Math310 : Exam 2 solution
Spring 2013

SHOW WORK: Unsupported answers will not receive credit.

Problem 1. (20pts) Prove or disprove

(1) $S = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x + y = 3 \right\}$ is a subspace of $\mathbb{R}^2$.

(2) $T = \{p(x) \in P_3 \mid p'(3) = 0\}$ is a subspace of P_3.

Problem 2. (30pts) Let $A = \begin{pmatrix} 1 & -2 & 1 & 0 & 3 \\ 0 & 0 & 2 & 2 & -1 \\ 1 & -2 & -1 & -2 & 0 \\ -1 & 2 & -2 & -1 & 2 \end{pmatrix}$.

a) Find a basis for the row space of A. What is the dimension of the row space of A?

b) Find a basis for $R(A)$. What is the dimension of $R(A)$?

c) Find a basis for $N(A)$. What is the dimension of $N(A)$?

d) Find rank and nullity of A.

Problem 3. (15pts) Let $\mathbf{u_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{u_2} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\mathbf{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbf{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

a) Find the transition matrix S corresponding to the change of basis from $\{\mathbf{u_1}, \mathbf{u_2}\}$ to $\{\mathbf{v_1}, \mathbf{v_2}\}$.

b) If $\mathbf{w} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$ find the coordinate representation of $\mathbf{w}$ with respect to $\{\mathbf{v_1}, \mathbf{v_2}\}$.

Problem 4. (20pts) Give an example (no explanation required) of

(1) a set in $\mathbb{R}^3$ that is linearly independent but not a basis of $\mathbb{R}^3$;

(2) a set in P_3 that is a spanning set for P_3 but not a basis of P_3;

(3) a basis in $\mathbb{R}^{2\times 2}$;

(4) a linear map from $\mathbb{R}^3$ to $\mathbb{R}^2$;

(5) a map from $\mathbb{R}^2$ to $\mathbb{R}^3$ that is not linear.

Problem 5. (15pts) Determine if $\{x^2 - x + 1,\ x + 3,\ x^2 - 5\}$ is a spanning set for P_3.

Problem 1.

(1) S is not a subspace of $\mathbb{R}^2$. To show that S is not a subspace of $\mathbb{R}^2$ it is enough to provide a counterexample : $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \in S$, but $2\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \notin S$

(2) T is a subspace of P_3. T is closed under addition: if $p, q \in T$ then $(p+q) \in T$ since $(p+q)'(3) = (p'+q')(3) = p'(3) + q'(3) = 0 + 0 = 0$. T is also closed under multiplication by a scalar: if $p \in T$ and r is a number then $(rp) \in T$ since $(rp)'(3) = rp'(3) = r \cdot 0 = 0$.

Problem 2. We first find $RREF(A)$:

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 3 \\ 0 & 0 & 2 & 2 & -1 \\ 1 & -2 & -1 & -2 & 0 \\ -1 & 2 & -2 & -1 & 2 \end{pmatrix} \underset{R_4+R_1}{\overset{R_3-R_1}{\longrightarrow}} \begin{pmatrix} 1 & -2 & 1 & 0 & 3 \\ 0 & 0 & 2 & 2 & -1 \\ 0 & 0 & -2 & -2 & -3 \\ 0 & 0 & -1 & -1 & 5 \end{pmatrix} \underset{2R_4+R_2}{\overset{R_3+R_2}{\longrightarrow}}$$

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 3 \\ 0 & 0 & 2 & 2 & -1 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 9 \end{pmatrix} \underset{\substack{R_4+9/4R_3 \\ -1/4R_3}}{\overset{\substack{R_1+3/4R_3 \\ R_2-1/4R_3}}{\longrightarrow}} \begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \underset{1/2R_2}{\overset{R_1-1/2R_2}{\longrightarrow}}$$

$$\begin{pmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

a) $\{(1,-2,1,0,0), (0,0,1,1,0), (0,0,0,0,1)\}$ is a basis of the row space of A. Dimension of the row space of A is 3.

b) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 0 \\ 2 \end{pmatrix} \right\}$ is a basis of $R(A)$. $dim\, R(A) = 3$.

c) To find $N(A)$ we solve the system $Ax = 0$. From the $RREF(A)$ we see that x_1, x_3, x_5 are lead and x_2, x_4 are free. Also

$$\begin{aligned} x_1 - 2x_2 - x_4 &= 0 \\ x_3 + x_4 &= 0 \\ x_5 &= 0 \end{aligned}$$

Thus $N(A) = \left\{ \begin{pmatrix} 2x_2 + x_4 \\ x_2 \\ -x_4 \\ x_4 \\ 0 \end{pmatrix} \mid x_2, x_4 \in \mathbb{R} \right\}$. So, $\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis of $N(A)$. $dim\, N(A) = 2$.

d) $rank(A) = dim\, R(A) = 3$ and $nullity(A) = dim\, N(A) = 2$.

Problem 3. Denote $U = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ and $V = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

a) The transition matrix corresponding to the change of basis from $\{\mathbf{u_1}, \mathbf{u_2}\}$ to $\{\mathbf{v_1}, \mathbf{v_2}\}$ is $S = V^{-1}U$.

$$\left(\begin{array}{cc|cc} 1 & -1 & 2 & 3 \\ 1 & 1 & 1 & 2 \end{array}\right) \overset{R_2 - R_1}{\longrightarrow} \left(\begin{array}{cc|cc} 1 & -1 & 2 & 3 \\ 0 & 2 & -1 & -1 \end{array}\right) \underset{1/2R_2}{\overset{R_1 + 1/2R_2}{\longrightarrow}} \left(\begin{array}{cc|cc} 1 & 0 & 3/2 & 5/2 \\ 0 & 1 & -1/2 & -1/2 \end{array}\right)$$

Thus, $S = \begin{pmatrix} 3/2 & 5/2 \\ -1/2 & -1/2 \end{pmatrix}$.

b) Write $\mathbf{w} = c_1\mathbf{v_1} + c_2\mathbf{v_2}$ and solve for c_1, c_2.

$$\left(\begin{array}{cc|c} 1 & -1 & 5 \\ 1 & 1 & 3 \end{array}\right) \overset{R_2 - R_1}{\longrightarrow} \left(\begin{array}{cc|c} 1 & -1 & 5 \\ 0 & 2 & -2 \end{array}\right) \underset{1/2R_2}{\overset{R_1 + 1/2R_2}{\longrightarrow}} \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -1 \end{array}\right)$$

Thus, $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ is the coordinate representation of $\mathbf{w}$ with respect to $\{\mathbf{v_1}, \mathbf{v_2}\}$.

Problem 4.

(1) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$ is a lineraly independent set in $\mathbb{R}^3$ and it is not a basis of $\mathbb{R}^3$;

(2) $\{1, x, x^2, 1 + x + x^2\}$ is a spanning set for P_3 and it is not a basis of P_3;

(3) $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ is the standard basis of $\mathbb{R}^{2\times 2}$;

(4) $L\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is a linear map from $\mathbb{R}^3$ to $\mathbb{R}^2$.

(5) $L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} x_1^2 \\ \sqrt{x_2} \\ 5 \end{pmatrix}$ is a map from $\mathbb{R}^2$ to $\mathbb{R}^3$ and it is not linear.

Problem 5. Using the basis $\{x^2, x, 1\}$ we replace polynomials $\{x^2-x+1,\ x+3,\ x^2-5\}$ with vectors $\left\{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix}\right\}$. Since $det\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 3 & 5 \end{pmatrix} = -9 \neq 0$, $\left\{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix}\right\}$ is a basis of $\mathbb{R}^3$, in particular, it is a spanning set for $\mathbb{R}^3$. Thus $\{x^2-x+1,\ x+3,\ x^2-5\}$ is a spanning set for P_3.