

HOMEWORK 2

This problem set is due Friday September 12. You may work on the problem set in groups; however, the final write-up must be yours and reflect your own understanding. In all these exercises assume that k is an algebraically closed field and R is a commutative ring with unit.

Problem 0.1. Consider the following five closed affine sets in \mathbb{A}^2 . Give a thorough discussion of which among them are isomorphic.

- (1) $X_1 = \{(x, y) \in \mathbb{A}^2 \mid x = y\}$
- (2) $X_2 = \{(x, y) \in \mathbb{A}^2 \mid x = y^{17}\}$
- (3) $X_3 = \{(x, y) \in \mathbb{A}^2 \mid x^2 = y^2\}$
- (4) $X_4 = \{(x, y) \in \mathbb{A}^2 \mid x^2 = y^3\}$
- (5) $X_5 = \{(x, y) \in \mathbb{A}^2 \mid x^2 = y^2 + y^3\}$

Problem 0.2. Prove the following statements:

- (1) If X is an affine variety, then any non-empty Zariski open subset U of X is dense in X .
- (2) If X is an affine variety, then any non-empty Zariski open subset U of X is irreducible.
- (3) Let $f : X \rightarrow Y$ be a regular, surjective map of closed affine sets. If X is irreducible, then Y is irreducible.

Problem 0.3. Show that any two ordered sets of $n + 2$ points in general position in \mathbb{P}^n are projectively equivalent. Show that two sets of four points in \mathbb{P}^1 are projectively equivalent if and only if their cross-ratios are equal. Harder: Characterize when $n + 3$ points in general linear position in \mathbb{P}^n are projectively equivalent.

Problem 0.4. Let Γ be a set of points in \mathbb{P}^n of cardinality d . Show that Γ can be expressed as the zero locus of polynomials of degree at most d . Show that if all the points in Γ do not lie on a line, then in fact Γ can be expressed as the zero locus of polynomials of degree $d - 1$ or less.

Problem 0.5. (1) Show that the Segre image of $\mathbb{P}^1 \times \mathbb{P}^1$ is a quadric hypersurface in \mathbb{P}^3 .
(2) Let L, M and N be three pairwise skew lines in \mathbb{P}^3 . Show that union of all the lines in \mathbb{P}^3 intersecting L, M and N is isomorphic to the Segre image of $\mathbb{P}^1 \times \mathbb{P}^1$.
(3) How many lines in \mathbb{P}^3 intersect the four lines $L_1 = (z_1 = z_2 = 0)$, $L_2 = (z_3 = z_4 = 0)$, $L_3 = (z_1 = z_2, z_3 = z_4)$ and $L_4 = (z_1 + 2z_2 = z_3 + z_4, z_1 + 2z_4 = z_2 + z_3)$?

Problem 0.6. Recall that the twisted cubic curve C is the image of the map $\nu : \mathbb{P}^1 \rightarrow \mathbb{P}^3$ given by $(x_0 : x_1) \mapsto (x_0^3 : x_0^2x_1 : x_0x_1^2 : x_1^3)$

- (1) Show that the homogeneous ideal is generated by $Q_1 : z_0z_2 = z_1^2, Q_2 : z_1z_3 = z_2^2, Q_3 : z_0z_3 = z_1z_2$.
- (2) Show that an alternative way to describe the twisted cubic is as the rank one locus of the matrix
$$\begin{pmatrix} z_0 & z_1 & z_2 \\ z_1 & z_2 & z_3 \end{pmatrix}$$
- (3) Show that $Q_i \cap Q_j$ for $i \neq j$ is C union a line.
- (4) More generally, show that the intersection of $\lambda_1Q_1 + \lambda_2Q_2 + \lambda_3Q_3$ and $\mu_1Q_1 + \mu_2Q_2 + \mu_3Q_3$, where $(\lambda_1 : \lambda_2 : \lambda_3) \neq (\mu_1 : \mu_2 : \mu_3)$ in \mathbb{P}^2 , is C union a line.