Lecture 3 (January 16, 2009) - Maschke's Theorem

Examples

For any field F, remember S_n acts on F^n by permuting the indices of some bases $e_1, ..., e_n$. Let $W_1 = \{\sum c_i\, e_i \mid \sum c_i = 0\}$. If $\sigma \in S_n$, $w \in W_1$, we claim that $\theta \cdot \omega \in W_1$. Then $\sigma(\sum c_i\, e_i) = \sum c_i\, \delta\, e_1 = \sum c_i\, e_{\sigma(i)}$ so that $\sum c_i$ doesn't change. Take $W_2 = \{\sum a_i e_i \mid a_1 = a_2 = ... = a_n\} = F \cdot \{e_1 + e_2 + ... + e_n\}$. Let $\sigma \in S_n$, $a \in F$. Then

$$\begin{aligned} \sigma \cdot a \cdot (e_1 + ... + e_n) &= a \cdot \sigma \cdot (e_1 + ... + e_n) \\ &= a \cdot \left(e_{\sigma(1)} + ... + e_{\sigma(n)}\right) = a \cdot (e_1 + ... + e_n). \end{aligned}$$

Recall two G-representations or FG-modules, V and W are called equivalent if there exists $V \xrightarrow{\psi} W$ which commutes with the action of G, or FG.

$$\begin{array}{ccc} & V \xrightarrow{\psi} W & \\ \rho(g) \downarrow & & \downarrow \rho_2(g) \\ & V \xrightarrow{\psi} W. & \end{array}$$

Theorem. *Let G be a finite group with* char $F \nmid |G|$. *Then any submodule of an FG-module is a direct summand, i.e., if V is an FG-module and $0 \neq U \subseteq V$ is an FG-submodule, then there exists W such that $V = U \oplus W$ as FG-modules.*

Corollary. *If* char $F \nmid |G|$, *then V is irreducible if and only if it is indecomposable.*

Proof. Iredducibility obviously implies indecomposability. If it is not irreducible, then if there exists $U \neq 0$, then there exists W such that $V = U \oplus W$. ☐

Corollary. *If* char $F \nmid |G|$, *then every FG-module is injective.*

Proof. (of Maschke's Theorem) The idea is to produce an FG-equivalent projection $\pi : V \to U$. Then FG-equivalent means if $x \in FG$, then $\pi(x \cdot v) = x \cdot \pi(v)$. Recall projective means $\pi : V \to U$ is surjective, and $\pi(\pi(V)) = \pi(V)$. This last part can be thought of as projection onto the Euclidean plane: if we project once and then project again, then that second action does nothing since it is already projected onto the plane.

Continuing, let $W = \ker \pi$. We claim $V = U \oplus W$. If $v \in U \cap W$, then $v \in \ker \pi$ and $v = \pi(g)$ for some y ($\pi(V) = U$). So $\pi(y) = \pi(\pi(y)) = \pi(v) = 0$ so indeed $\pi(y) = v$. If $v \in V$, write $v = (v - \pi(v)) + \pi(v)$ and we say $v - \pi(v) \in W = \ker \pi$, and $\pi(v) \in U$. Then, $\pi(v - \pi(v)) = \pi(v) - \pi\pi(v) = \pi(v) - \pi(v) = 0$. So, we have shown that $U \cap W = \{0\}$ and $U + W = V$. This then implies $V = U \oplus W$.

If π is FG-equivalent and $v \in \ker\, \pi = W$ and $x \in FG$, then we want to show $x \cdot v \in W = \ker \pi$. Indeed, $\pi(x \cdot v) = x \cdot \pi(v) = x \cdot 0 = 0$ so W is in fact FG-stable.

We need to make an appropriate π. Start with an arbitrary $\pi_0 : V \to U$, just a vector space projection. Now, if $g \in G$, $g\pi_0 g^{-1}u = u\ \forall u \in U$ [**Exercise.** convince yourself of this]. Now, we let

$$\pi = \frac{1}{|G|}\sum_{g\in G} g\pi_0 g^{-1},$$

and we can do this because char $F \not| \, |G|$, or otherwise $|G|$ would be 0 in π's domain. **Exercise.** Check that $\pi(u) = u$ and $\pi(\pi(u)) = \pi(u)$. ☐

Wedderburn's Theorem. *Let R be a non-zero ring with identity* (*not necessarily commutative*). *Then the following are equivalent:*

(1) *Every R-module is injective.*

(2) *Every R-module is projective.*

(3) *Every R-module is completely reducible.*

(4) *The ring R considered as a left R-module is a direct sum $R = L_1 \oplus ... \oplus L_n$, where each L_i is a simple module iwth $L_i = Re_i$ for some e_i's satisfying*

(i) $e_ie_j = 0$ *for* $i \neq j$ (ii) $e_i^2 = e_i$ (iii) $\sum e_i = 1$.

(5) *As rings, R is isomorphic to a direct product of matrix rings over division rings,*

$$R = R_1 \times ... \times R_r,$$

with each $R_j = M_{n_j}(\Delta_j)$ with Δ_j a division ring and each R_i a two-sided ideal in R. Further, r, n_j, Δ_j's are up to isomorphism uniquely determined.

Proof. Next time! ☐

Lecture 4 (January 21, 2009) - Maschke's Theorem

Definition. Let R be a ring and Q a module. Then Q is injective if one of the following holds:

(a) (R commutative) If $0 \to L \to M \to N \to 0$ is a short exact sequence, then

$$0 \to \mathrm{Hom}(N,Q) \to \mathrm{Hom}(M,Q) \to \mathrm{Hom}(L,Q) \to 0$$

is exact.

(b) If $0 \to L \to M$ is exact, then

$$\begin{array}{ccccc} 0 \to & L & \to & M \\ & f\downarrow & \swarrow & \\ & & Q. & \end{array}$$

(c) If Q is a submodule of any M, then Q is a direct summand. [Maschke's Thm]

(d) If I is a left-sided ideal of R, then any R-module homomorphism $I \to Q$ can be extended to $R \to Q$. [Baer's criterion]

Definition. Let R be a ring and P a module. Then P is projective if one of the following holds:

(a) (R commutative) If $0 \to L \to M \to N \to 0$ is short exact then

$$0 \to \mathrm{Hom}(P,L) \to \mathrm{Hom}(P,M) \to \mathrm{Hom}(P,N) \to 0$$

is exact.

(b) P is a direct summand of a free module.

(c) If $M \to N \to 0$ is exact, then

$$\begin{array}{ccc} 0 \to L \to M \\ f \uparrow \quad \nearrow \\ P. \end{array}$$

Corollary. (to Wedderburn's Theorem) *If char* $F \nmid |G|$, *then*

$$FG \cong M_{n_1}(\Delta_1) \times ... \times M_{n_r}(\Delta_r)$$

with $\Delta_1, ..., \Delta_r$ *division rings.*

Terminology. Such a ring is called semi-simple.

Modules over $M_n(\Delta)$ (Δ a division ring)

Definition. (1) *A non-zero element* e *is called idempotent if* $e^2 = e$.

(2) e_1, e_2 *are orthogonal if* $e_1 \cdot e_2 = e_2 \cdot e_1 = 0$.

(3) *An idempotent* e *is called primitive if it cannot be written as* $a + b$ *with* a, b *orthogonal idempotents.*

(Notice $(a+b)^2 = a^2 + b^2 + ab + ba = a^2 + b^2 = a + b$.)

(4) e *is called primitive central idempotent if it cannot be written as a sum of two orthognal idempotents in* $Z(R)$.

Proposition. *Let* $R = M_n(\Delta)$ *and* I *the identity matrix.*

(a) *The only two-sided ideals of* R *are* 0 *and* R.

(b) *The center of* R *is* $Z(R) \coloneqq \{\alpha I \mid \alpha \in Z(\Delta)\}$.

(c) $e_i = E_{ii}$ = *matrix with all zeros except in* ii.

(d) $L_i = Re_i$ *simple leftmodules.* $\forall i$, $L_i \cong L_1$ *and* $R = L_1 \oplus ... \oplus L_n$.

(e) *If* M *is a simple* R*-module, then* $M \cong L_1$.

Lecture 5 (January 23, 2009) -

Recall our proposition from last time.

Lemma. For R an abritrary non-zero ring,

(i) If M and N are simple R-modules, then if $\varphi : M \to N$ is a non-trivial R-module homomorphism, then φ is an isomorphism.

(ii) If M is simple, then $\mathrm{Hom}_R(M, M)$ is a division ring.

Remarks. Let $E_{ij} = (a_{rs})$, with

$$a_{rs} = \delta_{r_i} d_{s_j} = \begin{cases} 1 & \text{if } r = i, s = j \\ 0 & \text{elsewhere.} \end{cases}$$

(a) $E_{ij}A$ is the matrix whose ith row is equal to the jth row of A and zero's elsewhere. For example,

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}.$$

(b) AE_{ij} is the matrix whose jth column is equal to the ith column of A and else zero's.

(c) $E_{pq}AE_{rs}$ is the ps entry in a_{qr}.

Exercise. Verify the above explicitly.

Let J be a two-sided ideal and let $A \in J$. Then if some entry a_{qr} of A is non-zero, then $E_{pq}AE_{rs} \in J$. Then $E_{rs} = \frac{1}{a_{qr}} E_{pq}AE_{rs} \in J$. This means $\forall p, s$, $E_{pq} \in J$ implies $J = R$. Next, let $A \in Z(R)$. Then $E_{ij}A = AE_{ij}$. Then if $i \neq j$, then $a_{ij} = 0$. (verify!)

Next, $Re_i = RE_{ii}$. This is going to be ith column with 0's everywhere else. If A is non-zero, then $RA = R \cdot re_i \subseteq Re_i$. We then claim that $RA = Re_i$. If a_{p_i} is a non-zero enetry of A, then $E_{ii} = \frac{1}{a_{p_i}} E_{ip}A \in RA$, so that $Re_i \subseteq RA$. If $A \in Re_i$ and $A \neq 0$, then $RA = Re_i$ so that it is simple.

It's trivial that $L_i \cong L_j$.

Let M be any simple module. Since $1m = m \, \forall m$, $(\sum e_i)m = m$ so $\sum e_i m = m$. Given m, $\exists e_i$ such that $e_i m \neq 0$. We can then write a map $L_i \to M$. This map sends $re_i \overset{\varphi}{\mapsto} re_i m$. If $r = 1 \in \Delta$, then $\varphi(e_i) = e_i m \neq 0$.

Additionally, the e_i's are primitive. We see this as follows. Assume that $e_i = a + b$. Then $Re_i = Ra + Rb$. We claim $Ra \cap Rb = \{0\}$ with $ra = sb$. Then

$$ra = ra^2 = ra \cdot a = sb \cdot a = 0.$$

Homework. Look up the Dummit and Foote theorem on $R = R_1 \times ... \times R_r$! This module will be a simple module.

Lecture 6 (January 26, 2009) - Introduction to Characters

Recall that $\mathbb{C}G \cong M_{n_1}(\mathbb{C}) \times .... \times M_{n_r}(\mathbb{C})$. Then the regular representation will be

$$n_1 M_1 \oplus ... \oplus n_r M_r.$$

Furthermore, $M_{n_1}(\mathbb{C})$ will be n_1 copies of M_1, but it can also be realized as

$$M_1 \otimes M_1^* \cong \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \cong \mathbb{C}^{n_1^2}.$$

A regular representation will always be $\bigoplus M_i \otimes M_i^*$ (the so-called Peter-Weyl Theorem for compact groups).

Definition. A function $\varphi : G \to F$ is called a class function if $\varphi(gxg^{-1}) = \varphi(x)$ (orbits of conjugation) $\forall x\, \forall g$.

Definition. If φ is a representation of G on a vector space V over F, then we can define

$$\chi_\varphi(g) = \operatorname{tr} \varphi(g).$$

Example. For a regular representations of G, G acts on $\mathbb{C}_G$ by $g \cdot \sum \alpha_h h = \sum \alpha_h (gh)$. The basis vectors are precisely $\{h\}_{h \in G}$. If $g = 1$, then it sends $h \mapsto h$, so that the trace will be precisely $|G|$. If $g \neq 1$, then $h \mapsto gh \neq h$ (ever). So there won't be anything on the diagonal, and hence trace $= 0$. Hence,

$$\chi_{\text{reg}}(g) = \begin{cases} 0 & \text{if } g \neq 1 \\ |G| & \text{if } g = 1. \end{cases}$$

Example. Consider D_{2n} acting on $\mathbb{R}^2$ by

$$\rho : \begin{pmatrix} \cos \frac{2\pi}{n} & \sin \frac{2\pi}{n} \\ \sin \frac{2\pi}{n} & \cos \frac{2\pi}{n} \end{pmatrix} \text{ and } \sigma : \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and call this the "natural" class functions. Define $\chi_{\text{nat}}(\rho) = 2 \cos \frac{2\pi}{n}$ and $\chi_{\text{nat}}(\sigma) = 0$.
For any $\varphi : G \to GL(V)$, χ_φ is a class function:

$$\chi_\varphi(gxg^{-1}) = \operatorname{tr} \varphi(gxg^{-1}) = \operatorname{tr} \varphi(g)\varphi(x)\varphi(g)^{-1} = \operatorname{tr} \varphi(x) = \chi_\varphi(x).$$

Fact. $\varphi_1 \sim \varphi_2 \Leftrightarrow \chi_{\varphi_1} = \chi_{\chi_2}$ $[\varphi_1(g) = A\, \varphi_2(g)\, A^{-1},\ \operatorname{tr}\ \varphi_1(g) = \operatorname{tr}\ A\, \varphi_2(g)\, A^{-1} = \operatorname{tr} \varphi_2(g)]$

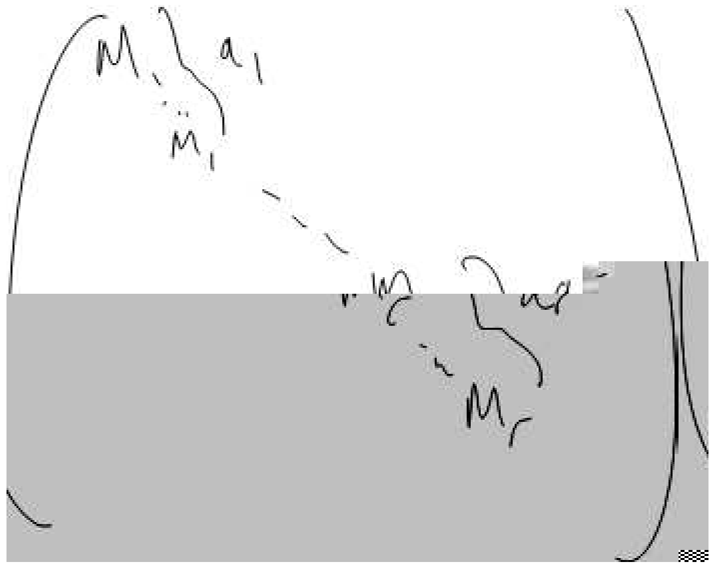
Consider $F\mathbb{C}$. Then χ_φ extends to $\mathbb{C}G$,

$$\chi_\varphi(\sum \alpha_g\, g) = \sum \alpha_g\, \chi_\varphi(g).$$

Recall the group algebra $\mathbb{C}G$ also acts on V. (if $f \in \mathbb{C}G$ can think of tr f) We write $\mathbb{C}G \cong M_{n_1}(\mathbb{C}) \times \ldots \times M_{n_r}(\mathbb{C})$ with $M_1, \ldots, M_r$ inequivalent. Every finite dimensional representation

$$M = a_1 M_1 \oplus \ldots \oplus a_r M_r \text{ with } a_i \geq 0.$$

If we write



then we easily see $\chi_M = a_1\chi_{M_1} + \ldots + a_r\chi_{M_r}$. Let's write $\chi_i = \chi_{M_i}$. On the other hand, every sum will be the character of a representation. Next, let

$$z_i = \left(0, 0, \ldots, 0, \overset{i\text{th}}{\overset{\downarrow}{1}}, 0, \ldots, 0 \right),$$

with $1 \in M_{n_i}(\mathbb{C})$ (id matrix). Then z_i's are linearly independent. Each character χ_j satisfies the following: $\chi_j(z_i) = 0$ for $i \neq j$ (b/c it acts by the zero matrix on M_j), and $\chi_j(z_j) = n_j$. Hence, the $\chi_1, \ldots, \chi_r$ are a dual basis to the independent set $z_1, \ldots, z_r$.

$$\chi_j(z_i) = n_j\,\delta_{ij} \implies \tfrac{x_j}{n_j}(z_i) = \delta_{ij} \implies x_j = n_j\,z_j \text{ (dual basis)}$$

$$\implies \chi_j\text{'s are linearly independent.}$$

Then if $M \cong B_1 M \oplus \ldots \oplus b_r M_r$, then $\chi_M = b_1\chi_1 + \ldots + b_r\chi_r = a_1\chi_1 + \ldots + a_r\chi_r$ naturally means $a_i = b_i$ (for $1 \leq i \leq r$).

Class functions

Let $\mathcal{O}_1, \ldots, \mathcal{O}_s$ be such that $f_{\mathcal{O}_1}(g) = \begin{cases} 1 & \text{if } s \in \mathcal{O}_1 \\ 0 & \text{elsewhere.} \end{cases}$ with $\chi_1, \ldots, \chi_r$ linearly independent class functions. We will show $r \leq s$ later.

Let θ, ψ be class functions. Define

$$(\theta, \psi) = \tfrac{1}{|G|}\sum_{g\in G}\theta(g)\,\overline{\psi(g)}.$$

Proposition. For $z_1, \ldots, z_r$ have $M_{n_1}(\mathbb{C}) \times \ldots \times M_{n_r}(\mathbb{C}) \cong \mathbb{C}G$. Then

$$z_i = \tfrac{\chi_i(1)}{|G|}\sum_{g\in G}\chi_i(g^{-1})g.$$

Proof. Write $z = z_i$ and $z = \sum \alpha_g\, g$. Recall

$$\chi_{\text{reg}}(g) = \begin{cases} 0 & \text{if } g \neq 1 \\ |G| & \text{if } g = 1. \end{cases}$$

Then Reg $= \bigoplus_1^r n_i M_i \implies \chi_{\text{reg}} = \sum_{i=1}^r n_i \chi_i$. We then claim

$$\chi_{\text{reg}}(zg^{-1}) = \alpha_g\,|G|.$$

Indeed,

$$\chi_{\text{reg}}(zg^{-1}) = \chi_{\text{reg}}(\sum \alpha_h hg^{-1}) = \sum \alpha_h\, \chi_{\text{reg}}(hg^{-1}) = |G|\alpha_g.$$

Next, if φ is the representation on M_j, then

$$\chi_j(zg^{-1}) = \text{tr}\ \varphi_j(zg^{-1}) = \text{tr}\ \varphi_j(z)\varphi_j(g^{-1}).$$

If $j \neq i$, then $\varphi_j(z_i) = 0$. Hence,

$$\varphi_j(zg^{-1}) = \begin{cases} 0 & j \neq i \\ \varphi_i(g^{-1}) & j = i, \end{cases}$$

so

$$\chi_j(zg^{-1}) = \chi_i(g^{-1})\delta_{ij}.$$

Finally,

$$\sum_{j=1}^r n_j\, \chi_j(g^{-1})\, \delta_{ij} = \alpha_g |G| \implies \alpha_g = \tfrac{1}{|G|} n_i\, \chi_i(g^{-1}).$$

However, $\chi_i(1) = \text{tr}\ \varphi_i(1) = n_i$. Then

$$\alpha_g = \tfrac{1}{|G|}\, \chi_i(1)\, \chi_i(g^{-1}),$$

so that

$$z_i = \tfrac{1}{|G|} \sum_i \chi_i(1)\, \chi_i(g^{-1})\, g. \ \square$$

Lecture 7 (January 28, 2009) -

Let $K_1, ..., K_s$ be distinct conjugacy classes, with $X_i = \sum_{g \in K_i} g$. Then $\{X_i\}$ form a basis for class functions $Z(\mathbb{C}G)$ (dim $Z(\mathbb{C}G) = r$, the number of simple modules).

Corollary. $r = s$

Proof. First, the X_i's are linearly independent. The next claim is that any element of $Z(\mathbb{C}G)$ is a linear combination of the X_i's. Let $\sum \alpha_g g \in Z(\mathbb{C}G)$. This means that for all $h \in G$, $h\alpha h^{-1} = \alpha$. Hence, $h(\sum \alpha_g g)h^{-1} = \alpha$ and $\sum \alpha_{h^{-1}gh} g = \sum \alpha_g g$. Hence, $\alpha_{h^{-1}gh} = \alpha_g\ \forall g \forall h$.

Take $z_1, ..., z_r$. Recall

$$z_i = \frac{\chi_i(1)}{|G|}\sum_{g\in G}\chi_i(g^{-1})g,$$

and notice

$$\begin{aligned} z_i\delta_{ij} = z_i z_j &= \frac{\chi_i(1)}{|G|}\frac{\chi_j(1)}{|G|}\sum_{g,h}\chi_i(g^{-1})\chi_j(h^{-1})gh \\ &= \frac{\chi_i(1)}{|G|}\frac{\chi_j(1)}{|G|}\sum_{g\in G}\left(\sum_{x\in G}\chi_i(xy^{-1})\chi_j(x^{-1})y\right) \end{aligned}$$

Hence $g^{-1} = xy^{-1}, g = yx^{-1}, h^{-1} = x^{-1}, h = x$. But

$$z_i\delta_{ij} = \delta_{ij}\sum_{y\in G}\frac{\chi_i(1)}{|G|}\chi_i(y^{-1})y.$$

Since the y are linearly independent elements, the corresponding coefficients must be equal:

$$\delta_{ij}\frac{\chi_i(1)}{|G|}\chi_i(y^{-1}) = \frac{\chi_i(1)}{|G|}\frac{\chi_j(1)}{|G|}\sum_{x\in G}\chi_i(xy^{-1})\chi_j(x^{-1}).$$

Hence,

$$\delta_{ij}\frac{|G|}{\chi_j(1)}\chi_i(y^{-1}) = \sum_{x\in G}\chi_i(xy^{-1})\delta_{ij}\frac{\chi_i(1)}{|G|}\chi_i(y^{-1})\chi_j(x^{-1}).$$

Put $y = 1$. Then

$$|G||\delta_{ij}\frac{\chi_i(1)}{\chi_j(1)} = \frac{1}{|G|}\chi_i(x)\chi_j(x^{-1}).$$

Notice in either case ($i = j$ and $i \neq j$), the left hand side is just δ_{ij}. Thus,

$$\delta_{ij} = \frac{1}{|G|}\sum_{x\in G}\chi_i(x)\,\chi_j(x^{-1}).$$

Lemma. $\chi_i(x^{-1}) = \overline{\chi_i(x)}$ for χ a character of any representation.

Proof. Look at $\varphi(x)$. Let φ be the representation associated to χ and look at $\varphi(x)$. Then $\varphi(x)^{|G|} = \varphi\big(x^{|G|}\big) = 1$. In fact, if $|x| = k$, then $\varphi(x)^k = \varphi(x^k) = \varphi(1) = 1$. Hence, $\varphi(x)$ satisfies the equation $X^k - 1 = 0$. Since the roots of $X^k - 1$ are distinct, the minimal polynomial of $\varphi(x)$ which divides $X^k - 1$ will have distinct roots and all will be roots of unity. Then $\varphi(x)$ must be diagonalizable,

$$\varphi(x) \sim \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \text{ with } |\lambda_i| = 1.$$

Then $\chi(x) = \sum \lambda_i$. Hence,

$$\varphi^{-1}(x) \sim \begin{pmatrix} \lambda_1^{-1} & & \\ & \ddots & \\ & & \lambda_n^{-1} \end{pmatrix} = \begin{pmatrix} \overline{\lambda_1} & & \\ & \ddots & \\ & & \overline{\lambda_n} \end{pmatrix},$$

so $\chi(x^{-1}) = \sum \overline{\lambda_i} = \overline{\sum \lambda_i} = \overline{\chi(x)}$. Thus,

$$\delta_{ij} = \tfrac{1}{|G|}\sum \chi_i(x)\,\chi_j(x^{-1}) = \tfrac{1}{|G|}\sum \chi_i(x)\overline{\chi_j(x)} = (\chi_i, \chi_j).$$

Theorem. *(First orthogonality relation) Let* $(\chi_1, \chi_j) = \delta_{ij}$*, e.g.* $M = \bigoplus_{i=n}^{r} a_i M_i$ *so that* $\chi_M = \sum a_i\chi_i$*. Then*

$$(\chi_M, \chi_M) = \left(\sum_i a_i\chi_i, \sum_j a_j\chi_j\right) = \sum_i\sum_j a_i a_j(\chi_i, \chi_j) = \sum_{i,j} a_i a_j \delta_{ij} = \sum_i a_i^2.$$

Corollary. *A representation* M *is irreducible if and only if* $(\chi_M, \chi_M) = 1$.

In other words, a regular representation

$$\rho_{\text{reg}} = \begin{cases} 0 & g \neq 1 \\ |G| & g = 1. \end{cases}$$

So $\langle \rho, \rho\rangle = \frac{1}{|G|}\sum\limits_{g\in G}\rho(g)\overline{\rho(g)} = \frac{1}{|G|}|G||G| = |G|$.

Corollary. *If* θ *is any class function,*

$$\theta(g) = \sum_i (\theta, \chi_i)\chi_i(g).$$

Theorem. (*Second orthogonality relation*)

$$\sum \chi_i(x)\,\overline{\chi_i(y)} = \begin{cases} |C_G(x)| & x \sim y \\ 0 & \textit{otherwise.} \end{cases}$$

(1) A complex number $\alpha \in \mathbb{C}$ is called an algebraic integer if

(i) it satisfies a monic equation with integral coefficients.

(ii) $\mathbb{Z}[\alpha]$ is a finitely generated $\mathbb{Z}$-module.

(2) If a rational number is an algebraic integer, then it must be an integer. [i.e., if $\sum \lambda_i$ is a sum of algebraic integers equal to p/q, then $q = 1$.]

Recall the collection of algebraic integers is a ring. Furthermore, if ψ is a character of some representation, then $\forall x \in G$, $\psi(x)$ is an algebraic integer. Then $\psi(x) = \sum \lambda_i$ (roots of unity).

Lecture 8 (January 30, 2009) - More Character Theory

Fact. If ψ is a character of a representation of G, then for all $x \in G$, $\psi(x)$ is an algebraic integer.

Notation. We will use the nonstandard notation $\overline{\mathbb{Z}}$ ot denote the integral closure of $\mathbb{Z}$ in $\overline{\mathbb{Q}}$ (algebraic integers).

Proposition. [19.1.3 Dummit and Foote] *Define complex-valued functions* w_i *on* G *by*

$$w_i(g) = \tfrac{|\textit{conjugacy class of } g|\,\chi_i(g)}{\chi_i(1)}.$$

Then the values $w_i(g) \in \overline{\mathbb{Z}}$ *(are algebraic integers).*

Proof. We claim that the values $w_i(g) \in \overline{\mathbb{Z}}$ (are algebraic integers). To see this, denote $K_1, \ldots, K_r$ to be the conjugacy classes. We'll prove

$$\sum_{g \in K_j} \varphi_i(g) = w_i(g)\, I$$

for any $g \in K_j$ (where φ_i is the representation for χ_i). Let $X = \sum_{g \in K_j} \varphi_i(g)$. Then X commutes with everything in the image of φ_i because

$$\varphi_i(h)\, X\, \varphi_i(h)^{-1} = \sum_{g \in K_j} \varphi_i(h)\, \varphi_i(g)\, \varphi_i(h)^{-1} = \sum_{g \in K_j} \varphi_i(hgh^{-1}) = \sum_{g \in K_j} \varphi_i(g) = X,$$

Hence, X commutes with everything, so $X \in Z(\varphi_i(G))$. That is, $X = \alpha I$ for some α. Notice

$$\chi_i(1)\alpha = \operatorname{tr} X = \sum_{g \in K_j} \operatorname{tr} \varphi_i(g) = \chi_i(g) \cdot |K_j|,$$

so that $\alpha = \frac{\chi_i(g)\, |K_j|}{\chi_i(1)}$. Now let $g \in K_s$. Define

$$a_{ijs} = \#\, \{(g_i, g_j) \mid g_i \in K_j,\ g_j \in K_j,\ g_i g_j = g\}.$$

First, notice a_{ijs} is an integer (it is counting something). Furthermore, it is independent of $g \in K_s$. Too see, first note if $g' \in K_s$, then $g' = xgx^{-1}$ for some x. Then

$$g = g_i g_j \Leftrightarrow xgx^{-1} = xg_ix^{-1} \cdot xg_jx^{-1}$$

(with $xg_ix^{-1} \in K_i$ and $xg_jx^{-1} \in K_j$). Since w_t is a class function, $w_t(K_1) = w_t(g)$ for any $g \in K_i$. So, what is $w_t(K_i)w_t(K_j)$? Well,

$$\begin{aligned} w_t(K_i)w_t(K_j)\, I &= \sum_{g_i \in K_i} \varphi_t(g_i) \cdot \sum_{g_j \in K_j} \varphi_t(g_j) = \sum_{\substack{g_i \in K_i \\ g_j \in K_j}} \varphi_t(g_i)\varphi_t(g_j) = \sum_{\substack{g_i \in K_i \\ g_j \in K_j}} \varphi_t(g_ig_j) \\ &= \sum_{s=1}^{r} \sum_{g \in K_s} \sum_{g_ig_j = g} \varphi_t(g) = \sum_{s=1}^{r} \sum_{g \in K_s} \varphi_t(g) \sum_{g_ig_j=g} 1 - \sum_{s=1}^{r} a_{ijs} \sum_{g \in K_s} \varphi_t(g) \\ &= \sum_{s=1}^{r} a_{ijs} w_t(K_j). \end{aligned}$$

Hence, the ring generated by $\mathbb{Z}[w_t(K_1), \ldots, w_t(K_r)]$ is in fact finitely generated as a $\mathbb{Z}$-module by $\omega_t(K_1), \ldots, w_t(K_r)$. Then $w_t(K_i)$ is an algebraic integer $\forall t, i$. $\square$

Corollary. For all i, $\chi_i(1)$ divides $|G|$.

Proof. With $g_j \in K_j$,

$$\begin{aligned} \frac{|G|}{\chi_i(1)} &= \frac{|G|}{\chi_i(1)} (\chi_i, \chi_i) = \frac{|G|}{\chi_i(1)} \frac{1}{|G|} \sum_g \chi_i(g)\overline{\chi_i(g)} = \frac{1}{\chi_i(1)} \sum_g \chi_i(g)\overline{\chi_i(g)} \\ &= \frac{1}{\chi_i(1)} \sum_{j=1}^{r} \sum_{g \in K_j} \chi_i(g)\overline{\chi_i(g)} = \frac{1}{\chi_i(1)} \sum_{j=1}^{r} |K_j| \chi_i(g_j)\overline{\chi_i(g_j)} \ \text{[for some } g_j \in K_j\text{]} \end{aligned}$$

$$= \sum_{j=1}^{r} w_i(g_j)\overline{\chi_i(g_j)}.$$

We saw that $w_i(g_j) \in \overline{\mathbb{Z}}$ and $\overline{\chi_i(g_j)}$ so the product must be in $\overline{\mathbb{Z}}$ and the sum of algebraic integers is also in $\overline{\mathbb{Z}}$. Hence, $|G|/\chi_i(1) \in \overline{\mathbb{Z}} \cap \mathbb{Q} = \mathbb{Z}$. □

Induced representations

If $H \leq G$ and ρ is a representation of G on a vector space V, we can restrict $\rho|_H : H \to \mathrm{GL}(V)$. This gives a functor $\mathcal{R}(G) \to \mathcal{R}(H)$. If φ is a representation of H and we have W a $\mathbb{C}H$-module, $V = W \otimes_{\mathbb{C}H} \mathbb{C}G$. Given (φ, W) a representation of H, we can define $\mathrm{Ind}_H^G(\varphi) = \{f : G \to W \mid f(hg) = \varphi(h)f(g)\}$. If we pick a set of representatives for G/H, then any such f is determined on G/H. Then in fact $\mathrm{Ind}_H^G(\varphi)$ is a G-representation, and $\gamma \in G$, $(\gamma f)(g) = f(g\gamma)$.

Fröbenius Reciprocity

Given a character χ of H, there is a representation φ, and we consider $\mathrm{Ind}_H^G \varphi$, which gives us $\mathrm{Ind}_H^G \chi$. Fröbenius reciprocity says the two inner products

$$(\text{restrict } \psi, \chi)_H = (\psi, \text{induced } \chi)_G.$$

Exercise. $\mathrm{Hom}_{\mathbb{C}H}(\text{res } \pi, \varphi) \cong \mathrm{Hom}_{\mathbb{C}G}(\pi, \text{ind } \varphi)$.

Lecture 9 (February 2, 2009) - Representation of Compact Groups

We are now following the book by Varadarajan.

Definition. *A topological group G is a topological space equipped with two continuous maps, $m : G \times G \to G$, and $i : G \to G$, so that the operation $x \cdot y := m(x, y)$ turns G into a group with $x^{-1} = i(x)$.*

Example. $G = \mathbb{R}$, with $m(x, y) = x + y$ and $i(x) = -x$.

Example. $G = \mathbb{R}^\times$ with $m(x, y) = xy, i(x) = x^{-1}$.

Example. $G = S^1 = \{z \in \mathbb{C}^\times \mid |z| = 1\}$ with $m(z, w) = zw$ and $i(z) = z^{-1}$.

The groups $\mathbb{R}$ and $\mathbb{R}^\times$ are not compact, and S^1 is compact. Furthermore,

$$SO(3) = \{g \in GL_3(\mathbb{R}) \mid g^T g = 1\}$$

is compact. (**Exercise.** Check this!)

However, $\mathbb{R}$ and $\mathbb{R}^\times$ *are* locally compact, and we can indeed define locally compact groups to be exactly as above but a locally compact space (this does not see the group structure at all).

These topological roups satisfy the second axiom of countability (i.e., every point has a countable basis of open neighborhoods).

Theorem. (Haar, Von Neumann) *Given a locally compact topological group G, there is a unique (up to scalar multiplication) Borel measure $d\mu_\ell$ such that for all sets A open, and all $x \in G, \mu_\ell(xA) = \mu_\ell(A)$. Similarly, there is a unique measure $d\mu_r$ such that $\mu_r(Ax) = \mu_r(A)$. (In general, $\mu_r \neq \mu_\ell$)*

Example. Over $\mathbb{R}$, this just means $\mu_{\text{Leb}}(x + A) = \mu_{\text{Leb}}(A)$.

Proof. (of Theorem) If C is a compact set, then given $\varepsilon > 0$, $\exists U$ open with $\overline{U}$ compact and $C \subseteq U$ such that $\mu(U \setminus C) < \varepsilon$. If V is open with $\overline{V}$ compact, then $\exists K \subseteq V$ with K compact, such that $\mu(V \setminus K) < \varepsilon$.

[A reference for this is Kelley-Srinivas(an?).] Suppose you have a group G acting on a topological space X and suppose X is locally compact. Then X carries a G-invariant measure if the action satisfies a certain "topological" property.

The uniqueness statement (in the above theorem) says if μ_1, μ_2 are two regular, non-trivial Borel measures such that $\mu_i(xA) = \mu_i(A)$ for $i = 1, 2$ then $\forall x, A$, there is a constant $c > 0$ such that $\mu_1 = c\mu_2$.

Now let μ_ℓ be a left invariant measure. Define for $x \in G$, $\mu_\ell^x(A) = \mu_\ell(Ax)$. Then $\forall y \in G$, $\mu_\ell^x(yA) = \mu_\ell(yAx) = \mu_\ell(Ax) = \mu_\ell(A)$. There exists a scalar $\delta(x)$ such that $\mu_\ell^x = \delta(x)\mu_\ell$, with $\delta(x) > 0$ and $\delta(xy) = \delta(x)\delta(y)$. Since μ_ℓ is regular, one can show that $\delta(x)$ is continuous.

In other words, let G be compact. Then since $\delta(x)$ is continuous, $c < \delta(x) < C$. Let $x \in G$ be such that $\delta(x) \neq 1$. Then if $\delta(x) > 1$, $\delta(x^N) \to +\infty$ unless for some $N > 0$, $x^N = 1$. Similarly, if $\delta(x) < 1$, apply the same to $\delta(x^{-1}) > 1$. If for some $x \in G$, $\delta(x) > 1$, then $\delta(x^N) \to +\infty$, which contradicts the fact that δ is bounded, unless $x^N = 1$ for some N (it becomes periodic), in which case $\delta(x)^N = 1$ so that $\delta(x) = 1$ which contradicts $\delta(x) > 1$.

If G is compact, $\delta(x) = 1$, $\mu_\ell^x = \delta(x)\,\mu_\ell = \mu_\ell$, $\mu_\ell^x(A) = \mu_\ell(A)$ implies $\mu_\ell(Ax) = \mu_\ell(A)$. $\square$ **Corollary.** *μ_ℓ is also right-invariant.*

If G is compact, we write $\mu = \mu_r = \mu_\ell$. Since $\mu(G) < +\infty$, we noramlize our measure so that $\mu(G) = 1$. We can normalize our measure

$$\mu'(A) = \tfrac{1}{\mu(G)}\mu(A).$$

Locally compact groups with $\mu_r = \mu_\ell$ are called unimodular, e.g., all compact groups, $GL_n(\mathbb{R})$, $SL_n(\mathbb{R}), SO_n(\mathbb{R})$, etc.

Next time, we will show that $\left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \subseteq SL_2(\mathbb{R})$ is not unimodular.

Remark. Let F be any non-discrete locally compact topological field. Weil used this to classify $\delta_\times$ multiplicatively for all such F structures.

Lecture 11 (February 6, 2009) -

Assume the vector spaces we will be working on are all $\mathbb{C}$-vector spaces. If G is a compact group and $\pi : G \to GL(V)$ is a finite dimensional representation of G, we set $\theta_\pi(g) = \text{tr } \pi(g)$. Furthermore, $\theta_\pi : G \to \mathbb{C}$ satisfies $\pi_\pi(yxy^{-1}) = \theta_\pi(x)$ and $\theta_\pi(1) = \dim V = \text{id}(\pi)$ (convince yourself of this!).

Exercise. Show that $\theta_\pi \cong \theta_{\pi'}$ if and only if $\pi \cong \pi'$. In other words, θ_π is a function of only the equivalence class of π (denote this by ω, so we can talk about θ_ω).

Exercise. Show that $\theta_{\pi\oplus\pi'} = \theta_\pi + \theta_{\pi'}$.

If χ, χ' are two functions on G, then we defined

$$(\chi, \chi') := \textstyle\int_G \chi(g)\overline{\chi'(g)}\, dg$$

to be a measure that is normalized so that vol $G = 1$.

If ω, ω' are two equivalence classes of irreducible unitary representations of G, then $(\theta_\omega, \theta_{\omega'}) = \delta_{\omega\omega'}$.

Definition. We let $\widehat{G}$ denote the collection of classes of irreducible unitary representations.

If $\omega = \bigoplus_{1\le i\le r} m_i\omega_i$ with $\omega_1, ..., \omega_r \in \widehat{G}$ distinct, then

$$(\theta_\omega, \theta_\omega) = \big(\theta_{\oplus m_i\omega_i}, \theta_{\oplus m_j\omega_j}\big) = \sum_{i,j} m_i m_j \big(\theta_{\omega_i}, \theta_{w_j}\big) = \sum m_i m_j\, \delta_{ij} = \sum m_i^2.$$

Example. If we have an $n \times n$ matrix, then this will look like (a_{ij}). What will $(a_{ij})e_k$ look like (where e_k is the column vector with 1 only in the kth spot)? Well, it will be $(Ae_k, e_l) = a_{lk}$ (we call these matrix coefficients). It essentially selects the kth column.

In general, if $\pi : G \to GL(V)$ is a unitary representation, a matrix coefficient of π is a function of the form

$$\mathrm{f}_{vv'} : x \mapsto (\pi(x)v, v').$$

This follows from the identity

$$\textstyle\int_G (\pi(x)v_1, v_1')(\pi'(x)v_2, v_2')dx = d(\pi)^{-1}(v_1, v_2)(v_1', v_2')\delta_{[\pi][\pi']},$$

where $d(\pi)$ is the dimension of the representation π. This is true as follows. First, think of $(\pi(x)v_1, v_1)$ as a function from $V \times V \to \mathbb{C}$. However, first let's consider (v_1, v_2). This is a function

$$(v_1, v_2) \overset{F}{\mapsto} \textstyle\int_G (\pi(x)v_1, v_1')(\pi'(x)v_2, v_2')\, dx.$$

We claim $F(\pi(y)v_1, \pi'(y)v_2) = F(v_1, v_2)$. [Holly notes that v_1, v_2 are fixed.] The identity then follows because

$$F_{v_1', v_2'}(\pi(y)v_1, \pi'(y)v_2) = \textstyle\int_G (\pi(x)\pi(y)v_1, v_1')(\pi'(x)\pi'(y)v_2, v_2')\, dx =$$
$$\textstyle\int_G (\pi(xy), v_1, v_1')(\pi'(xy)v_2, v_2)\, dx = \int_G (\pi(x)v_1, v_1')(\pi'(x)v_2, v_2')\, dx = F_{v_1', v_2'}(v_1, v_2),$$

where the penultimate equality follows from the fact this is a Haar measure, so that we can collapse accordingly. In other words, we have $V \times V \to \mathbb{C}$ inducing a map $V \to V^*$.

For fixed v_1', v_2', define a map $\psi_{v_1',v_2'} : V \to V^*$ by $\psi_{v_1',v_2'}(v_1)(v_2) = F_{v_1',v_2'}(v_1, v_2)$ (**note**: Be careful with complex conjugation!). Notice $\psi_{v_1',v_2'}$ is G-equivariant with G acting by π on V and $(\pi')^*$ on V^*. Since representations are unitary, $V \cong V^*$ and $\pi' \cong (\pi')^*$ identified by the Hermitian product. Hence,

$$\psi_{v_1',v_2'} \in \mathrm{Hom}(V_\pi, V_{\pi'}),$$

which is a scalar by Schur's Lemma (possibly zero). That is, if $[\pi] \neq [\pi']$, then $\psi_{v_1',v_2'} \equiv 0$.

On the other hand, if you have $[\pi] = [\pi']$, then there is a unique Hermitian pairing on V invariant under π. This means $F_{v_1',v_2'}(v_1, v_2) = C \cdot (v_1, v_2)$, where C is some constant. Now, fix v_1, v_2. We claim that $(v_1', v_2') \mapsto F_{v_1',v_2'}(v_1, v_2)$ is also G-equivariant. This is because Haar measure is right invariant. Indeed,

$$\begin{aligned} F_{\pi(y)v_1',\pi(y)v_2'}(v_1, v_2) &= \int_G (\pi(x)v_1, \pi(y)v_1')\overline{(\pi(x)v_2, \pi(y)v_2')}\, dx \\ &= \int_G (\pi(y^{-1})\pi(x)v_1, \pi(y^{-1})\pi(y)v_1')\overline{(\pi(y^{-1})\pi(x)v_2, \pi(y^{-1})\pi(y)v_2')}\, dx, \end{aligned}$$

because Haar measure is invariant. Continuing,

$$= \int_G (\pi(y^{-1}x)v_1, v_1')\overline{(\pi(y^{-1}x)v_2', v_2')} dx.$$

Now do a left change of variables. Hence, $F_{v_1',v_2'}(v_1, v_2) = C \cdot (v_1', v_2')(v_1, v_2)$. Then we can just compute the constant which is independent of v_1, v_2, v_1', v_2'. We have

$$\int_G (\pi(x)v_1, v_2)\overline{(\pi(x)v_1', v_2')}\, dx.$$

Aaaand....why is this constant? Ramin will figure it out by Monday!

If (e_i) (with $1 \leq i \leq d(\pi)$) is an orthonormal basis for the space of π, then

$$f_{ij,\omega} : x \mapsto d(w)^{1/2}(\pi(x)e_i, e_j)$$

will be an orthonormal basis for the space $A(\omega) :=$ linear span of all matrix coefficients of π.

- $A(\omega) \perp A(\omega')$

- dim $A(\omega) < \infty$

- $A(\omega) \subseteq L^2(G)$.

By the Peter-Weyl Theorem, $L^2(G) = \overline{\bigoplus_{w\in\widehat{G}} A(\omega)}$ (the completion).

Lecture 12 (January 9, 2009) -

Essentially, we want to show that any irreducible invariant subspace of $L^2(G)$ is finite dimensional, and conversely (the completeness theorem).

You have an action G on $L^2(G)$, written

$$(\rho(x)f)(y) = f(yx).$$

Suppose we have an operator $K : L^2(G) \to L^2(G)$ with an "eigenspace" V. Then there is λ such that $f \in V$ gives $Kf = \lambda f$. Then we claim V is also invariant under ρ if K and ρ commute (that is, $\rho(x) \circ K = K \circ \rho(x)$). Indeed, then if $f \in V$,

$$\rho(x)(Kf) = K \circ (\rho(x)f).$$

Then $\rho(x)(\lambda f) = K(\rho(x)f)$ yields $\lambda(\rho(x)f) = K(\rho(x)f)$, so that $\rho(x)f \in V$.

In this way, the goal of the Peter-Weyl Theorem is to show there are many such operators.

Compact self-adjoint operators (Review of functional analysis)

This material is found in the appendix of Varadarajan's book. Let V be a Banach space. Suppose $A : D(A) \to V$ is a linear operator, with $D(A) \subseteq V$ a dense subspace. A is a closed *closed operator* if its graph is closed in $V \oplus V$. The resolvant set

$$\rho(A) = \{\lambda \in \mathbb{C} : (\lambda I - A)^{-1} \text{ exists as a bounded operator}\}.$$

In other words, $\lambda I - A$ is a bijection of $D(A)$ with dense image, and $(\lambda I - A)^{-1}$ extends to a bounded operator on V. Then we call the spectrum

$$\sigma(A) = \mathbb{C} \backslash \rho(A).$$

This is in fact a closed set. If A is a bounded operator, it's non-empty. If dim $V < \infty$, we simply get eigenvalues.

Now, let H be a Hilbert space. Then we have an inner product for which the Banach space norm is given by $||x|| = (x,x)^{1/2}$. A linear operator $A : H \to H$ is self-adjoint if $(Au, v) = (u, Av)$ for all $u, v \in H$.

Spectral theory of self-adjoint operators

Let X be a space and let B be a σ-algebra of subsets of X. Define $\int_R \lambda\, P_\lambda \, d\mu(\lambda)$ such that

(a) $P(\emptyset) = \emptyset$ (b) $P(X) = 1$ (c) if $E = \bigcup_n E_n$, then $P(E)v = \sum_n P(E_n)v$.

Then we have spectral integrals

$$A(f) = \int_X f(x)\, dP(x).$$

Now we think of the $A(f)$'s as operators on H.

If X is second countable, then there is a smallest set C s.t. $P(C) = 1$.

-supp of $P = 0\,(A)$

- $P(E)$ is a spectral projection

- Images of $P(E)$ are spectral subspaces

- $\lambda_0 \in \mathbb{R}$ is an eigenvalue if and only if $P(\{\lambda_0\}) \neq 0$.

Definition. *An operator A is called compact if it maps sets with bounded norm to sets with compact closure.*

Notice K$(x,y) = \overline{K(y,x)}$. This tells us "what kind of groups" we should be looking for.

Lecture 13 (February 11, 2009) -

Let's consider the right regular action $(\rho(x)f)(y) = f(yx)$. The idea is to look for compact self-adjoint operators that commute with $\rho(x)$ for all $x \in G$.

Eigenspaces will be finite-dimensional. Furthermore, these eigenspaces are invariant under ρ. Take a kernel function $K(x,y)$ such that (1) $K(x,y) = \overline{K(y,x)}$. Furthermore, (2) $K(xg, yg) = K(x,y)$. The operator given by $A_K f(x) = \int_G K(x,y) f(y)\, d\varphi(y)$. Then by (1), A_K is self-adjoint, and by (2), A_K commutes with $\rho(x)$ for all x. Since G is compact, Supp $K(x,y)$ will be compact. Thus, A_K is a compact operator. Now,

$$K(x,y) = K(xy^{-1}, yy^{-1}) = K(xy^{-1}, 1).$$

Hence, there is a function a such that $K(x,y) = a(xy^{-1})$. In order to get $K(x,y) = \overline{K(y,x)}$, we need $a(xy^{-1}) = \overline{a(yx^{-1})}$. Then set $xy^{-1} = z$ so $yx^{-1} = z^{-1}$, and then we can just try to get $a(z) = \overline{a(z^{-1})}$. However, for any continuous function a on G satisfying $a(x) = \overline{a(x^{-1})}$ we have an associated integral operator

$$A_a f(x) = \int_G a(xy^{-1}) f(y)\, dy$$

that is compact, self-adjoint, and commutes with ρ. Each eigenspace is finite-dimensional, except possibly the kernel of A_a. Every $f \in L^2(G)$ belongs to a stable finite dimensional representation of G unless $f \in \ker A_a \; \forall a$.

Lemma. If for all a as above, $A_a f = 0$, then $f = 0$.

Proof. Notice $\int_G a(xy^{-1}) f(y)\, d\mu(y) = 0$. This is the convolution $a * f(x) = 0$. Now, there exists a sequence a_n with $n \geq 1$ of functions such that

(1) a_n is real, continuous, ≥ 0.

(2) $\int a_n = 1$.

(3) $a_n(x) = a_n(x^{-1})$.

(4) supp $a_n \to 0$.

This is called a delta/Dirac sequence. Construct a sequence satisfying (1), (2), and (4), call it b_n. Let $a_n(x) = \frac{1}{2}(b_n(x) + b_n(x^{-1}))$. Hence,

$$\int_G a_n(x)\, d\mu(x) = \tfrac{1}{2}\int_G b_n(x)\, d\mu(x) + \tfrac{1}{2}\int_G b_n(x^{-1})\, d\mu(x).$$

We claim

$$\int_G f(x^{-1})\, d\mu(x) = \int_G f(x)\, d\mu(x)$$

for any integrable f. Define a measure

$$\mu'(A) = \mu(\{a^{-1} \mid a \in A\}).$$

We claim μ' is left and right invariant.

$$\mu'(rA) = \mu\big(\{(ra)^{-1} \mid a \in A\}\big) = \mu(\{a^{-1}v^{-1} \mid a \in A\}) = \mu(\{a^{-1} \mid a \in A\}r^{-1}) = \mu(\{a^{-1} \mid a \in A\}) = \mu'(A)$$

which implies $\mu' = c\mu$ so $\mu'(G) = c\mu(G)$ implies $c = 1$ and thus $\mu' = \mu$. Now,

$$\int_G f(x^{-1})\, d\mu(x) = \int_G f(x)\, d\mu(x^{-1}) = \int_G f(x)\, d\mu'(x) = \int_G f(x)\, d\mu(x).$$

Hence,

$$\int a_n = \tfrac{1}{2}(\int b_n + \int b_n) = 1.$$

Notice for all n, we are assuming $0 = a_n * f \to f$ as $n \to \infty$. Thus $f = 0$ in $L^2(G)$. $\square$

Theorem. (Peter-Weyl) The irregular representations of G are all finitely-dimensional and they separate the points of G. The irreducible characters form a basis for $L^2(G)^{\text{inv}}$ and $L^2(G)$ is the orthogonal direct sum of matrix coefficients.

$$L^2(G)^{\text{inv}} = \bigoplus_{\omega \in \widehat{G}} \mathbb{C}\,\theta_\omega.$$

$$L^2(G) = \bigoplus_{\omega \in \widehat{G}} A(\omega) \text{ (matrix coeffs of } \omega\text{)}.$$

For any $\pi \in \omega \in \widehat{G}$, $\{e_i\}_{1 \le i \le d(\omega)}$ an orthonormal basis for the space of π, let

$$v_{ij,\omega}(x) = d(\omega)^{1/2}(\pi(x)e_j, e_i).$$

Then $(v_{ij,w})_{1 \le i,j \le d(\omega), \omega \in \widehat{G}}$ is an orthonormal basis for $L^2(G)$. $\square$

Lecture 15 (February 16, 2009) -

Lemma. *If $\{e_i\}$ is an orthonormal basis for a vector space V and $A, B : V \to V$, then*

$$\sum_{i,j} (Ae_i, e_j)(Be_j, e_i) = \text{tr}(AB).$$

Proof. This is obvious. Write the matrices of $A = (a_{ij})$ and $B = (b_{ij})$ in terms of the basis $\{e_i\}$. Then

$$(Ae_i, e_j) = a_{ij} \text{ and } (Be_j, e_i) = b_{ji},$$

so that

$$\sum_{i,j} (Ae_i, e_j)(Be_j, e_i) = \sum_{i,j} a_{ij}\, b_{ji} = \sum_i \left(\sum_j a_{ij}\, b_{ji}\right) = \sum_i (AB)_{ii} = \text{tr}(AB). \ \square$$

Last time, we got the identity

$$\sum_\omega \sum_{i,j} |(f, v_{i,j,\omega})|^2 = d(\omega)\, \text{tr}\Big(\pi\Big(\overline{f}^T\Big)\pi\big(\overline{f}\big)\Big).$$

We claim $\text{tr}\Big(\pi\Big(\overline{f}^T\Big)\pi\big(\overline{f}\big)\Big)$ is equal to $\text{tr}(\pi(f^T * f))$, where $f(x)^T = \overline{f(x^{-1})}$. Let f and g be two functions. Then let's see what $\pi(f)\pi((g)v)$ is. Recall

$$\pi(g)v = \int_G g(x)\, \pi(x)\, v\, dx.$$

Then

$$\pi(f)\pi(g)v = \int_G f(y)\pi(y)\left(\int_G g(x)\pi(x)v\,dx\right)dy = \int_G\int_G f(y)\,g(x)\,\pi(y)\pi(x)\,v\,dx\,dy = \int_G\int_G f(y)g(x)\,\pi(yx)\,v\,dx\,dy = [\text{send}\,y\text{ to }yx^{-1}]\int_G\int_G f(yx^{-1})g(x)\,\pi(y)\,v\,dx\,dy = \int_G\left(\int_G f(y^{-1}x)\,g(x)\,dx\right)\pi(y)v\,dy = \int_G (f*g)(y)\pi(y)\,v\,dy,$$

with

$$(f*g)(y) = \int_G f(yx^{-1})g(x)dx.$$

Thus,

$$\pi(f)\pi(g) = \pi(f*g)$$

which implies

$$\text{tr}(\pi(f)\,\pi(g)) = \text{tr}(\pi(f*g)) = \theta_w(f*g).$$

Then we get

$$\text{tr}\Big(\pi\Big(\overline{f}^T\Big)\pi\big(\overline{f}\big)\Big) = \text{tr}\Big(\pi\Big(\overline{f}^T * \overline{f}\Big)\Big).$$

We claim this equals tr $\pi(f^T * f)$. Well, $\overline{f}^T(x) = \overline{\overline{f}(x^{-1})} = f(x^{-1})$. We verified last time that

$$\pi(f^T) = \pi(f)^{\text{adj}}.$$

We need to show the trace is a real number, since $\text{tr}\big(AA^{\text{adj}}\big) \in \mathbb{R}$. Hence, it easily follows

$$||f||^2 = \sum_\omega \sum_{i,j} |(f, v_{i,j,\omega})| = \sum_{\omega\in\widehat{G}} d(\omega)\,\theta_\omega(f^T * f).$$

Lemma. $||f||^2 = f^T * f(1)$.

Proof. Well,

$$(f^T * f)(1) = \int_G f^T(1\cdot x^{-1})f(x)\,dx = \int_G \overline{f}(x)\,f(x)\,dx = ||f||^2.$$

Hence,

$$(f^T * f)(1) = \sum_{w\in\widehat{G}} d(\omega)\,\theta_\omega(f^T * f).\ \square$$

So if we let $h = f^T * f$ we get the so-called **positive functions**. In other words, a positive function satisfies

$$h(1) = \sum_{\omega\in\widehat{G}} d(\omega)\,\theta_\omega(h),$$

and in fact $\theta_\omega(h)$ will be $c_n e^{in\theta}$ for a constant c_n, reminding us of Fourier expansion (and this is where this originates from).

Lecture 17 (February 23, 2009) -

Last time we talked about tangent spaces. Given a differentiable manifold M and a point $x \in M$, we defined the tangent space $T_x M$ to be the collection of maps

$$X : C^\infty(M) \to \mathbb{R}$$

the behave like derivative, i.e., $X(fg) = f(x)X(g) + g(x)X(f)$, and they are also local at x in the sense that if $f = g$ in a neighborhood of x, then $X(f) = X(g)$.

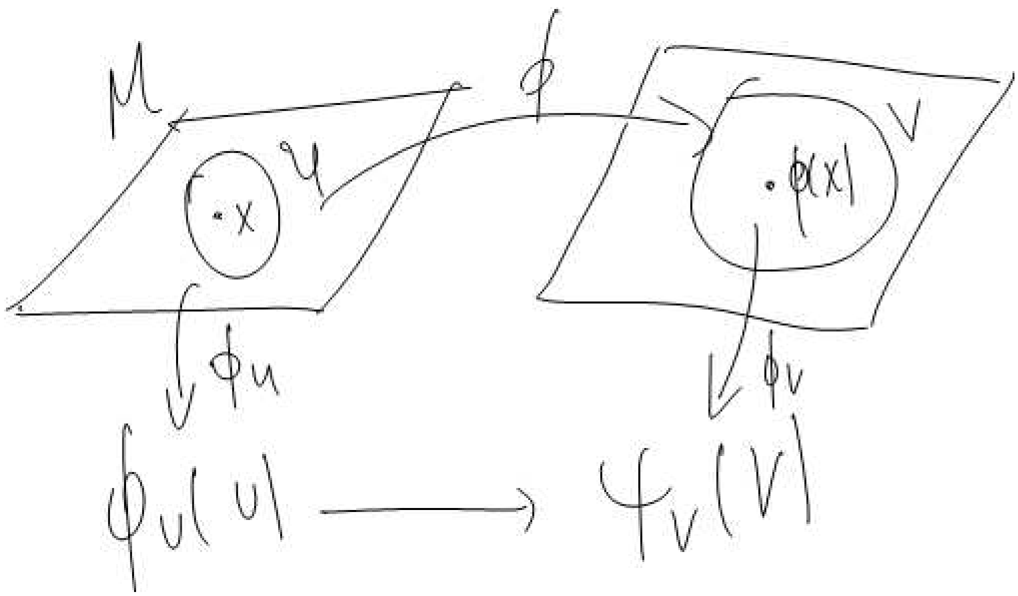
If $x_1, ..., x_n$ is a local coordinate system at x with a distinguished set of tangent vectors $X_1, ..., X_n$, then we can always write the derivative

$$X(f) = \sum_{k=1}^n a_k X_k(f)$$

with $X_k = \frac{\partial}{\partial x_k}$ in the Euclidean setting. We can then say $\dim T_x M = n$.

Differentials of smooth maps between manifolds

If M, N are manifolds, a function $\phi : M \to N$ is smooth if for all $x \in M$, there is a neighborhood



such that

$$\psi_V \circ \phi \circ \phi_{U_n}^{-1} : \phi_U(U) \to \phi_V(V)$$

with $\phi_U(U) \subseteq \mathbb{R}^n$ and $\phi_V(V) \subseteq \mathbb{R}^m$.

We can now define a differential of a smooth map. Given a ϕ smooth as above, we can define

$$d_x\phi : T_x M \to T_{\phi(x)} N$$

to be the function $d_x\phi(X)(g) = X(g \circ \phi)$ for $g \in C^\infty(N)$. This is clearly linear.

Example. Let $\phi : \mathbb{R}^n \to \mathbb{R}^m$ be a collection of functions $(\phi_1, ..., \phi_m)$. Then

$$d_x\phi : \mathbb{R}^n \to \mathbb{R}^m$$

so this is simply a generalization of a Jacobian.

Chain rule

Consider $M \xrightarrow{\Phi} N \xrightarrow{\Psi} P$ with $x \in M$. Then $\Psi \circ \Phi : M \to P$ and

$$d_x(\Psi \circ \Phi) = d_{\Phi(x)}\psi \circ d_x\Phi.$$

Then

$$T_xM \xrightarrow{d_x\Phi} T_{\Phi(x)}N \xrightarrow{d_{\Phi(x)}\psi} T_{\psi(\Phi(x))}P$$

with $d_x(\Psi \circ \Phi) : T_xM \to T_{\psi(\Phi(x))}P$.

Vector fields

A vector field is a map X that takes to each point $x \in M$ a tangent vector $X \in TM$ "in a smooth fashion." For each $x \in M$, $X_x \in T_xM$ and as such, a map $C^\infty(M) \to \mathbb{R}$. So, do the following. Fix an $f \in C^\infty(M)$. Then we get a function $M \to \mathbb{R}$ given by $x \mapsto X_x(f)$. A vector field X is smooth if for all $f \in C^\infty(M)$, $x \mapsto X_n(f)$ ($M \to \mathbb{R}$) is smooth.

Given any coordinate chart U, $x_1, ..., x_n$, then a vector field can be described as

$$X_x(f) = \sum a_k \frac{\partial f}{\partial x_k}$$

as before (see blue above). We require $a_k(x)$ to be a smooth function.

For any $f \in C^\infty(M)$, we associate with it another function

$$X(f) \in C^\infty(M) \text{ given by } x \mapsto X_f(f).$$

So we can think of a vector field as an R-linear map $X : C^\infty(M) \to C^\infty(N)$ that satisfies

$$X(fg) = fX(g) + gX(f).$$

That is, a vector field is a differential operator on $C^\infty(M)$.

The tangent bundle TM of M is the union $\bigcup_{x \in M} T_xM$ topologized in such a way that for all smooth vector fields X, the map $x \mapsto X_x$ is continuous (and indeed smooth). In other words, we take the "simplest" / "nicest" topology. We want open sets to be all subsets such that if we have any of these maps $x \mapsto X_x$ and if we pull them back onto the manifold, we want those to be open.

If X, Y are vector fields and XY is not one, then

$$\begin{gathered} XY(fg) = X(Y(fg)) = X(fY(g) + gY(f)) = \\ X(f)Y(g) + fX(Y(g)) + X(g)Y(f) + gX(Y(f)) = \\ fXY(g) + gXY(f) + X(f)Y(g) + X(g)Y(f) \end{gathered}$$

so if we consider

$$\begin{gathered} (XY - YX)(fg) = fXY(g) + gXY(f) + X(f)Y(g) + X(g)Y(f) - \\ fYX(g) - gYX(f) - X(f)Y(g) - X(g)Y(f) = \\ f(XY - YX)(g) + g(XY - YX)(f). \end{gathered}$$

So if we define

$$[X,Y] := XY - YX$$

then $[X,Y]$ is a vector field. This vector field satisfies the condition

$$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0,$$

the Jacobi identity. Furthermore, $[X,Y] = -[Y,X]$.

Definition. *Let L be a real vector space. If L is equipped with a bilinear map*

$$[\,\cdot\,,\,\cdot\,] : L \times L \to L$$

satisfying the Jacobi identity and is anti-symmetric, then it is called a Lie algebra.

Lecture 18 (February 25, 2009) -

Lie groups

A Lie group is a smooth manifold that's also a group. That is, we have $\mathcal{M}$ with maps $* : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ with $i : \mathcal{M} \to \mathcal{M}$ such that both are smooth, and then $(\mathcal{M}, *, i)$.

Examples

- $\mathbb{R},\ +\ ,\ -$
- $\mathbb{R}^\times,\ \times\ , x \mapsto \frac{1}{x}$
- $GL_2(\mathbb{R}) \subseteq \mathbb{R}^4$ (a (Zariski) open set in $\mathbb{R}^4$) so $GL_2(\mathbb{R})$ will be a manifold

Let's look at the last one in detail. What is matrix multiplication?

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

Inversion is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \tfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \text{ smooth.}$$

Furthermore, $SL_2(\mathbb{R})$ is also a Lie group.

Lie algebra

From here on, we will let G mean a Lie group. If G is a Lie group, fix $g \in G$. Then there exists a map

$$L_g : G \to G$$

given by $h \mapsto gh$. This will be smooth. As such, it makes sense to talk about the derivative of this function. Thus, we have a map $d_h L_g : T_h G \to T_{gh} G$.

We say a vector field X on G is called invariant if $d_h L_g(X_h) = X_{gh}$ for all $g, h \in G$. On $\mathbb{R}^2$, for example, we can look at the constant vector field.

In general, if $u \in T_e G$, we define a vector field X^u on G by

$$X^u_g := d_e L_g(u),$$

where e is the identity. We claim X^u is invariant. Hence, we need to check that

$$d_h L_{gh^{-1}}(X_h^u) = X_g^u.$$

Now, we need to check

$$d_h L_{gh^{-1}}\, d_e L_h(u) = d_e L_g(u), \text{ that is,}$$

$$G \xrightarrow{L_h} G \xrightarrow{L_{gh^{-1}}} G$$

with the first G mapping to the last G by L_g (so $e \mapsto h \mapsto g$). But, checking that this is true is simply the chain rule.

Hence, there is a one-to-one correspondence betweeen invariant vector fields and $T_e G$.

Next, if X and Y are invariant vector fields, then so is $[X, Y]$. (check this explicitly just for "fun" :)) Let ψ be the map from $T_e G$ to invariant vector fields.

Definition. The Lie algebra of a Lie group G is $\mathfrak{g} = T_e G$ with Lie bracket

$$[u, v] = \psi^{-1}([X^u, X^v]),$$

or what is the same, $[X^u, X^v]_e$.

Example. Let $G = (\mathbb{R}, +)$. Then $L_g : x \mapsto x + g$. The derivative $d_e L_g$: identity. What is an invariant vector field? The tangent space on 0 is $\mathbb{R}$. So, it will be

$$X_h^u = u$$

with $u, h \in \mathbb{R}$. Now, let's look at what brackets are. If X and Y are two vector fields, and $f : \mathbb{R} \to \mathbb{R}$ is a smooth function, then

$$X_h f = f'(h) X_h$$

"the value of X on f at the point h will be the derivative of f at h times X at h" (notice $X_h \in \mathbb{R}$). So, now,

$$\begin{gathered}(XY - YX)f = XYf - YXf = X(Yf) - Y(Xf) = \\ X(Y \cdot f') - Y(X \cdot f') = X(Y)f' + Y \cdot X(f') - Y(X)f' - X \cdot Y(f') = \\ X(Y)f' + Y \cdot X \cdot f'' - Y(X)f' - X \cdot Y \cdot f'' = X(Y)f' - Y(X)f' = \\ (X(Y) - Y(X))f' = (X \cdot Y' - Y \cdot X')f'.\end{gathered}$$

For invariant ("constant") vector fields, their derivatives are going to be zero. Hence,

$$[X, Y] = 0$$

for invariant vector fields. Hence, $\mathfrak{g}_{\mathbb{R}} = \{\mathbb{R}, [X, Y] = 0\}$.

Lecture 19 (February 27, 2009) -

Exponential map

If X is a vector field on any manifold and $\gamma : (a, b) \to \mathcal{M}$ is a smooth curve, then γ is called an integral curve for X if $\forall t \in (a, b),\ \frac{d\gamma}{dt} = X_{\gamma(t)}$.

Now, we can think of (a, b) as a 1-dimensional manifold. Hence, for any $t \in (a, b)$, it makes sense to talk about

$$d_t\gamma : d_t(a, b) \to T_{\gamma(t)}\mathcal{M}.$$

This is naturally a linear map. What is a linear map from $R \to V$ where V is a vector space? This is a choice of a vector $v \in V$. Then, $d_t\gamma$ gives you a vector, usually denoted by $d\gamma/dt$. Then

$$\frac{d\gamma}{dt}(f) = \frac{df(\gamma(t))}{dt}\bigg|_{t_0}.$$

Theorem 1. *Given a vector field X and an $m \in \mathcal{M}$, there exists an $\varepsilon > 0$ and a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = m$ and γ is an integral curve for X.*

Theorem 2. *Given a vector field X and an $m \in \mathcal{M}$, if $\gamma_1 : I_1 \to \mathcal{M}$ and $\gamma_2 : I_2 \to \mathcal{M}$ are two solutions to the above differential equation (are integrals curves), then*

$$\gamma_1(0) = \gamma_2(0) = m$$

implies $\gamma_1 = \gamma_2$ on $I_1 \cap I_2$.

If for all $m \in \mathcal{M},$ any integral curve as above for X can be extended to $\mathbb{R}$, then X is called complete. **Fact.** Any vector field on a compact manifold is complete.

Example. Take the upper half-plane on $\mathbb{R}^2$ not including $y = 0$. Let X be the constant vector fields with unit vectors pointing south. Then integral curves will be ones pointing straight down. However, this won't work, because we "run into a wall".

We define flow as follows. When we have a complete vector field X on $\mathcal{M}$, we have a notion of flow on $\mathcal{M}$: A family of maps $\Phi_t : \mathcal{M} \to \mathcal{M}$ for each real $t \in \mathbb{R}$ can then be given as $\Phi_t(m) = \gamma(t)$ where γ is the integral curve for X that satisfies $\gamma(0) = m$.

Fact. *If G is a Lie group, then every left-invariant vector field is complete.*

Definition. (exponential) *Let G be a Lie group, and let $\mathfrak{g} = T_eG$. For each $v \in \mathfrak{g}$, let X^v be the associated invariant algebra and Φ_t^v the flow. Then let*

$$\exp : \mathfrak{g} \to G \quad \textit{with} \quad \exp(v) := \Phi_1(e).$$

Properties. (1) exp : $\mathfrak{g} \to G$ is smooth.

(2) $d\ \exp_0 : \mathfrak{g} \to \mathfrak{g}$ will be the identity.

(3) By the Implicit Function Theorem, exp is a local diffeomorphism.

Lemma. *If $\Phi : G \to H$ is a Lie group homomorphism, then*

$$d\Phi_e : \mathfrak{g} \to \mathfrak{h}$$

is a Lie algebra homomorphism, and the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{d\Phi_e} & \mathfrak{h} \\ \exp \downarrow & & \downarrow \exp \\ G & \xrightarrow[\Phi]{} & H. \end{array}$$

Example. Let $G = GL_n(\mathbb{R})$. Then $\mathfrak{g} = M_{n\times n}(\mathbb{R})$ (n by n matrices). Then the map $\exp : \mathfrak{g} \to G$ is the classical exponential,

$$\exp X = \sum_{n=0}^{\infty} \frac{X^n}{n!}.$$

In particular, let H be a Lie subgroup of $GL_n(\mathbb{R})$ (e.g. $SL_n(\mathbb{R})$). Let $\Phi : H \hookrightarrow GL_n(\mathbb{R})$ be the embedding. Then $d\Phi_e : \mathfrak{h} \hookrightarrow \mathfrak{g}$ will be an embedding. If $X \in \mathfrak{h}$, we want to know what $\exp X$ is. Well,

$$\Phi_e(\exp_H X) = \exp_G X = \sum_{n=0}^{\infty} \frac{X^n}{n!}.$$

Adjoint group

Given $g \in G$, we have a map (conjugation)

$$\mathrm{ad}_g : G \to G \;\text{ with }\; x \mapsto gxg^{-1} \;\text{ and } e \mapsto e.$$

Then $\mathrm{Ad}_g := d_e\mathrm{ad}_g : T_eG \to T_eG$. Hence, $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$. Hence,

$$g \in G \;\text{ means }\; \mathrm{Ad}_g \in \mathrm{End}\,\mathfrak{g},$$

and in fact $\mathrm{Ad}_{g^{-1}} = (\mathrm{Ad}_g)^{-1}$, with $\mathrm{Ad}_g \in GL(\mathfrak{g})$. Then

$$\mathrm{Ad} : G \to GL(\mathfrak{g}),$$

so Ad is a representation of G onto its Lie algebra! This is called the **adjoint representation**. A group G is called of adjoint type if Ad is faithful.

Fact. Commutative groups are not adjoint.

Lecture 20 (March 2, 2009) -

Tensor algebra (tensor products and exterior products)

Let V and W be two real vector spaces. Let $F(V,W)$ be the free vector space generated by elements of the form $v \times w$, for $v \in V$, $w \in W$. Let $I(V,W)$ be the sub-vector space generated by elements of the form

$$\begin{gathered} (v_1+v_2,w)-(v_1,w)-(v_2,w), \\ (v,w_1+w_2)-(v,w_1)-(v,w_2), \\ (av,w)-a(v,w), \\ (v,aw)-a(v,w). \end{gathered}$$

We let $V \otimes W = F(V,W)/I(V,W)$. Then we have maps

$$V \times W \to F(V,W) \to V \otimes W \quad \text{(that is, } (u,v) \mapsto u \otimes v\text{))}$$

with $\psi : V \times W \to V \otimes W$ the natural bilinear embedding. We can characterize tensor products by a universal property. If U is a vector space and φ is a bilinear mapping $\varphi : V \times W \to U$, then there is a *unique* linear map $\phi : V \otimes W \to U$ with $\varphi = \phi \circ \psi$.

Properties. (a) $V \otimes W \cong W \otimes V$.

(b) $V \otimes (W \otimes U) \cong (V \otimes W) \otimes U$.

(c) $V^* \otimes W \cong \text{Hom}(V, W)$.

Exercise. Prove these. (Later edit by Robert: Well, I guess there is nothing to prove except part (c))

Let $\varphi \in V^* \otimes W$. How would we associate a $\tilde{\varphi} \in \text{Hom}(V, W)$? Well, we don't. We define a map

$$V^* \times W \to \text{Hom}(V, W) \qquad (v^*, w)(v) \mapsto v^*(v)w.$$

Then using the above property of tensor products, we get $V^* \times W \to V^* \otimes W$ so get can take $V^* \otimes W \to \text{Hom}(V, W)$.

Exercise. Do the above computations and show the map is invertible.

In particular, dim $V \otimes W =$ dim Hom$(V^*, W) =$ dim $V^* \cdot$ dim $W =$ dim $V \cdot$ dim W.

Notation. We will call

$$\underbrace{V \otimes .... \otimes V}_{r} \otimes \underbrace{V^* \otimes ... \otimes V^*}_{s}$$

(r, s)-tensors.

Definition. The wedge product

$$\bigwedge^k V = \overbrace{V \otimes ... \otimes V}^{k\text{-times}} / I_k(V).$$

where

$$I_k(V) = \{\text{sub-vector spaces generated by symbols } v_1 \otimes ... \otimes v_k \text{ s.t. for some } i \neq j,\ v_i = v_j\}$$

So in $\bigwedge^2 V$, $v \otimes v = 0$. For example, in $\bigwedge^2 V$, $(v + w) \otimes (v + w) = 0$, but we can multiply, so

$$v \otimes v + v \otimes w + w \otimes v + w \otimes w = v \otimes w + w \otimes v = 0.$$

Hence, $v \otimes w = -w \otimes v$. In order to avoid confusion, in $\bigwedge^k V$, we will denote the image of $v_1 \otimes ... \otimes v_k$ by $v_1 \wedge ... \wedge v_k$. This, in particular, implies that

$$v_1 \wedge ... \wedge v_k = -v_2 \wedge v_1 \wedge v_3 \wedge .... \wedge v_k.$$

If $\sigma \in S_k$, then $v_{\sigma(1)} \wedge ... \wedge v_{\sigma(n)} =$ sgn $\sigma\, (v_1 \wedge ... \wedge v_k)$. Because of this, there is a well-defined isomorphism $\left(\bigwedge^k V\right)^* \cong \bigwedge^k V^*$. The pairing is defined as follows:

If $\xi_1, ..., \xi_k \in V^*$ and $v_1, ..., v_k \in V$, we define

$$(\xi_1 \wedge ... \wedge \xi_k)(v_1 \wedge ... \wedge v_k) = \det\, (\xi_i(v_j)),$$

and note this is well-defined. Note that this is good, because elements of the form $v_1 \wedge ... \wedge v_k$ form a set of generators for $\bigwedge^k V$.

Remark. If $\{e_1, ..., e_n\}$ is a basis for V, then the collection $e_{i_1} \wedge ... \wedge e_{i_k}$, with $i_1 < ... < i_k$ forms a basis for $\bigwedge^k V$.

If $u \in \bigwedge^k V$, and $v \in \bigwedge^\ell V$, then $u \wedge v = (-1)^{k\ell} v \wedge u$. **Exercise.** Prove by induction.

N.B. If $k > n$, then $\bigwedge^k V = 0$. If $k = n$, then $\bigwedge^n V = 1$.

Let $\mathcal{M}$ be an n-dimensional manifold. Let

$$\bigwedge^k T^* M = \bigcup_{x \in \mathcal{M}} \bigwedge^k (T_x \mathcal{M})^*.$$

We'll call a map

$$\omega : \mathcal{M} \to \bigwedge^k T^* \mathcal{M}$$

a differential form of degree k if $\forall x \in \mathcal{M}$, $\omega(x) \in \bigwedge^k (T_x \mathcal{M})^*$ ("stands above x"). Second, for all vector fields $X_1, ..., X_k$, we want the function $\mathcal{M} \to \mathbb{R}$ given by

$$x \mapsto \omega(x)\,(X_1(x) \wedge ... \wedge X_k(x))$$

to be smooth. And, as usual, you give $\bigwedge^k T^* M$ the structures that make all differentiable forms smooth.

If (U, φ) is a coordinate chart on $\mathcal{M}$, and $y_1, ..., y_m$ a coordinate system, then take $\{\partial / \partial y_i\}$ as a basis for $T_x \mathcal{M}$. The dual basis will be denoted by $\{dy_i\}$. So a differential form locally looks like:

$$\omega = \sum_{i_1 < ... < i_k} f_{i_1, ..., i_k}\, dy_{i_1} \wedge ... \wedge dy_{i_k}.$$

The above smoothness requirements then becomes the requirements that the real-valued functions $f_{i_1, ..., i_k}$ are smooth. Then

$$(dy_{i_1} \wedge ... \wedge d_{y_k}) \left(\frac{\partial}{\partial y_{j_1}} \wedge ... \wedge \frac{\partial}{\partial y_{j_k}} \right) = \det_{1 \le p, q \le k} \left(\delta_{i_p j_q} \right) = \begin{cases} 1 & \text{if } i_p = j_p \\ 0 & \text{otherwise.} \end{cases}$$

So, we can "pick out" functions one at a time using this superization of the Kronecker delta. ☐

Lecture 21 (March 4, 2009) -

Let $\omega \in A^k(\mathcal{M})$ with ω a k-alternating multilinear form on the space of vector fields. Then we can evaluate

$$\omega(X_1, ..., X_k)(m) := \omega(m)(X_1(m), ..., X_k(m)) \quad (\mathcal{M} \to \mathbb{R}).$$

Exterior derivatives

If $f \in C^\infty(\mathcal{M})$, $f : \mathcal{M} \to \mathbb{R}$, we have an associated $d_x f : T_x \mathcal{M} \to \mathbb{R}$. Effectively, we can think of $df : TM \to \mathbb{R}$. If X is any vector field, we can define the value

$$df(X)(x) = d_x f(X_x) \in \mathbb{R}$$

for any $x \in \mathcal{M}$. Hence, we can think of df as a one-form on $\mathcal{M}$, that is, $df \in A^1(\mathcal{M})$. It is then standard to think of $C^\infty(\mathcal{M}) = A^0(\mathcal{M})$ (the zero-forms). Then

$$d : A^0(\mathcal{M}) \to A^1(\mathcal{M}).$$

For each k, there is a unique map $d : A^k(\mathcal{M}) \to \mathrm{A}^{k+1}(\mathcal{M})$ such that $d^2 = 0$, that is,

$$A^k(\mathcal{M}) \xrightarrow{d} A^{k+1}(\mathcal{M}) \xrightarrow{d} A^{k+2}(\mathcal{M})$$

with $d(f) = df$. In local coordinates, this is given by the following:

$$d(f\, dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k}.) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\, dx_i \wedge dx_{i_1} \wedge ... \wedge dx_{i_k}.$$

Properties. (a) If $\omega = \eta$ in a neighborhood of a point p, then $d\omega = d\eta$ in a neighborhood of p (so it's local).

(b) If $\omega_1 \in A^r(\mathcal{M})$ and ω_2 is any form, then $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge d\omega_2 + (-1)^r \omega_1 \wedge d\omega_2$.

Pullbacks

If v and w are two vector spaces over $\mathbb{R}$, then a map $f : V \to W$ induces a map $f^* : W^* \to V^*$. What does $f^*(W^*)$ do? It's supposed to be in V^*. So, you should be able to take $f^*(w^*)(v)$ for all $v \in V$. Well, we can just define

$$f^*(w^*)(v) = w^*(f(v)).$$

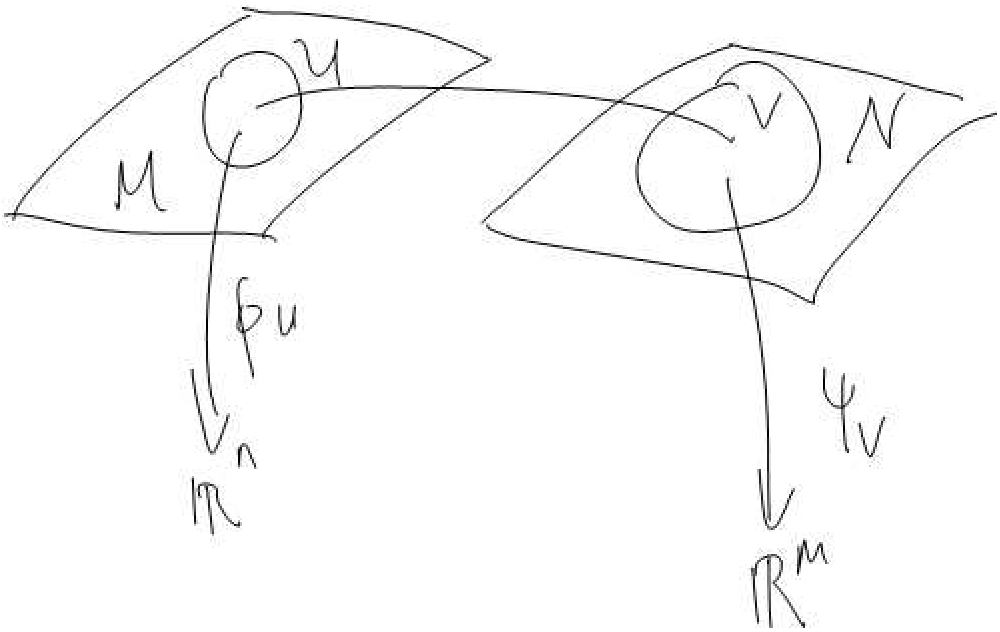
If $\varphi : \mathcal{M} \to \mathcal{N}$ is a map of differentiable manifolds, for any k we'll define a map $\delta\varphi : A^k(\mathcal{N}) \to A^k(\mathcal{M})$. For any x, there is a map

$$d_x\varphi : T_x\mathcal{M} \to T_{\varphi(x)}\mathcal{N} \qquad (d_x\varphi)^* : \left(T_{\varphi(x)}\mathcal{N}\right)^* \to (T_x\mathcal{M})^*.$$

This introduces a map

$$\bigwedge^k (d_x\varphi)^* : \bigwedge^k \left(T_{\varphi(x)}\mathcal{N}\right)^* \to \bigwedge^k (T_x\mathcal{M})^*.$$

Classically, we would do this as follows.



We have $\psi_V \circ \varphi \circ \psi_U^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ given by $(x_1, ..., x_m) \mapsto (h_1(x), ..., h_n(x))$ and a differential form

$$\omega = \sum_{i_1,...,i_k} f_{i_1....i_k}\, dy_{i_1} \wedge ... \wedge dy_{i_k}$$

$$\omega \mapsto \sum_{i_1,...,i_k} f_{i_1....i_k}(h_1(x), ..., h_n(x))\, d(h_{i_1}(x)) \wedge ... \wedge d(h_{i_k}(x))$$

where $d(h_{i_r}) = \sum_{j=1}^{m} \frac{\partial h_{i_r}}{\partial x_j}\, dx_j$.

Example. Consider the map $\mathbb{R}^2 \to \mathbb{R}^3$ given by $(x, y) \mapsto (xy^2, x^3, y^2)$. Call these (x_1, y_1, z_1) and consider the two-form

$$x_1\, dx_1 \wedge dy_1 + y_1 z_1\, dx_1 \wedge dz_1 + z_1^3\, dy_1 \wedge dz_1.$$

Now we want to pullback

$$(xy^2) d(xy^2) \wedge d(x^3) + x^3y^2 d(xy^2) \wedge d(y^2) + y^6 d(x^3) \wedge d(y^2). \quad (*)$$

Now,

$$d(xy^2) = y^2\, dx + 2xy\, dy,\ d(y^2) = 2y\, dy,\ \text{and } d(x^3) = 3x^2\, dx.$$

So,

$$d(xy^2) \wedge d(x^3) = (y^2 dx \wedge 3x^2 dx) + (2xy\, dy \wedge 3x^2 dx) = -6x^3 y\, dx \wedge dy,$$

$$d(xy^2) \wedge d(y^2) = 2y^3\, dx \wedge dy, \text{ and}$$

$$d(x^3) \wedge d(y^2) = 6x^2 y\, dx \wedge dy.$$

Hence, (*) becomes

$$(xy^2)(-6x^3 y)\, dx \wedge dy + (x^3y^2)(2y^3) dx \wedge dy + (y^6)(6x^2 y) dx \wedge dy =$$
$$(-6x^4y^3 + 2x^3y^5 + 6x^2y^7) dx \wedge dy.$$

Thus, that is the pullback of the above differential form. ☐

Properties. (a) δ and d commute. That is, $\delta(d\psi(\omega)) = \delta\psi(d\omega)$.

(b) $\delta\psi(\omega)(X_1, ..., X_k)(m) = \omega_{\psi(m)}(d_m\psi(X_{1,m}), ..., d_m\psi(X_{k,m}))$ with $m \in \mathcal{M}$, where the $X_{i,m}$ notation means to evaluate X_1 at m.

Lie groups

We call a form ω on G left-invariant if

$$\delta \mathrm{L}_g \, \omega = \omega$$

[the pullback of the left invariant] for all $g \in G$.

Fact 1. If ω is a left-invariant k-form in G and $X_1, ..., X_k$ are left-invariant vector fields on G then $\omega(X_1, ..., X_k)$ is constant.

Fact 2. If a 1-form ω, and two vector fields X, Y are left invariant, then

$$d\omega(X, Y) = -\omega([X, Y]).$$

Proof. Exercise. Hint: Prove that if ω is a p-form on a manifold $\mathcal{M}$, and $Y_0, ..., Y_p$ are vector fields, then

$$d\omega(Y_0, ..., Y_p) = \sum_{i=0}^{p} (-1)^i Y_i \, \omega\Big(Y_0, ..., \widehat{Y}_i, ..., Y_p\Big) + \sum_{i<j} (-1)^{i+j} \omega\Big([Y_i, Y_j], Y_0, ..., , \widehat{Y}_i, ..., \widehat{Y}_j, ..., Y_p\Big).$$

Then use Fact 1. ☐

One other fact. If $\varphi : G \to H$ is a Lie group homomorphism, then $\delta\varphi$ sends invariant forms to invariant forms. Simple:

$$\delta L_g \, \delta\varphi(\omega) = \delta(\varphi \circ L_g)\omega \overset{\text{why?}}{=} \delta\big(L_{\varphi(g)} \circ \varphi\big)\omega = \delta\varphi \, \delta\big(L_{\varphi(g)}\big)\omega = \delta\varphi(\omega).$$

Lecture 23 (March 9, 2009) -

Integration on chains

For each $p \geq 1$, let $\Delta^p = \{(a_1, ..., a_p) \in \mathbb{R}^p \,|\, \sum a_i \leq 1, \text{ each } a_i \geq 0\}$. If $p = 0$, we denote $\Delta^0 = \{0\}$. If $\mathcal{M}$ is a manifold, a differentiable singular p-chain simplex σ in $\mathcal{M}$ is a map $\sigma : \Delta^p \to \mathcal{M}$ which extends to a smooth map from a neighborhood of Δ^p to $\mathcal{M}$.

A p-chain in $\mathcal{M}$ is $c = \sum a_i \sigma_i$, where the σ_i's are p-simplices, and the $a_i \in \mathbb{R}$. We define a collection of maps $k_i^p : \Delta^p \to \Delta^{p+1}$ for $0 \leq i \leq p + 1$. For $p = 0$, we have

$$k_0^0(0) = 0 \quad \text{and} \quad k_1^0(0) = 1.$$

For $p \geq 1$, we want to define a map $\Delta^p \to \Delta^{p+1}$. So, let's send

$$k_0^p(a_1, ..., a_p) = \left(1 - \sum_{i=1}^{p} a_i, a_1, ..., a_p\right)$$

$$k_i^p(a_1, ..., a_p) = (a_1, ..., a_{i-1}, 0, a_i, ..., a_p)$$

for $1 \leq i \leq p + 1$.

If σ is a p-simplex in $\mathcal{M}$ ($p \geq 1$), we define its ith face ($0 \leq i \leq p$) to be the simplex $\sigma_i = \sigma \circ k_i^{p-1}$, with

$$\Delta^{p-1} \xrightarrow{k_i^{p-1}} \Delta^p \xrightarrow{\sigma} \mathcal{M}.$$

We define the boundary of σ

$$\partial\sigma = \sum_{i=0}^{p} (-1)^i \sigma^i.$$

If we have a p-chain $\sigma = \sum a_j \sigma_j$, then $\partial\sigma = \sum a_j \, \partial\sigma_j$.

Example. Let $p = 2$. Then

$$k_0^1 : \Delta^1 = [0,1] \to \Delta^2 \text{ with } a \mapsto (1 - a, a) \ (\sigma^0)$$
$$k_1^1 : \Delta^1 \to \Delta^2 \text{ with } a \mapsto (0, a) \ (\sigma^1)$$
$$k_2^1 : \Delta^1 \to \Delta^2 \text{ with } a \mapsto (a, 0) \ (\sigma^2).$$

Then

$$\partial\Delta^2 = (-1)^0 \sigma^0 + (-1)^1 \sigma^1 + (-1)^2 \sigma^2.$$

[Draw a picture. :)]

Example. Let $p = 3$. Then

$$k_0^2 : (a_1, a_2) \mapsto (1 - a_1 - a_2, a_1, a_2)$$
$$k_1^2 : (a_1, a_2) \mapsto (0, a_1, a_2)$$
$$k_2^2 : (a_1, a_2) \mapsto (a_1, 0, a_2)$$
$$k_3^2 : (a_1, a_2) \mapsto (a_1, a_2, 0).$$

Theorem. $\partial \circ \partial = 0$.

Proof. Easy exercise.

If $p = 0$, we want to integrate differential p-forms on p-chains, and ω is a 0-form, i.e., a function $\int_\sigma \omega = \omega(\sigma(0))$. If $p \geq 1$, then $\int_\sigma \omega = \int_{\Delta^p} \delta\sigma(\omega)$, because

$$\sigma : \Delta^p \to M \quad \longleftrightarrow \quad \sigma : U \to \mathcal{M} \ \ (U \subseteq \mathbb{R}^4).$$

If $\sigma = \sum a_i \, \sigma_i$, then $\int_\sigma \omega := \sum a_i \int_{\sigma_i} \omega$.

Stokes's Theorem (First version). *If σ is a p-chain ($p \geq 1$) in a differentiable manifold $\mathcal{M}$ and ω is a smooth $(p-1)$-form defined on a neighborhood of the image of σ, then $\int_{\partial\sigma} \omega = \int_\sigma d\omega$.*

Corollary (Fundamental Theorem of Calculus). *Set $p = 1$ and $\mathcal{M} = \mathbb{R}$.*

Lecture 27 (March 20, 2009) -

The torus is a Lie group isomorphic to $\mathbb{R}^k/\mathbb{Z}^k$ (homeomorphic to $S^1 \times ... \times S^1$ (k times)).

Definition. *A subgroup T of a compact Lie group G is called a maximal torus if there is no subgroup $T' \supset T$ where T' is a torus.*

A maximal torus always exists. There exists a one-parameter subgroup in G, so it must be compact. $T \supset *$. If T does not exist, then id (identity) is a maximal torus. Then take T and look for T' such that $T' \supset T$, with dim $T' >$ dim T. Hence, this search stops. Hence, it exists.

Example. Look at $SU(2)$. If G acts on a set X transitively, then

$$X \cong G/\{\text{stab of any point}\}.$$

Moreover, if G is Lie and X is a topological space, and action is continuous, then $\cong$ above is a homeomorphism. (**Exercise**)

Naturally, $SU(2)$ acts on $\mathbb{C}^2$: $((x_1, y_1), (x_2, y_2)) = x_1\overline{x_2} + y_1\overline{y_2}$. Hence it preserves sets of vectors such that $(\overline{v}, v) = 1$.

$$(\overline{v}, v) = \{(x, y_1) \,|\, x_1\overline{x_1} + y_1\overline{y_1} = 1\}.$$

But here x_1, y_1 are complex numbers. Hence this is a sphere of dim 3 (our X). It is easy to check this action is transitive. Take f.r.t on S^3 and find stabilizer.

$$\left(A\overline{A}^T = 1\right) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix} = 1\right\}.$$

This means $a\overline{a} + b\overline{b} = 1$, $a\overline{c} + b\overline{d} = 0, c\overline{a} + d\overline{b} = 0, c\overline{c} + d\overline{d} = 1, ad - bc = 1.$ The "$= 0$" conditions are equivalent. We can determine them uniquely. Let's find stabilizer: stabilizer of $(1, 0)$ is the identity. The stabilizer leaves invariant orthogonal complement to $(1, 0)$ which is 1-dimensional. Stabilizer is $SU(n-1)$. But $SU(1) = \{1\}$, so the stabilizer here is just the identity, so it is equivalent to S^3: $SU(2) \cong S^3$.

What are the maximal tori here? We can find at least one torus: diagonal matrices

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix},$$

where $\alpha \in S^n = \{z : \, |z| = 1\}$. Then

$$\overline{A} = \begin{pmatrix} \overline{\alpha} & 0 \\ 0 & \overline{\alpha}^{-1} \end{pmatrix}$$

and $\alpha\overline{\alpha} = |\alpha|^2 = 1$. The dimension of this torus is 1, and of the group is 3.

Exercise: Show this torus is maximal.

Another example: consider matrices of the form

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \alpha + \alpha^{-1} \\ 0 & \alpha \end{pmatrix}.$$

Conjugation gives different subgroup.

Main Theorem. *All maximal tori in a compact Lie group are conjugate.*

Definition. *Weyl subgroup of torus T is $\{g + G \mid gTg^{-1} = T\}$.*

Proposition. *Weyl group is finite.*

Example. A subgroup of a compact group does not have to be compact. Consider

$$\left\{e^{2\pi i n\sqrt{2}} \mid n \in \mathbb{Z}\right\} \subset S^1.$$

Everywhere dense. (**Exercise.**)

We claim Aut $T^n = GL(n, \mathbb{Z})$. Indeed,

$$0 \to \mathbb{Z}^n \to \mathbb{R}^n \to T^n \to 0.$$

Therefore, if we have a map $\varphi : T^n \to T^m$ then we have a map $\mathbb{R}^n \to T^n$ and we can factor through into T^m. Now we need to check φ preserves operation. Why is it that $\tilde{\varphi}$ also preserves operation?

Proof. [of finiteness of Weyl group] Let N be normalizer and N_0 a connected component. First notice N is a closed subgroup ($gTg^{-1} = T$). This implies compact, and so $N|_{N_0}$ is discrete, compact (and so finite). We want to show that $N_0 = T$. We have

$$N \to \text{Aut}\, T = GL_n(\mathbb{Z})$$

with $N_0 \to$ id, i.e., any element of N_0 commutes with all elements in $T,\ N_0 \supseteq T$. If we show that N_0 is torus then we are done. If $\alpha : \mathbb{R} \to N_0$ is a 1-parameter subgroup of N_0, then $\alpha(\mathbb{R})$ is connected subgroup containing $T \implies \alpha(\mathbb{R}) \subset T$.

Lecture 28 (March 30, 2009) -

Maximal tori and Weyl groups

A torus is a Lie group isomorphic to $\mathbb{R}^k/\mathbb{Z}^k$ (in fact, any k-dim lattice Λ will do). Then the torus will be isomorphic to $\prod_k S^1$. If G is a Lie group, then a subgroup $T \leq G$ is called a maximal torus if it is a torus that is maximal (not contained in any other torus).

$$\left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| \, x \in \mathbb{R} \right\}$$

has no maximal tori, and neither does $\mathbb{R}^\times$.

Algebraic Geometry Definition. *A torus ovedr a field k is a k-form of $\mathbb{G}_m$.* (i.e., this will be $(G/k)(\overline{k}) = \overline{k}^\times$, so for example for S^1, this just says $S^1 = \{x^2 + y^2 = 1\}$.)

Definition. *For a Lie group G and a maximal torus T, define the normalizer to be*

$$N = \{g \in G \mid gTg^{-1} = T\}.$$

Further, let $W = N/T$, called the Weyl group.

Theorem. *W is finite.*

Main Lemma. *Let G be a compact, connected Lie group, and let T be a maximal torus. Then the map $g : G/T \times T \to G$ given by $(g, t) \mapsto gtg^{-1}$.*

Notice $\tilde{g} : G \times T \to G$ is given by $(g, t) \mapsto gtg^{-1}$. This map has mapping degree $|W|$. In particular, q is surjective.

Definition. (Mapping degree) Let M, N be compact, connected oriented n-dimensional manifolds, and let $f : M \to N$. Then there exists deg$(f) \in \mathbb{Z}$ (mapping degree) such that for all $\alpha \in A^n(N)$,

$$\int_M f^*\alpha = (\deg f) \int_N \alpha.$$

If deg $f \neq 0$, then f is surjective.

Lecture 29 (April 1, 2009) -

Main Lemma. *If G is compact, connected, and T a maximal torus in G, then the map*

$$q : G/T \times T \to G$$
$$(gT, t) \mapsto gtg^{-1}$$

has mapping degree $|W| \neq 0 \implies q$ is surjective.

Definition. (Mapping degree) Let M, N be compact, connected oriented n-dimensional manifolds, and let $f : M \to N$. Then there exists deg$(f) \in \mathbb{Z}$ (mapping degree) such that for all $\alpha \in A^n(N)$,

$$\int_M f^*\alpha = (\deg f) \int_N \alpha.$$

If deg $f \neq 0$, then f is surjective.

We need to know q^*deg (pullback of this differential form). Notice G acts on $\mathfrak{g}$ using the following: $g \in G$

$$c(g) : G \to G \quad x \mapsto gxg^{-1}.$$

Clearly, $c(g)e = e$. Then $d_e c(g) : T_eG \to T_eG$. Hence $g \mapsto$ Ad(g) (recall Ad$(g) : \mathfrak{g} \to \mathfrak{g}$) is a representation of the compact group G on the finite dimensional vector space $\mathfrak{g}$. Hence there exists an inner product (metric) or $\mathfrak{g}$ invariant under Ad.

$$(u, v) = \int_G \ (\pi(g)u, \pi(g)v)_0 \, dg.$$

Write $\mathfrak{g} = t \oplus t^\perp$. Set $L(G/T) = t^\perp$. So what we are really saying is that

$$\mathfrak{g} = t \oplus L(G/T).$$

If we take $\text{Ad}\,|_T$, this will act trivially on t since T is commutative (but nontrivially on $L(G/T)$). Hence, we obtain a map $\text{Ad}_{G/T} : T \to \text{Aut}(L(G/T))$. Then we have differential forms $d(gT),\ dt$, and dg. On the other hand, we have a projection

$$\pi : G \to G/T,$$

which introduces a map $d_e\pi : \mathfrak{g} \to T_{eT}(G/T)$. Notice that $d_e\pi(t) = 0$. We can write

$$d_e\pi : t \oplus L(G/T) \to T_{eT}(G/T),$$

so $d_e\pi$ induces a map $L(G/T) \cong T_{eT}(G/T)$. If $n = \text{dim}\, G$ and $k = \text{dim}\, T$, then

$$\pi : G \to G/T$$

gives us a pullback $\pi^* \, d(gT) \in A^{n-k}(G)$. [Btw all of this is in Broker.] There is an orthogonal projection $\text{pr} : \mathfrak{g} \to t$ along $L(G/T)$. Consider $\text{pr}^* \ dt$. We have $dt_e \in \Lambda^k \, t^*$ (with $e \in T$). Then we can take

$$\text{pr}^* dt_e \in \Lambda^k \, \mathfrak{g}^*.$$

Extend this to a G-invariant differential form on G of degree k, denoted dt (abuse of notation!). We can consider

$$\pi^*(dgT) \wedge dt.$$

Both of those are G-invariant, and their wedge is a G-invariant n-form. As such, it would be $c \cdot dg$. In fact, $dg = \pi^*(dgT) \wedge dt$. (**Exercise**. Show the integral of left is 1, and right is 1. We know the dt part is 1. So just show the pullback of π is as well.)

On the other hand, $G/T \times T$ has a differential form given by the following:

$$\text{pr}_1^* \, d(gT) \wedge \text{pr}_2^* \ dt$$

where

$$G/T \ \overset{\text{pr}_1}{\leftarrow} G/T \times T \ \overset{\text{pr}_2}{\rightarrow} T.$$

Thus

$$L(G/T \times T) = T_{eT}(G/T) \oplus t \cong L(G/T) \oplus t = \mathfrak{g},$$

where the leftmost thing has identity (eT, e). Then $\alpha_{(eT,e)} = dg_e$.

$$q : G/T \times G \to G$$

$$q^* dg = \text{det}\, q \cdot \alpha.$$

We define $\text{det}\, q$ by this equation. Question is: what is $(\text{det}\, q)(gT, t)$?

Proposition. $(\det q)(gT, t) = \det\big(\text{Ad}_{G/T} t^{-1} - E_{G/T}\big)$ where $E_{G/T}$ is the identity on $L(G/T)$.

Proof. The forms $dg, d(gT)$ are left invariant under G, dt is left invariant under T. Then

$$G \times T \to G \times T \to G \to G$$

$$(x,y) \overset{l_a}{\mapsto} (x,ty) \overset{\tilde{q}}{\mapsto} (gx)(ty)(gx)^{-1} \overset{l_b}{\mapsto} (gt^{-1}g^{-1})(gx)(ty)(gx)^{-1}.$$

with $a = (g,t)$ (where l_a and l_b mean "left translation by a" and "left translation by b", respectively). Notice

$$(gt^{-1}g^{-1})(gx)(ty)(gx)^{-1} = gt^{-1}xtyx^{-1}g = c(g)(c(t^{-1})xyx^{-1}),$$

where $c(g) : G \to G$ is conjugation. So the point $(e,e) \mapsto (e,e)$. We claim that det q is the determinant of the differential of this map at the point (e,e) restricted to the subspace $L(G/T) \oplus t \subseteq \mathfrak{g} \oplus t$. $\square$

Lecture 30 (April 3, 2009) -

We have a map $\pi : G \to G/T$ where T is a maximal torus. Then

$$dg = \pi^* \, d(gT) \wedge dt$$

on T, we have $dt \in A^k T$, where $k = \dim\, T$. We can evaluate $dt_e \in \bigwedge^k \mathfrak{t}^*$. We had $\mathfrak{t} \subseteq \mathfrak{g}$. This gives us a projection map $\mathfrak{g} \subseteq \mathfrak{t}$, orthogonal projection. As such,

$$\delta^* : \mathfrak{t}^* \to \mathfrak{g}^*$$

gives a map

$$\bigwedge^k \delta^* : \bigwedge^k \mathfrak{t}^* \to \bigwedge^k \mathfrak{g}^*.$$

Then we get

$$\bigwedge^k \delta^*(dt_e) \in \bigwedge^k \mathfrak{g}^*.$$

Extend this to a left-invariant differential form of degree k on G. Another differential form is

$$\alpha = \mathrm{pr}_1^* \, d(gT) \wedge \mathrm{pr}_2^* \, dt$$

where

$$G/T \overset{\mathrm{pr}_1}{\leftarrow} G/T \times T \overset{\mathrm{pr}_2}{\rightarrow} T.$$

So dg is on G and α is on $G/T \times T$. Now, we have a map $q : G/T \times T \to G$ which is given by $q(gT,t) = gtg^{-1}$. So we can pullback $q^* dg$ and this will be a top-degree differential form, and we have a top-degree differential form α already, and since the space of top-degree differential forms is 1-dimensional, we have

$$q^* dg = (\det q)\alpha.$$

We start by writing a map

$$\begin{array}{c} G \times T \to G \\ \downarrow \ \nearrow \\ G/T \times T. \end{array}$$

Then the exact map is $(x,y) \mapsto (gx, ty) \mapsto (gx)(ty)(gx)^{-1} \mapsto (gt^{-1}g^{-1})(gx)(ty)(gx)^{-1}$ which is $(x,y) \mapsto c(g)(c(t^{-1})x \cdot y \cdot x^{-1})$

Exercise. Given $G \times G \to G$ by $(h_1, h_2) \mapsto h_1 h_2$, verify

$$d_{(e,e)}\mu : \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g} \quad (X,Y) \mapsto X + Y.$$

Then with this exercise, we know the derivative of $(x,y) \mapsto c(t^{-1})x \cdot y \cdot x^{-1}$ is

$$(X,Y) \mapsto \mathrm{Ad}_{G/T}(t^{-1})X + Y - X, \text{ then}$$

$$\begin{pmatrix} X \\ Y \end{pmatrix} \mapsto \begin{pmatrix} \mathrm{Ad}_{G/T}(t^{-1}) - E_{G/T} & \\ & E_T \end{pmatrix}$$

so

$$\det q = \det_{L(G/T)}\big(\mathrm{Ad}_{G/T}(t^{-1}) - {}_{G/T}\big),$$

$$\text{where } t \text{ is of the form } t = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Lemma. *If $t \in T$ is such that $\langle t \rangle$ is dense in T, then*

(i) $q^{-1}(t)$ *has* $|W|$ *points, and*

(ii) $\det q > 0$ *at each of these.*

Weyl Integration Formula

Let G be compact connected and T a maximal torus, and take f continuous on G. Then

$$\int_G f(g)\,dg = \tfrac{1}{|W|}\int_T \left[\det\big(E_{G/T} - \mathrm{Ad}_{G/T}(t^{-1})\big)\int_{G/T} f(gtg^{-1})\,dg\right] dt.$$

We have

$$q : G/T \times T \to G \xrightarrow{f} \mathbb{R} \quad \text{given by} \quad (gT, t) \mapsto gtg^{-1}.$$

In general, when we integrate over conjugacy classes

$$\int_{G/T} f(gtg^{-1})\,dg,$$

the integral is 0 unless the conjugacy class is of maximal dimension.

For now, let $G = U(n)$, and let $D \subset G$ be the diagonal group. Then

$$D = \left\{ \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix};\ |t_i| = 1 \right\}$$

is a maximal torus, and $D \cong S^1 \times \ldots \times S^1$ (n-copies). In this case, the Weyl group W will be isomorphic to S_n (permutation group on n letters), and the action will be by permuting the diagonal entries. If γ is a conjugacy class in G, since we know every element of G is diagonalizable since it is compact, we know that $\gamma \cap D \neq \emptyset$. In fact, this will be a single W-orbit in D. If $t \in \gamma \cap D$, then we claim $W \cdot t \subseteq \gamma \cap D$. If you remember, $W = N(T)/T$ (the normalizer of T, modulo T). The way it acts is that

$(n, t) \mapsto ntn^{-1}$, which means that $ntn^{-1} \in \gamma$. Then $w \cdot t \in \gamma \cap D$ (since we assume we are working with things that are normalizing the torus). Now if f is a class function on G, then $f_D \coloneqq f|_D$ means that f and f_D determine each other. So f_D will be a function which is W-invariant on D (with no further restrictions). The idea is that if we have a representation π of G, this will have a character χ_π which is a class function. Then $\chi_\pi|_D$. Second, $\pi|_D$ will be $\sum m_j x_j$.