HOMEWORK 1

This problem set is due Friday September 5. You may work on the problem set in groups; however, the final write-up must be yours and reflect your own understanding. In all these exercises assume that k is an algebraically closed field and R is a commutative ring with unit.

Problem 0.1. *Show that the following conditions on a ring R are equivalent.*

(1) *Every ascending chain of ideals $I_1 \subset I_2 \subset \cdots$ stabilizes.*
(2) *Every ideal is finitely generated.*
(3) *Every non-empty set of ideals contains a maximal element.*

Rings satisfying these conditions are called Noetherian rings. Show that the ring of continuous real valued functions on the unit interval is not Noetherian.

Problem 0.2. *Prove that if a ring R is Noetherian, then the formal power series ring $R[[x]]$ over R is also Noetherian.*

Problem 0.3.
(1) *Show that the union of the coordinate axes in $\mathbb{A}^3_k$ is a closed algebraic set. Determine generators for its ideal.*
(2) *Consider the curve in $\mathbb{A}^3_k$ given in parametric form $C = \{(t, t^2, t^3) \in \mathbb{A}^3 \mid t \in k\}$. Determine generators for the ideal of C.*
(3) *Consider the set $\{(0,0), (1,1), (0,1), (1,0)\}$ of four points in $\mathbb{A}^2_{\mathbb{C}}$. Find generators for its ideal.*
(4) *Consider the set $\{(0,0), (1,1), (2,2), (1,0)\}$ of four points in $\mathbb{A}^2_{\mathbb{C}}$. Find generators for its ideal. How does your answer differ from the previous part? What is special about these four points?*

Problem 0.4. *Consider the set $V = \{(t^3, t^4, t^5) | t \in k\}$ in $\mathbb{A}^3_k$. Show that V is an affine variety. Find generators of its ideal. How many generators do you need? Can V be described as the zero locus of two polynomials?*

Problem 0.5. *Let f be a polynomial in $k[x_1, \dots, x_n]$. Show that $\mathbb{A}^n - V(f)$ can be realized as the affine variety $V(x_{n+1} f - 1)$ in $\mathbb{A}^{n+1}$. Conclude that the general linear group $GL(n,k)$ (invertible $n \times n$ matrices with entries in k under usual matrix multiplication) can be realized as an affine variety in $\mathbb{A}^{n^2+1}_k$.*

Problem 0.6. *Let $S = \mathbb{A}^2_k - \{(0,0)\}$ be the complement of the origin in $\mathbb{A}^2_k$. Find $I(S)$, the set of polynomials vanishing on S. What is $V(I(S))$? Can S be an affine variety?*

Problem 0.7. *Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be two affine varieties. Prove that $X \times Y \subset \mathbb{A}^{n+m}$ is an affine variety.*

Problem 0.8. *Show that any two ordered sets of $n+2$ points in general position in $\mathbb{P}^n$ are projectively equivalent. Show that two sets of four points in $\mathbb{P}^1$ are projectively equivalent if and only if their cross-ratios are equal. Harder: Characterize when $n+3$ points in general linear position in $\mathbb{P}^n$ are projectively equivalent.*

Problem 0.9. *Let Γ be a set of points in $\mathbb{P}^n$ of cardinality d. Show that Γ can be expressed as the zero locus of polynomials of degree at most d. Show that if all the points in Γ do not lie on a line, then in fact Γ can be expressed as the zero locus of polynomials of degree $d-1$ or less.*