ in $\mathbb{A}^n$ is normal.*

Problem 0.5. *A projective variety $X \subset \mathbb{P}^n$ is called projectively normal if its homogeneous coordinate ring $S(Y)$ is integrally closed. Show that if Y is projectively normal, then Y is normal. Show that the converse does not in general hold by considering the quartic curve C given as the image of*

$$(t, u) \mapsto (t^4, t^3u, tu^3, u^4)$$

in $\mathbb{P}^3$. Show that C is normal, but not projectively normal.