## HOMEWORK 8

This problem set is due Friday November 13. You may work on the problem set in groups; however, the final write-up must be yours and reflect your own understanding.

Problem 0.1. Calculate the degree of the Grassmannian of lines in $\mathbb{P}^{n}$ in its Plücker embedding. (Hint: You may want to use Pieri's formula and learn about Catalan numbers.)

Problem 0.2. Show that a variety $X$ and a general hyperplane section $X \cap H$ have the same degree. Calculate the degree of the surface scroll in $\mathbb{P}^{4}$ defined by the rank one locus of the matrix

$$
\left(\begin{array}{lll}
z_{0} & z_{1} & z_{2} \\
z_{2} & z_{3} & z_{4}
\end{array}\right)
$$

Problem 0.3. Let $a \leq b$ be two positive integers. The surface scroll $S_{a, b}$ is constructed as follows. Pick two disjoint linear spaces $\mathbb{P}^{a}$ and $\mathbb{P}^{b}$ in $\mathbb{P}^{a+b+1}$. Fix rational normal curves $C_{a}$ and $C_{b}$ of degrees a and $b$ in the $\mathbb{P}^{a}$ and $\mathbb{P}^{b}$, respectively, and an isomorphism $\phi: C_{a} \rightarrow C_{b} . S_{a, b}$ is the surface obtained by taking the union of the lines joining $p \in C_{a}$ with $\phi(p)$ in $C_{b}$. Prove that $S_{a, b}$ is an algebraic surface. Describe in detail $S_{1,1}$. Show that the surface in the previous exercise is $S_{1,2}$. We can also allow $a=0$ and $\phi$ to be the constant map. In that case the surface is a cone over a rational normal curve. Calculate the degree of the surface scroll $S_{a, b}$.

Problem 0.4. Let $X$ be a non-degenerate (i.e., not contained in any hyperplanes) projective variety of degree $d$ and dimension $k$ in $\mathbb{P}^{n}$. Show that $d \geq n-k+1$. Show that for rational normal curves, the Veronese surface $v_{2}\left(\mathbb{P}^{2}\right)$ in $\mathbb{P}^{5}$ and surface scrolls $S_{a, b}$ equality holds. Challenge: Classify the varieties where equality holds.

Problem 0.5. Prove that the every automorphism of projective space $\mathbb{P}^{n}$ is induced by an automorphism $\phi \in G L(n+1)$ of $k^{n+1}$. In other words, the automorphism group of $\mathbb{P}^{n}$ is $\mathbb{P} G L(n+1)$.

Problem 0.6. Prove that an irreducible, non-degenerate curve of degree $n$ in $\mathbb{P}^{n}$ is the rational normal curve of degree $n$.

