

HOMEWORK 9

This problem set is due Monday November 3. You may work on the problem set in groups; however, the final write-up must be yours and reflect your own understanding.

Problem 0.1. Calculate the degree of the Grassmannian of lines in \mathbb{P}^n in its Plücker embedding. (Hint: You may want to use Pieri's formula and learn about Catalan numbers.)

Problem 0.2. Show that a variety X and a general hyperplane section $X \cap H$ have the same degree. Calculate the degree of the surface scroll in \mathbb{P}^4 defined by the rank one locus of the matrix

$$\begin{pmatrix} z_0 & z_1 & z_2 \\ z_2 & z_3 & z_4 \end{pmatrix}$$

Problem 0.3. Let $a \leq b$ be two positive integers. The surface scroll $S_{a,b}$ is constructed as follows. Pick two disjoint linear spaces \mathbb{P}^a and \mathbb{P}^b in \mathbb{P}^{a+b+1} . Fix rational normal curves C_a and C_b of degrees a and b in the \mathbb{P}^a and \mathbb{P}^b , respectively, and an isomorphism $\phi: C_a \rightarrow C_b$. $S_{a,b}$ is the surface obtained by taking the union of the lines joining $p \in C_a$ with $\phi(p)$ in C_b . Prove that $S_{a,b}$ is an algebraic surface. Describe in detail $S_{1,1}$. Show that the surface in the previous exercise is $S_{1,2}$. We can also allow $a = 0$ and ϕ to be the constant map. In that case the surface is a cone over a rational normal curve. Calculate the degree of the surface scroll $S_{a,b}$.

Problem 0.4. Let X be a non-degenerate (i.e., not contained in any hyperplanes) projective variety of degree d and dimension k in \mathbb{P}^n . Show that $d \geq n - k + 1$. Show that for rational normal curves, the Veronese surface $v_2(\mathbb{P}^2)$ in \mathbb{P}^5 and surface scrolls $S_{a,b}$ equality holds. Challenge: Classify the varieties where equality holds.

Problem 0.5. Prove that the every automorphism of projective space \mathbb{P}^n is induced by an automorphism $\phi \in GL(n+1)$ of k^{n+1} . In other words, the automorphism group of \mathbb{P}^n is $\mathbb{P}GL(n+1)$.