Answers to Sample Problems on Chapters 5 and 6

Problem 1.

a) $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$\lambda = 0$, $Span((-1,0,1)^T, (-1,1,0)^T)$; $\lambda = 3$, $Span((1,1,1,)^T)$. A is diagonalizable.

$$A = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1}$$

b) $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 5 & -1 \end{pmatrix}$

$\lambda = 1$, $Span((1,0,0)^T)$; $\lambda = -2$, $Span((1,1,-5)^T)$; $\lambda = 4$, $Span((1,1,1,)^T)$. A is diagonalizable.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -5 & 1 \end{pmatrix}^{-1}$$

c) $A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$

$\lambda = 2$, $Span((1,1)^T)$. A is defective.

d) $A = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}$

$\lambda = 2+i$, $Span((1,1+i)^T)$; $\lambda = 2-i$, $Span((1,1-i)^T)$. A is diagonalizable.

$$A = \begin{pmatrix} 1 & 1 \\ 1+i & 1-i \end{pmatrix} \begin{pmatrix} 2+i & 0 \\ 0 & 2-i \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1+i & 1-i \end{pmatrix}^{-1}$$

Problem 2.

a) $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$$A^6 = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 729 \end{pmatrix} \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1}$$

$$B = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt[3]{3} \end{pmatrix} \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1}$$

$$e^{At} = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{3t} \end{pmatrix} \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} = \frac{1}{3} \begin{pmatrix} e^{3t}+2 & e^{3t}-1 & e^{3t}-1 \\ e^{3t}-1 & e^{3t}+2 & e^{3t}-1 \\ e^{3t}-1 & e^{3t}-1 & e^{3t}+2 \end{pmatrix}$$

b) $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 5 & -1 \end{pmatrix}$

$$A^6 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 64 & 0 \\ 0 & 0 & 4096 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -5 & 1 \end{pmatrix}^{-1}$$

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\sqrt[3]{2} & 0 \\ 0 & 0 & \sqrt[3]{4} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -5 & 1 \end{pmatrix}^{-1}$$

$$e^{At} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -5 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{4t} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -5 & 1 \end{pmatrix}^{-1} =$$

$$\frac{1}{6} \begin{pmatrix} 6e^t & -6e^t + e^{-2t} + 5e^{4t} & -e^{-2t} + e^{4t} \\ 0 & e^{-2t} + 5e^{4t} & -e^{-2t} + e^{4t} \\ 0 & -5e^{-2t} + 5e^{4t} & 5e^{-2t} + e^{4t} \end{pmatrix}$$

d) $A = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}$

$$A^6 = \begin{pmatrix} 1 & 1 \\ 1+i & 1-i \end{pmatrix} \begin{pmatrix} (2+i)^6 & 0 \\ 0 & (2-i)^6 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1+i & 1-i \end{pmatrix}^{-1}$$

$$B = \begin{pmatrix} 1 & 1 \\ 1+i & 1-i \end{pmatrix} \begin{pmatrix} (2+i)^{1/3} & 0 \\ 0 & (2-i)^{1/3} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1+i & 1-i \end{pmatrix}^{-1}$$

$$e^{At} = \begin{pmatrix} 1 & 1 \\ 1+i & 1-i \end{pmatrix} \begin{pmatrix} e^{(2+i)t} & 0 \\ 0 & e^{(2-i)t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1+i & 1-i \end{pmatrix}^{-1} =$$

$$\begin{pmatrix} e^{2t}(\cos t - \sin t) & e^{2t}\sin t \\ -2e^{2t}\sin t & e^{2t}(\cos t + \sin t) \end{pmatrix}$$

Problem 3.

a) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$; $e^{At} = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}$

b) $A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$; $e^{At} = \begin{pmatrix} 1+t & t \\ -t & 1-t \end{pmatrix}$

Problem 4.

a) $Y_{gen}(t) = c_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$Y(t) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + e^{3t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{3t} \\ e^{3t} - 1 \\ e^{3t} + 1 \end{pmatrix} \text{ or}$$

$$Y(t)=e^{At}\begin{pmatrix}1\\0\\2\end{pmatrix}=\frac{1}{3}\begin{pmatrix}e^{3t}+2 & e^{3t}-1 & e^{3t}-1\\ e^{3t}-1 & e^{3t}+2 & e^{3t}-1\\ e^{3t}-1 & e^{3t}-1 & e^{3t}+2\end{pmatrix}\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}e^{3t}\\ e^{3t}-1\\ e^{3t}+1\end{pmatrix}$$

b) $Y_{gen}(t)=d_1\begin{pmatrix}e^{2t}\cos t\\ e^{2t}(\cos t-\sin t)\end{pmatrix}+d_2\begin{pmatrix}e^{2t}\sin t\\ e^{2t}(\cos t+\sin t)\end{pmatrix}$

$Y(t)=-\begin{pmatrix}e^{2t}\cos t\\ e^{2t}(\cos t-\sin t)\end{pmatrix}+3\begin{pmatrix}e^{2t}\sin t\\ e^{2t}(\cos t+\sin t)\end{pmatrix}=\begin{pmatrix}e^{2t}(3\sin t-\cos t)\\ e^{2t}(4\sin t+2\cos t)\end{pmatrix}$ or

$Y(t)=e^{At}\begin{pmatrix}-1\\2\end{pmatrix}=\begin{pmatrix}e^{2t}(\cos t-\sin t) & e^{2t}\sin t\\ -2e^{2t}\sin t & e^{2t}(\cos t+\sin t)\end{pmatrix}\begin{pmatrix}-1\\2\end{pmatrix}=\begin{pmatrix}e^{2t}(3\sin t-\cos t)\\ e^{2t}(4\sin t+2\cos t)\end{pmatrix}$

c) $Y_{gen}(t)=e^{At}\begin{pmatrix}c_1\\c_2\end{pmatrix}=\begin{pmatrix}1+t & t\\ -t & 1-t\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}=\begin{pmatrix}c_1+c_1t+c_2t\\ c_2-c_1t-c_2t\end{pmatrix}$

$Y(t)=e^{At}\begin{pmatrix}3\\5\end{pmatrix}=\begin{pmatrix}3+8t\\5-8t\end{pmatrix}$

d) $Y_{gen}(t)=e^{At}\begin{pmatrix}c_1\\c_2\end{pmatrix}=\begin{pmatrix}e^t & te^t\\ 0 & e^t\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}=\begin{pmatrix}c_1e^t+c_2te^t\\ c_2e^t\end{pmatrix}$

$Y(t)=e^{At}\begin{pmatrix}2\\3\end{pmatrix}=\begin{pmatrix}2e^t+3te^t\\ 3e^t\end{pmatrix}$

Problem 5.

$$Z'=AZ,\ \text{where } A=\begin{pmatrix}0&0&1&0\\0&0&0&1\\2&-1&0&1\\1&-2&1&0\end{pmatrix}\ \text{and } Z(t)=\begin{pmatrix}z_1(t)\\z_2(t)\\z_3(t)\\z_4(t)\end{pmatrix}=\begin{pmatrix}y_1(t)\\y_2(t)\\y_1'(t)\\y_2'(t)\end{pmatrix}.$$

Problem 6.

a) $\theta=\arccos(1/(2\sqrt{5}))$, $\alpha=1/2$, $proj_v u=1/4(-1,0,1,1)^T$.

b) $\theta=\arccos(-13/14)$, $\alpha=-13/(2\sqrt{21})$, $proj_v u=-13/14(x-2)$.

Problem 7.

a) $\left\{\frac{1}{5}\begin{pmatrix}4\\2\\1\\2\end{pmatrix},\frac{1}{3}\begin{pmatrix}0\\-2\\2\\1\end{pmatrix},\frac{1}{15\sqrt{41}}\begin{pmatrix}-36\\7\\-34\\82\end{pmatrix}\right\}$ is an orthonormal basis of S;

$p=<b,u_1>u_1+<b,u_2>u_2+<b,u_3>u_3=\frac{1}{41}\begin{pmatrix}92\\14\\55\\164\end{pmatrix}$, $b-p=\frac{1}{41}\begin{pmatrix}-51\\68\\68\\0\end{pmatrix}$,

$dist=17/\sqrt{41}$.

b) $\left\{\frac{1}{\sqrt{2}}, \frac{\sqrt{6}}{2}x, \frac{3\sqrt{10}}{4}(x^2 - 1/3)\right\}$ is an orthonormal basis of S; $p = \frac{3}{5}x$, $dist = \frac{8}{175}$.

Problem 8.

$S^{\perp} = Span((-3, 4, 4, 0)^T)$; $q = 17/41(-3, 4, 4, 0)^T$, $dist = 17/\sqrt{41}$.

Problem 9.

$\hat{x} = (109/41, -86/41, 204/41)^T$; $p = A\hat{x} = 1/41(92, 14, 55, 164)^T$.

Problem 10.

$y = 0.55 + 1.65x + 1.25x^2$.