Math310: Sample Problems from Chapters 5 and 6

The following topics from Chapters 5 and 6 will be covered on the final exam

- Inner product spaces: finding angles, scalar and vector projections; deciding whether a set of vectors is an orthonormal/orthogonal set; Gram-Schmidt orthogonalization process; projecting a vector onto a subspace.
- $\mathbb{R}^n$ with the standard scalar product: deciding whether two subspaces are orthogonal; finding orthogonal complement; least squares problems (projecting a vector onto a subspace).
- Eigenvalues and eigenvectors: finding eigenvalues and eigenspaces of a matrix; deciding whether a matrix is diagonalizable; finding a representation $A = XDX^{-1}$ when possible; computing rational powers of a matrix and matrix exponential e^A; solving systems of linear differential equations.

Problem 1. Find the eigenvalues and the corresponding eigenspaces for each of the following matrices. Decide whether each of the following matrices is diagonalizable or not. If possible find a representation $A = XDX^{-1}$, where D is a diagonal matrix.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 5 & -1 \end{pmatrix}, \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}$$

Problem 2. For each diagonalizable matrix from Problem 1 compute A^6, find B such that $B^3 = A$, evaluate e^{At}.

Problem 3. Use the definition of the matrix exponential to compute e^{At} for each of the following matrices:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

Problem 4. Find the general solution to $Y' = AY$. Solve the initial value problem.

a) $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, $Y(0) = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

b) $A = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}$, $Y(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

c) $A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$, $Y(0) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

d) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $Y(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Problem 5. Translate the second order system

$$\begin{aligned} y_1'' &= 2y_1 - y_2 + y_2' \\ y_2'' &= y_1 - 2y_2 + y_1' \end{aligned}$$

into a first order system of linear differential equations.

Problem 6. For the given vectors u and v in the given inner product space compute the angle between u and v and find the scalar and vector projections of u onto v.

a) $\mathbb{R}^4$ with the weighted inner product, $w = (1, 2, 1, 2)^T$ and $u = (1, -1, 0, 1)^T$, $v = (-1, 0, 1, 1)^T$.

b) $C[0, 1]$ with the inner product defined by $< f, g >= \int_0^1 f(x)g(x)dx$ and $u = x + 1$, $v = x - 2$.

Problem 7. For the given inner product space use the Gram-Schmidt orthogonalization process to find an orthonormal basis of S . Check your answer: show that the set you found is orthonormal.

Find the projection p of b onto S. Then the distance from b to S is the norm of $b - p$. Determine the distance from b to S.

a) $\mathbb{R}^4$ with the standard scalar product, $S = \text{Span} \left\{ \begin{pmatrix} 4 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix} \right\}$ and $b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$.

b) $C[-1, 1]$ with the inner product defined by $< f, g >= \frac{1}{2} \int_{-1}^1 f(x)g(x)dx$, $S = Span\{1, x, x^2\}$ and $b = x^3$.

Problem 8. Let $S = \text{Span}\left\{ \begin{pmatrix} 4 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix} \right\}$ and $b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$.

Find the orthogonal complement $S^{\perp}$ of S. Find the projection q of b onto $S^{\perp}$. Then the distance from b to S is the length of q. Find the distance from b to S and compare your answer with Problem 7a).

Problem 9. Find the least squares solution to the system $Ax = b$,

where $A = \begin{pmatrix} 4 & 4 & 0 \\ 2 & 0 & -1 \\ 1 & 3 & 1 \\ 2 & 3 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$.

Determine the projection p of b onto $R(A)$. Compare your answer to Problem 7a).

Problem 10. Find the beast least squares fit to the points $(-1, 0)$, $(0, 1)$, $(1, 3)$, $(2, 9)$ by a quadratic polynomial.