Sample problems for Exam 2

Problem 1. Prove or disprove that set S is a subspace of vector space V.

(1) $V = \mathbb{R}^2$, $S = \{(x, y)^T \,|\, x = y\}$
(2) $V = \mathbb{R}^2$, $S = \{(x, y)^T \,|\, x^2 = y^2\}$
(3) $V = P_3$, $S = \{p \in P_2 \,|\, p(2) = 3\}$
(4) $V = P_3$, $S = \{p(x) = ax^2 + bx + c \,|\, a + b + c = 0\}$
(5) $V = \mathbb{R}^{2\times2}$, $S = \left\{ A \in \mathbb{R}^{2\times2} \,|\, A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$
(6) $V = \mathbb{R}^{2\times2}$, $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,|\, a + b + c + d = 2 \right\}$

Problem 2. Let S be the subspace of $\mathbb{R}^4$ spanned by

$$\mathbf{w_1} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{w_2} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{w_3} = \begin{pmatrix} 2 \\ 1 \\ 4 \\ 1 \end{pmatrix}, \quad \mathbf{w_4} = \begin{pmatrix} 3 \\ 2 \\ 6 \\ 1 \end{pmatrix}.$$

a) Does $b = (1, 0, 0, 0)^T$ belong to S?

b) Do $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}, \mathbf{w_4}$ span $\mathbb{R}^4$?

c) Are $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}, \mathbf{w_4}$ linearly independent?

d) Do $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}, \mathbf{w_4}$ form a basis of $\mathbb{R}^4$?

e) Find a basis of S. What is the dimension of S?

Problem 3. Let S be the subspace of P_3 spanned by $\mathbf{p_1} = x^2 + 2x + 3$, $\mathbf{p_2} = 2x^2 + 3x + 1$, $\mathbf{p_3} = 3x^2 + x + 2$, $\mathbf{p_4} = x^2 + x + 1$.

a) Does x belong to S?

b) Do $\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4}$ span P_3?

c) Are $\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4}$ linearly independent?

d) Do $\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4}$ form a basis of P_3?

e) Find a basis of S. What is the dimension of S?

Problem 4. Let S be the subspace of $\mathbb{R}^{2\times 3}$ spanned by

$$\mathbf{B_1} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad \mathbf{B_2} = \begin{pmatrix} 6 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix}, \quad \mathbf{B_3} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

a) Does $\begin{pmatrix} 7 & 4 & 1 \\ -2 & -5 & -8 \end{pmatrix}$ belong to S?

b) Do $\mathbf{B_1}, \mathbf{B_2}, \mathbf{B_3}$ span $\mathbb{R}^{2\times 3}$?

c) Are $\mathbf{B_1}, \mathbf{B_2}, \mathbf{B_3}$ linearly independent?

d) Do $\mathbf{B_1}, \mathbf{B_2}, \mathbf{B_3}$ form a basis of $\mathbb{R}^{2\times 3}$?

e) Find a basis of S. What is the dimension of S?

Problem 5. Let $A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 6 & 8 & 10 \\ 3 & 6 & 9 & 11 & 15 \\ 4 & 7 & 11 & 16 & 0 \end{pmatrix}$.

a) Find a basis for the row space of A.

b) Find a basis for $R(A)$.

c) Find a basis for $N(A)$.

d) Find rank and nullity of A.

Problem 6. For the given vector space V and given bases F and G of V find the transition matrix corresponding to the change of basis from G to F. Also for the given vector w in V find $[w]_G$ and $[w]_F$.

(1) $V = \mathbb{R}^3$, $F = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$, $G = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$,
$w = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

(2) $V = P_2$, $F = \{2x - 3, 5x - 7\}$, $G = \{x + 8, 2x + 15\}$, $w = 1$.

Problem 7. Prove or disprove that a given map is a linear transformation. See homework problems from Section 4.1 for examples.

Answers

Problem 1.

(1) S is a subspace of $\mathbb{R}^2$.
(2) S is not a subspace of $\mathbb{R}^2$.
(3) S is a not subspace of P_3.
(4) S is a subspace of P_3.
(5) S is a subspace of $\mathbb{R}^{2\times 2}$.
(6) S is a not subspace of $\mathbb{R}^{2\times 2}$.

Problem 2.

a) No, b does not belong to S.

b) No, $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}, \mathbf{w_4}$ do not span $\mathbb{R}^4$.

c) No, $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}, \mathbf{w_4}$ are linearly dependent.

d) No, $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}, \mathbf{w_4}\}$ is not a basis of $\mathbb{R}^4$.

e) $\{\mathbf{w_1}, \mathbf{w_2}\}$ is a basis of S. $dim\, S = 2$.

Problem 3.

a) Yes, x belongs to S.

b) Yes, $\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4}$ span P_3.

c) No, $\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4}$ are linearly dependent.

d) No, $\{\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4}\}$ is not a basis of P_3.

e) $\{\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}\}$ is a basis of S. $dim\, S = 3$.

Problem 4.

a) Yes, $\begin{pmatrix} 7 & 4 & 1 \\ -2 & -5 & -8 \end{pmatrix}$ belongs to S.

b) No, $\mathbf{B_1}, \mathbf{B_2}, \mathbf{B_3}$ do not span $\mathbb{R}^{2\times 3}$.

c) No, $\mathbf{B_1}, \mathbf{B_2}, \mathbf{B_3}$ are linearly dependent.

d) No, $\{\mathbf{B_1}, \mathbf{B_2}, \mathbf{B_3}\}$ is not a basis of $\mathbb{R}^{2\times 3}$.

e) $\{\mathbf{B_1}, \mathbf{B_2}\}$ is a basis of S. $dim\, S = 2$.

Problem 5.

a) $\{(1,2,3,4,5), (0,1,1,0,20), (0,0,0,1,0)\}$ is a basis for the row space of A.

b) $\{(1,2,3,4)^T, (2,4,6,7)^T, (4,8,11,16)^T\}$ is a basis for $R(A)$.

c) $\{(-1,-1,1,0,0)^T, (35,-20,0,0,1)^T\}$ is a basis for $N(A)$.

d) $rank(A) = 3$ and $nullity(A) = 2$.

Problem 6.

(1) $S = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}$, $[w]_G = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $[w]_F = \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$.

(2) $S = \begin{pmatrix} -47 & -89 \\ 19 & 36 \end{pmatrix}$, $[w]_G = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $[w]_F = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$.