Some Mathematics for Essay 3

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Recall you are examining the function

$$flat(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

and want to prove that the Maclaurin series of $flat$ does not converge to $flat$. Do this by showing that $flat$ and all its derivatives have value 0 at $x = 0$ and conclude the Maclaurin series is the identically 0 series, which converges to the identically 0 function and not to $flat$. Below is an outline of a suggested proof.

Lemma 1. *For $x \neq 0$, $flat^{(n)}(x) = flat(x)p_n(\frac{1}{x})$ for all $n = 0, 1, 2, \ldots$ where p_n is a polynomial (of degree 3 greater than that of p_{n-1} when $n \geq 1$).*

Proof outline. Use mathematical induction on n, the degree of the derivative.

Base case ($n = 0$): $flat^{(0)}(x) = flat(x) = flat(x) \cdot 1$ and 1 is a polynomial.

Inductive step: Show $flat^{(k)}(x) = flat(x)p_k(\frac{1}{x}) \Rightarrow flat^{(k+1)}(x) = flat(x)p_{k+1}(\frac{1}{x})$, where p_{k+1} is a polynomial with degree 3 more than the degree of the polynomial p_k. Start by differentiating as follows:

$$\begin{aligned} flat^{(k+1)}(x) &= \frac{d}{dx} flat^{(k)}(x) \\ &= \frac{d}{dx}\left(flat(x)p_k(\frac{1}{x})\right) \\ &= flat(x)\left(\frac{2}{x^3}p_k(\frac{1}{x}) - p_k'(\frac{1}{x}) \cdot \frac{1}{x^2}\right) \\ &= flat(x)p_{k+1}(\frac{1}{x}), \text{ where } p_{k+1}(\frac{1}{x}) = \frac{2}{x^3}p_k(\frac{1}{x}) - p_k'(\frac{1}{x}) \cdot \frac{1}{x^2}. \end{aligned}$$

You will need to supply more details and argue that the degree of p_{k+1} is 3 more than that of p_k.

Lemma 2. *$flat^{(n)}(0) = 0$ for all $n = 0, 1, 2, \ldots$.*

Proof outline. Again use induction on n.

Base case ($n = 0$): $flat(0) = 0$ by definition.

Inductive step: Use the definition of the derivative as the limit of a difference quotient, along with the inductive assumption that $flat^{(k)}(0) = 0$ to show $flat^{(k+1)}(0) = 0$.

$$\begin{aligned} flat^{(k+1)}(0) &= \lim_{h \to 0} \frac{flat^{(k)}(0+h) - flat^{(k)}(0)}{h} \\ &= \lim_{h \to 0} \frac{flat^{(k)}(h)}{h} \\ &= \lim_{h \to 0} \frac{flat(h)p_k(\frac{1}{h})}{h}, \text{ and now letting } u = \frac{1}{h}, \\ &= \lim_{u \to \pm\infty} \frac{up_k(u)}{e^{u^2}} = 0. \end{aligned}$$

Again, supply more details. You should now be able to conclude, using the definition of Maclaurin series (perhaps cite a calculus book), the following:

Proposition. *The Maclaurin series of flat does not converge to flat except at $x = 0$.*

For suggested topic 1, you can use an approach like:

$$\begin{aligned} f(x) &:= \begin{cases} flat(x+a)flat(x+b) & \text{for } -b < x < -a \\ 0 & \text{otherwise,} \end{cases} \\ g(x) &:= \int_{-\infty}^{x} f(t)\,dt, \\ h(x) &:= \frac{g(x)}{g(0)} \text{ and} \\ bump(x) &:= h(x)h(-x). \end{aligned}$$

Though a formal proof in not required, give an argument for why *bump* has the desired properties of being a $\mathbf{C}^\infty$ bump function (at 0).

For suggested topic 2, you can show the ideal $I_n = \langle flat, flat', \ldots, flat^{(n)} \rangle$ does not contain $flat^{(n+1)}$ so that $I_n \subsetneq I_{n+1}$. Use the fact from Lemma 1 that the degree of p_{n+1} is 3 more than that of p_n. This means that the ring of $\mathbf{C}^\infty(\mathbb{R})$ functions contains an infinite proper ascending chain of ideals $I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_n \subsetneq \ldots$, thus showing $\mathbf{C}^\infty(\mathbb{R})$ fails to satisfy the ascending chain condition. The ascending chain condition (ACC) states that any such chain must be finite, and Noetherian rings must satisfy the ACC [2], so $\mathbf{C}^\infty(\mathbb{R})$ is not Noetherian.

References

1. Ideal (ring theory). (2010, August 24). In Wikipedia, The Free Encyclopedia. Retrieved 15:47, October 22, 2010, from
`http://en.wikipedia.org/w/index.php?title=Ideal_(ring_theory)&oldid=380799258`.

2. Noetherian ring. (2010, October 13). In Wikipedia, The Free Encyclopedia. Retrieved 19:36, October 21, 2010, from
`http://en.wikipedia.org/w/index.php?title=Noetherian_ring&oldid=390563549`.