Lecture 3 (January 16, 2009) -

We continue by showing transitivity of homotopy (so that it is indeed an equivalence relation).

Assume $f \cong f'$ and $f' \cong f''$ by F and F'. Then we can let

$$F''(s,t) = \begin{cases} F(2s,t) & s \in [0,1/2] \\ F'(2s-1,t) & s \in (1/2,1] \end{cases}.$$

This is an explicit continuous function between f and f'', so $f \cong f''$. □

Now, let's show that give $\varphi : I \to I$ with $\varphi(0) = 0$ and $\varphi(1) = 1$, $\alpha \cong \alpha \circ \varphi$. Indeed,

$$\alpha_s(t) = \alpha((1-s)t + s\varphi(t)).$$

Exercise. Show associativity of the fundamental group using a direct parametrization.

Lecture 4 (January 21, 2009) -

Read pages 162-165, 1-4, 25-37, and 5-7 in Hatcher.

Let (X, x_0) be a pointed space.

Lecture 5 (January 23, 2009) -

Lifting Theorem

Take $\varphi : R \to S_1$ with $\varphi(t) = e^{2\pi i t}$.

Lifting Theorem. If X is star shaped from $\vec{0} \in X \subset \mathbb{R}^n$, X is compact, $f : X \to S_1$, $t_0 \in \mathbb{R}$, $\varphi(t_0) = f(x_0) \Longrightarrow \exists \tilde{f} : X \to \mathbb{R}$ s.t. $\varphi \circ \tilde{f} = f$, $\tilde{f}(x_0) = t_0$. In other words, there is a lift.

Proof. Take $\varphi : \left(-\frac{1}{2}, \frac{1}{2}\right) \to S^1 - \{-1\}$ a homeomorphism with inverse $\psi : S^1 - \{-1\} \to \left(-\frac{1}{2}, \frac{1}{2}\right)$. Then $\exists \delta > 0$ such that $\forall x, x' \in X$, $||x - x'|| < \delta$ implies that $|f(x) - f(x')| < 2$. Now choose n such that $\frac{||x||}{n} < \delta$ $\forall x \in X$. Then $f(0), , \left(\frac{1}{n}x\right), \ldots,$ $f\left(\frac{n-1}{n}x\right), f(x)$ form a consecutive partition of non-antipodal points.

Lecture 6 (January 26, 2009) -

Degrees of paths

For a path α, we define deg $\alpha = \tilde{\alpha}(1)$ (where $\tilde{\alpha}$ is a lift for α). If $\alpha \sim \beta$, then deg $\alpha =$ deg β. If $H : I \times I \to S^1$, we get $\tilde{H} : I \times I \to \mathbb{R}$ with $\tilde{H}(0,0) = 0$ such that $\tilde{H}(0,t) = \tilde{\alpha}(t)$ (by unique lifting), $\tilde{H}(s,0) = 0$ (by unique lifting), and $\tilde{H}(1,0) = \tilde{\beta}(0)$ and $H(1,t) = \beta(t)$. Then $\tilde{H}(1,t) = \tilde{\beta}(t)$ by unique lifting. Finally, $\tilde{H}(s,1)$ is constant in S. Hence, $\tilde{\alpha}(1) = \tilde{\beta}(1)$. This means indeed deg $\alpha =$ deg β.

Now, we want to show deg : $\pi_1(S,1) \to \mathbb{Z}$ is an isomorphism. Recall that $[\alpha]*[\beta] = [\alpha * \beta]$. If $\tilde{\alpha}(0) = 0, \tilde{\alpha}(1) = m =$ deg α, then call $m + \tilde{\beta}(t)$ the path in $\mathbb{R}$ from m to $m +$ deg β. Then $\tilde{\alpha} * \left(m + \tilde{\beta}\right)$ covers $\alpha * \beta$, and $\tilde{\alpha} * \left(m + \tilde{\beta}\right)$. Then the lift of $\alpha * \beta$ gives

$$(\widetilde{\alpha * \beta}(1)) = \text{deg}\,\alpha + \text{deg}\,\beta.$$

Then ker deg $= [c_1]$. If deg $\alpha = 0$, this means $\tilde{\alpha}(1) = 0$ so $\tilde{\alpha}$ is a directed path: $\tilde{\alpha}(0) = 0 = \tilde{\alpha}(1)$. Then define $\tilde{\alpha}_s(t) = s\tilde{\alpha}(t)$ ($\tilde{\alpha}_0(t) = 0$, $\tilde{\alpha}_0 = c_0$, $\tilde{\alpha}_1(t) = \tilde{\alpha}$, $c_0 \sim \tilde{\alpha}$. Then we show $c_1 \sim \alpha$.

Of course, $\pi_1(D^2, 1) \cong 1$. For $\alpha : I \to D^2 \subset \mathbb{C}$ with $\alpha(0) = \alpha(1) = 1$, just take

$$\alpha_s(t) = s\alpha(t) + (1 - s) \cdot 1.$$

Then $\alpha_0(t) = c_1(t)$ and $\alpha_1(t) = \alpha(t)$.

We claim there is no retract on the above: $\nexists r$ continuous s.t. $S^1 \xrightarrow{\iota} D^2 \xrightarrow{r} S^1$ with $r \circ \iota = \iota_{S^1}$. This follows from the fact π_1 is a topological invariant: we would need

$$\pi_1(S^1, 1) \xrightarrow{\iota_\#} \pi_1(D^2, 1) \xrightarrow{r_\#} \pi_1(S^1, 1)$$

such that $(r \circ i)_\# = r_\# \circ i_\#$. Of course, this is impossible, since we would need $\mathbb{Z} \to 1 \to \mathbb{Z}$. [this r has to be continuous b/c of the stuff we talked about lifting all used continuity]

If $f : D^2 \to D^2$ then f has a fixed point. Suppose f has no fixed points.

$$r(x) = tx + (1 - t)f(x) \text{ for } t \geq 1$$

Then $||r(x)|| = 1$. **Exercise.** Check r is continuous.

Todo. Read notes he gave.

Lecture 7 (January 28, 2009) -

Consider the quotient space D^n/S^{n-1} with $x \sim y$ if $x = y$ or $x, y \in$ equiv class $\{[x]\} = X/A$.

Definition. A map p is a quotient map if $V \subset X/A$ is open $\Leftrightarrow p^{-1}(V)$ is open in X.

Lecture 9 (February 2, 2009) -

See handout on singular homology.

Lecture 13 (February 11, 2009) - δ

There is an identity map $\mathbf{1} : \Delta^n \to \Delta^n$ with $\Delta_n(\Delta^n) \to \Delta_n(X)$ where we send $\mathbf{1} \to \sigma$. Then we can have the boundary operator $\Delta_n(\Delta^n) \xrightarrow{\partial} \Delta_{n-1}(\Delta^n)$. We also have $\Delta_n(X) \xrightarrow{\partial} \Delta_{n-1}(X)$, with $\Delta_{n-1}(\Delta^n) \to \Delta_{n-1}(X)$ (this gives us a square diagram).

For X a set, define

$$A_X = \{f : X \to \mathbb{Z} \mid f(x) \neq 0 \text{ for only finitely many } x\}.$$

Let

$$(f+g)(x) = f(x) + g(x).$$

Then A_X forms an abelian group.

Lecture 15 (February 16, 2009) - Reduced homology

Suppose $A \subset X$ with $A \xrightarrow{\iota} X \xrightarrow{r} A$ with $r \circ i = 1_A$. Then A is a retract of X and r is said to be a retraction.

Lecture 17 (February 20, 2009) -

If $t : S^0 \to S^0$ and $t(x) = -x$, $t_1 : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ given by $t(x_1, ..., x_{n+1}) = (-x_1, x_2, ..., x_{n+1})$, then $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$. Further,

$$D^n = \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0, |x| \leq \pi\}.$$

Now, we have $f : D^n \to S^n$. We want

$$f(x) = \begin{cases} -(\cos|x|)e_{n+1} + \frac{\sin|x|}{|x|}x & x \neq 0 \\ -e_{n+1} & x = 0. \end{cases}$$

Then $f \circ t_1 = t_1 \circ f$, and f is equivariant.

$$f(\partial D^n) = \{e_{n+1}\} = p \subset S^n$$

Furthermore,

$$\tilde{H}_0(S^0) \xrightarrow{\pm 1} \tilde{H}_0(S^0)$$

with $H_1(D^1, S^0) \underset{\cong}{\overset{\partial y}{\to}} \tilde{H}_0(S^0)$ and $H_1(D^1, S^0) \underset{\cong}{\overset{\partial x}{\to}} \tilde{H}_0(S^0)$. Finally,

$$\tilde{H}_1(S^1, p) \xrightarrow{f_*} \tilde{H}_1(D^1, S^1) \text{ and } \tilde{H}_1(S^1, p) \xrightarrow{f_*} H_1(D^1, S^0),$$

with $H_1(D^1, S^0) \xrightarrow{t_*} \tilde{H}_1(D^1, S^1)$ and $H_11(S^1, p) \xrightarrow{t_*} \tilde{H}_1(S^1, p)$, where t_* is simply multiplication by -1. Finally, we have

$$\tilde{H}_1(S^1) \underset{\cong}{\to} H_1(S^1, p) \text{ and } \tilde{H}_1(S^1) \underset{\cong}{\to} \tilde{H}_1(S^1, p),$$

and $\tilde{H}_1(S^1) \xrightarrow{t_*} \tilde{H}_1(S^1)$. (this has to be re-diagrammized...)

Given a map $f : S^n \to S^n$ or $f : (D^n, S^{n-1}) \to (D^n, S^{n-1})$, we can define a degree deg f by $f_*(\alpha) = (\text{deg } f)\alpha$. Then α generates $\tilde{H}_n(S^n)$. The degree has two properties, namely

(1) deg $\mathbb{I} = 1$.

(2) deg $f \circ g = (\text{deg } f)(\text{deg } g)$.

(3) $f \cong g \implies$ deg $f =$ deg g.

(4) If f is a homotopy equivalence, then deg f is ± 1.

(5) If $f : (D^n, S^{n-1}) \to (D^n, S^{n-1})$, then deg $f =$ deg $(f \mid S^{n-1})$.

Then

$$t_j(x_1, ..., x_{n+1}) = (x_1, ..., x_{j-1}, -x_j, x_{j+1}, ..., x_{n+1}), \text{ and}$$
$$s(x_1, ..., x_{n+1}) = (x_j, x_2, ..., x_{j-1}, x_1, x_{j+1}, ..., x_{n+1}).$$

Clearly, $s \circ s = \mathbb{I}$ and $t_j = s \circ t_1 \circ s$, so that deg$(t_i) =$ deg $t_1 = -1$.

Antipodal map

$a(x_1, ..., x_{n+1}) = (-x_1, ..., -x_{n-1})$, and $a(\vec{x}) = -\vec{x}$. Then $a = a \mid S^n$, and deg $a = (-1)^{n+1}$. Suppose $f(x) \neq x$ for all x, with $f : S^n \to S^n$. That is, each x goes to a point different from x. If deg $f \neq (-1)^{n+1}$, then f has a pixed point. If

$$L\ f\ \ = 1 + \ -1 \qquad\qquad f$$

If $X = U \cup V$ where U and V are open, then the Mayer-Vietoris sequence

$$\ldots \overset{\Delta}{\to} H_q(U \cap V) \overset{i}{\to} H_q(U) \oplus H_q(V) \overset{j}{\to} H_q(X) \overset{\Delta}{\to} H_{q-1}(U \cap V) \to \ldots$$

is exact where $i(\alpha) = (i_{1*}\alpha, i_{2*}\alpha)$ and $j(\beta, \gamma) = j_{1*}\beta - j_{2*}\gamma$ and where

$$\begin{array}{ccc} U \cap V & \overset{i_1}{\to} & U \\ i_2 \downarrow & & \downarrow j_1 \\ V & \underset{j_2}{\longrightarrow} & U \cup V. \end{array}$$

The boundary map Δ is defined to be the composition

$$H_q(U \cup V) \to H_q(U \cup V, U) \overset{=}{\leftarrow} H_q(V, U \cap V) \overset{\partial_*}{\to} H_{q-1}(U \cap V)$$

where the middle map is excision. Excision and the exact sequence of a pair give the diagram

$$\begin{array}{ccccccccc} H_q(U \cap V) & \overset{i_{2*}}{\longrightarrow} & H_q(V) & \longrightarrow & H_q(V, U \cap V) & \overset{\partial_*}{\longrightarrow} & H_{q-1}(U \cap V) & \overset{i_{2*}}{\longrightarrow} & \\ i_{1*} \downarrow & & j_{2*} \downarrow & & = \downarrow & & i_{1*} \downarrow & & \\ H_q(U) & \underset{j_{1*}}{\longrightarrow} & H_q(U \cup V) & \longrightarrow & H_q(U \cup V, U) & \underset{\partial_*}{\longrightarrow} & H_{q-1}(U) & \underset{j_{1*}}{\longrightarrow} & \end{array}$$

The proof from here is algebra. In simpler notation, the Barrett-Whitehead lemma states that given a commutative diagram with exact rows and where c is an isomorphism,

$$\begin{array}{ccccccccc} A_q & \overset{f}{\to} & B_q & \overset{g}{\to} & C_q & \overset{\partial}{\to} & A_{q-1} & \overset{f}{\to} & B_{q-1} \\ a \downarrow & & b \downarrow & & c \downarrow \; = & & \downarrow a & & \downarrow b \\ \overline{A}_q & \underset{\overline{f}}{\to} & \overline{B}_q & \underset{\overline{g}}{\to} & \overline{C}_q & \underset{\overline{\partial}}{\to} & \overline{A}_{q-1} & \underset{\overline{f}}{\to} & \overline{B}_{q-1} \end{array} .$$

there is a long exact sequence

$$\ldots \overset{\Delta}{\longrightarrow} A_q \overset{i}{\longrightarrow} \overline{A}_q \oplus B_q \overset{j}{\longrightarrow} \overline{B}_q \overset{\Delta}{\longrightarrow} A_{q-1} \overset{i}{\longrightarrow} \ldots$$

with $i(\alpha) = (a(\alpha), f(\alpha)),\ j(\overline{\alpha}, \beta) = \overline{f}(\overline{\alpha}) - b(\beta)$, and $\Delta = \partial \circ c^{-1} \circ \overline{g}$.

Lecture 19 (February 25, 2009) -

Proposition. [pg 169 Hatcher] Let $A \subset S^n$ such that $A \cong D^k$. Then $\tilde{H}_q(S^n - A) = 0$.

Proof. We will use induction. When $k = 0$, we have $S^n - D^0 = \mathbb{R}^n$, which is homotopic to a point. Now, $D^k \cong D^{k-1} \times I$. Assume $\tilde{H}_q(S^n - A) \neq 0$. Then take

$$h : D^{k-1} \times I \overset{\approx}{\to} A.$$

Let

$$A' = h\big(D^{k-1} \times \big[0, \tfrac{1}{2}\big]\big),\ A'' = h\big(D^{k-1} \times \big[\tfrac{1}{2}, 1\big]\big)$$

with $A' \cup A'' = A$ and $A' \cap A'' \cong D^{k-1}$. Then $S^n - A = (S^n - A^1) \cap (S^n - A)$. Hence, we get the Mayer-Vietoris sequence

$$\tilde{H}_q(S^n - A) \to \tilde{H}_q(S^n - A') \oplus \tilde{H}_q(S^n - A'') \to \tilde{H}_q(S^n - A' \cap A'').$$

Here, $y_i \in \tilde{H}_q(S^n - A)$ is mapped to $y_i \neq 0$ in $\tilde{H}_q(S^n - A_1)$ where $A_1 = A'$ or A''. Then

$$y_0 \mapsto y_1 \mapsto \ldots \mapsto y_i \in \tilde{H}_q(S^n - A_i).$$

Now, since $\bigcup S^n - A_i = S^n - h(D^{k-1} \times \{t\}) = S^n - \bigcap A_i$. So,

$$\tilde{H}_q(S^n - A_i) \to \tilde{H}_q(S^n \cap A_i) = 0 \text{ (by induction)}$$

with $y = 0$ in $\tilde{H}_q(S^n \cap A_i)$. We get a string of the y_i's which are all non-zero but end up mapping to something zero, so we have a contradiction (that $y_0 \neq 0$). ☐

Jordan-Brouwer Separation Theorem (still page 169--part b of the proposition)

Let Σ^m, S^n be such that $\Sigma^m \cong S^m$. Then

$$\tilde{H}_q(S^n - \Sigma^m) = \begin{cases} \mathbb{Z} & q = n - m - 1 \\ 0 & q \neq n - m - 1. \end{cases}$$

Proof. We proceed by induction on m. First, $m = 0$. Then

$$\tilde{H}_q(S^n - S^0) = \tilde{H}_q(S^{n-1}) = \begin{cases} \mathbb{Z} & q = n - 1 \\ 0 & q \neq n - 1 \end{cases}$$

(recall the reduced homology is precisely that for the S^{n-1} sphere). Now, notice

$$\Sigma^m \cong S^m = D^m_- \cup D^m_+ \text{ [where } D^m_- \text{ and } D^m_+ \text{ are two hemispheres]}$$

with $S^n - D^m_-$, $S^n - D^m_+$. Then the Mayer-Vietoris sequence is

$$0 \to \tilde{H}_{q+1}((S^n - D^m_-) \cup (S^n - D^m_+)) \overset{\partial}{\underset{\cong}{\to}} \tilde{H}_q(S^n - \Sigma^m) \to 0,$$

where

$$\tilde{H}_{q+1}((S^n - D^m_-) \cup (S^n - D^m_+)) = \tilde{H}_{q+1}(S^n - (D^m_- \cap D^m_+)) = H_q(S^n - \Sigma^{m-1}).$$

Hence,

$$\tilde{H}_q(S^n - \Sigma^m) \overset{\cong}{\leftarrow} \tilde{H}_{q+1}(S^n - \Sigma^{m-1}) \overset{\cong}{\leftarrow} \ldots \leftarrow \tilde{H}_{q+m}(S^n - \Sigma^0) = \tilde{H}_{q+m}(S^{n-1})$$

but we know the last guy is just

$$\begin{cases} \mathbb{Z} & q = n - m - 1 \\ 0 & \text{otherwise.} \end{cases}$$ ☐

Using the above theorem, we know for example that $\tilde{H}_0(S^2 - \Sigma^1) = \mathbb{Z}$.

Now, consider $S^1 \hookrightarrow S^3$.

Lecture 20 (February 27, 2009) -

Jordan-Brouwer

Recall last time we computed

$$\tilde{H}_q(S^n - \Sigma^m) = \begin{cases} \mathbb{Z} & \text{if } \varphi = n - m - 1 \\ 0 & \text{otherwise.} \end{cases}$$

If X is path connected and given by $g : \dot{I} \to X$, then $\exists f : I \to X$ such that

$$\begin{array}{l} \dot{I} \to X \\ \uparrow \ \nearrow \\ I. \end{array}$$

Further, X is connected if any map $h : X \to \dot{I}$ is constant.

Lemma. *If X is path connected, then X is connected.*

Proof. Suppose $h : X \to \dot{I}$ is onto. Then there exists an x_0 such that $h(x_0) = 0$, and there exists an x_1 such that $h(x_1) = 1$. Define $g(0) = x_0$ and $g(1) = x_1$. First, notice that $h \circ f(0) = 0$ and $h \circ f(1) = 1$. Hence, we can't have such a continuous function. ☐

This shows $H_0(S^n - \Sigma^{n-1}) = \mathbb{Z} \oplus \mathbb{Z}$, so that it has two path components, say U and V (so $S^n - \Sigma^{n-1} = U \cup V$).

Proposition. *Let $\Sigma^{n-1} \subset S^n$. Then $\Sigma^{n-1} = \partial U = \partial V$, with $\partial U = \overline{U} - \mathring{U}$, where $\mathring{U}$ is the interior.*

Proof. Assume U and V are open with $\overline{U} \subset S^n - V$. Take $S^n = U \cup V \cup \Sigma$. Then $U = (S^n - V) - \Sigma$ and $\overline{U} \subset S^n - V$. We have

$$\partial U = \overline{U} - U \subset (S^n - V) - U = S^n - (U \cup V) = \Sigma.$$

Hence, $\partial V \subset \Sigma$. We want to show $\Sigma \subset \partial U$ and $\Sigma \subset \partial V$, that is, $\Sigma \subset \overline{U} \cap \overline{V}$. Let $x \in \Sigma$ and let N be any open neighborhood of x in S^n. Let $A \subset \Sigma^{n-1} \cap N$ with $\Sigma^{n-1} - A \cong D^{n-1}$. By the Lemma,

$$\tilde{H}_0(S^n - (\Sigma^{n-1} - A)) = 0$$

(the reduced homology of the complement of the disk is 0). Since $S^n - (\Sigma^{n-1} - A)$ is path connected, if $p \in U$ and $q \in V$, then there is a path w in $S^n - (\Sigma^{n-1} - A)$ from p to q which meets Σ^{n-1}, and hence must meet A.

Lecture 22 (March 4, 2009) - Real projective space

Attaching a cell to X

We have a map $g : S^{n-1} \to X$ and $Z = X \cup_g D^n := X \amalg D^n / x \sim g(x)$ with $x \in S^{n-1}$ and $g(x) \in X$ (with $\amalg$ disjoint union).

Example. Take $X = \{*\}$ a one-point space. Then $Z = S^n$.

We can define $\mathbb{R}P^n := S^n / x \sim -x$. But we don't need the whole sphere, we can take one hemisphere with just antipodal points on the boundary identified: that is D^n with $\mathbb{R}P^{n-1}$. Now, we can do the following. Take $S^{n-1} \xrightarrow{g} \mathbb{R}P^{n-1}$ with $g(x) = g(-x)$. Then

$$\mathbb{R}P^n = \mathbb{R}P^{n-1} \cup_g D^n.$$

Remember $\mathbb{Z}/2$ acts on S^n by the antipodal map:

$$\mathbb{Z}/2 \to \text{Homeo}(S^n)$$

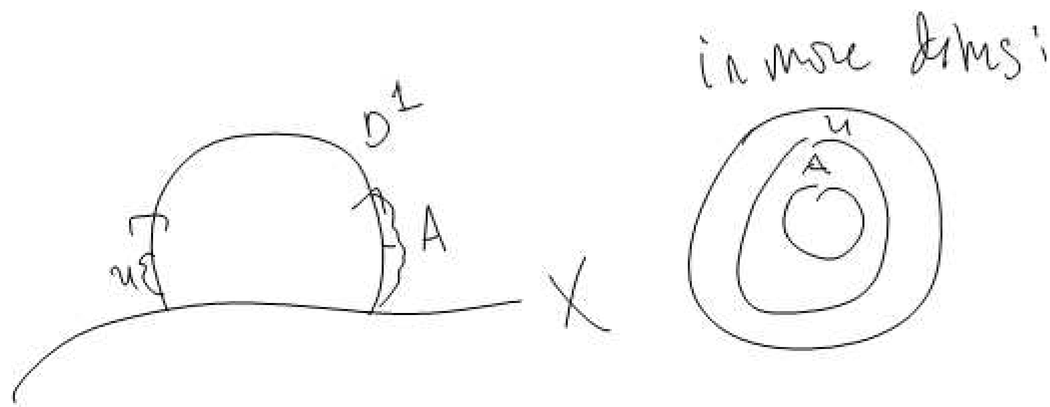
with $S^n/(\mathbb{Z}/2)$ the set of orbits. Define a projection map $S^n \xrightarrow{p} \mathbb{R}P^n$ such that $V \subset \mathbb{R}P^n$ is open if and only if $p^{-1}(V)$ is open. Consider

$$H_q(D^n, S^{n-1}) \overset{\equiv}{\to} H_q(Z, X)$$

where $Z = \mathbb{R}P^{n-1} \cup_g D^n$ where $g : S^{n-1} \to \mathbb{R}P^{n-1}$. We then get the diagram

$$\begin{array}{ccc} (D^n, S^{n-1}) & \xrightarrow{f} & (Z, X) \\ \downarrow \cong & & \downarrow \cong \\ (D^n, A) & \to & (Z, A') \\ \uparrow = & & \uparrow = \\ (D^n - U, A - U) & \to & (Z - U', A' - U') \end{array}$$

such that $f|S^{n-1} = g$ with $A' = X \cup f(A)$ and $U' = X \cup f(U)$.



Now, we know that

$$H_q(D^n, S^{n-1}) = \begin{cases} Z & \text{if } q = n \\ 0 & \text{if } q \neq n. \end{cases}$$

Now let's look at the exact sequence of the pair $(\mathbb{R}P^n, \mathbb{R}P^{n-1})$.

$$0 \to H_n(\mathbb{R}P^{n-1}) \to H_n(\mathbb{R}P^n) \to H_n(\mathbb{R}P^n, \mathbb{R}P^{n-1}) \xrightarrow{\partial} H_{n-1}(\mathbb{R}P^{n-1}) \to H_{n-1}(\mathbb{R}P^n) \to 0$$

$$0 \to \tilde{H}_q(\mathbb{R}P^{n-1}) \to \tilde{H}_q(\mathbb{R}P^n) \to 0 \text{ if } q \neq n, n-1.$$

For example, for $\mathbb{R}P^0 = \{*\}$, $\tilde{H}_q(\mathbb{R}P^0) = 0$.

We then <u>claim</u> that $\tilde{H}_q(\mathbb{R}P^n) = 0$ if $q > n$. *Proof.* It's true if $n = 0$, and true if $n = 1$, and true for $n > 1$ by induction.

Recall the diagram

$$\begin{array}{ccccccccc} & & H_n(D^n, S^{n-1}) & = & H_{n-1}(S^{n-1}) & & \\ & & \|\, f & & \downarrow p_* & & \\ 0 \to \tilde{H}_n(\mathbb{R}P^n) & \to & H_n(\mathbb{R}P^n, \mathbb{R}P^{n-1}) & \to & \tilde{H}_{n-1}(\mathbb{R}P^{n-1}) & \to & \tilde{H}_{n-1}(\mathbb{R}P^n) \to 0. \end{array}$$

with $S^{n-1} \xrightarrow{p} \mathbb{R}P^{n-1}$ and notice

$$H_n(\mathbb{R}P^n, \mathbb{R}P^{n-1}) = H_n(D^n, S^{n-1}) = H_{n-1}(S^{n-1}) = \mathbb{Z}.$$

Theorem. [page 144 of Hatcher]

$$H_q(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & \text{if } q = 0 \\ \mathbb{Z}/2 & \text{if } q = \text{ is odd and } 1 \le q < n \\ \mathbb{Z} & \text{if } q = n \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We have shown the $q = 0$ case. Assume it's true for some $k = n - 1$ with n odd. Look at the above diagram. Assume we've already done the induction from 1 to 2 and know that $H_{n-1}(\mathbb{R}P^{n-1}) = 0$ since $n - 1$ is even. We don't know anything about $\tilde{H}_q(\mathbb{R}P^n)$ or $H_{n-1}(\mathbb{R}P^n)$. However, if you think about it $\tilde{H}_{n-1}(\mathbb{R}P^n) = 0$ and $\tilde{H}_n(\mathbb{R}P^n) = \mathbb{Z}$. We get the diagram (look above and use what we learned):

$$0 \to H_n(\mathbb{R}P^{n-1}) \to \mathbb{Z} \to 0 \text{ (by induction)} \to H_{n-1}(\mathbb{R}P^n) \to 0.$$

Now consider $n = 2$. Then $\tilde{H}_{n-1}(\mathbb{R}P^{n-1}) \cong \mathbb{Z}$. We want to show $\tilde{H}_n(\mathbb{R}P^n) = 0$ and $\tilde{H}_{n-1}(\mathbb{R}P^n) = \mathbb{Z}/2$. What is p_*? (Below, a is the antipodal map and $\mathbb{I}$ the identity)

$$\begin{array}{ccccc} (S^{n-1}, x_0) & \overset{\subseteq}{\to} & (S^{n-1} \vee S^{n-1}, x_0) & \xrightarrow{\mathbb{I} \circ a} & (S^n, x_0) \\ \downarrow p & & & & \downarrow \cong \\ (\mathbb{R}P^{n-1}, y_0) & \to & (\mathbb{R}P^{n-1}, \mathbb{R}P^{n-2}) & \longrightarrow & (\mathbb{R}P^{n-1}/\mathbb{R}P^{n-2}, \mathbb{R}P^{n-2}/\mathbb{R}P^{n-2}). \end{array}$$

Recall

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{p} & \mathbb{R}P^{n-1} \\ \xi \downarrow & & \downarrow \\ S^{n-1} \vee S^{n-1} & \underset{\mathbb{I} \circ a}{\to} & \mathbb{R}P^{n-1}/\mathbb{R}P^{n-2} \cong S^{n-1}. \end{array}$$

where ξ collapses equator to a point, and $\mathbb{I} \circ a$ is the antipodal map followed by the identity map. Now, recall deg$(\mathbb{I} \circ a) = 1 + (-1)^n$, because deg$(\mathbb{I}) = 1$ and deg$(a) = (-1)^n$.

$$H_q(\mathbb{R}P^{n-1}/\mathbb{R}P^{n-2}) = H_q(\mathbb{R}P^{n-1}, \mathbb{R}P^{n-2}).$$

Lecture 24 (March 9, 2009) -

We know $H_q(C, L) = H_q(C \otimes L)$ and $H_q(X; L) = H_q(\Delta(X); L)$. Let R be a PID. A set

$$F_A = \{F : A \to R \mid \text{f.g., nonzero}\}$$

with $f(a) \neq 0$ for only finite # of points. Then we can make F_A into a left R-module

$$(f+g)(a) = f(a) + g(a) \quad \text{and} \quad (rf)(a) = rf(a).$$

Take $i : A \to F_A$ such that $i(a)b = 1$ if $a = b$ and 0 if $a \neq b$. Then

$$f = \sum f(a)\, i(a).$$

Then $F_A \xrightarrow{\varphi} F_B$. Then $\varphi(i(a)) = \sum_{b \in B} c_a^b j(b)$, with $c_a^b = \varphi(i(a))(b)$. Then

$$\varphi(f)(b) = \sum_a c_b^a f(a) \varphi(i(a)) = \sum_{b \in B} c_b^a j(b).$$

Fix an R-module L, with $t(F_a) = \coprod_{a \in A} L = \{g : A \to L\}$ finitely nonzero. Unfortunately, this only works for free modules with a basis.

An R-module is free if $F \cong F_A$ for some A. Put $F_A \xrightarrow{\varphi} F_B$ and $t(F_A) \xrightarrow{t(\varphi)} t(F_B)$ with $t(\varphi)(g) = \sum_c \sum_b c_a^b$. Now,

$$\varphi(f) = \varphi\left(\sum_a f(a) i(a)\right) = \sum_a f(a) \varphi(i(a)) = \sum_a \sum_b f(a) c_b^a j(b).$$

So the matrix for φ is $\sum\limits_a f(a) c_b^a$. Take the matrix for $t\varphi$,

$$\sum_a f(a) c^a.$$

Hence,

$$t(\varphi)(g) = \sum_c \sum_b c_a^b \varphi.$$

Now, t is strongly additive,

$$t(f+g) = t(f) + t(g), \text{ and}$$
$$t(\textstyle\coprod_{i \in I} F_i) = \coprod_{i \in I} t(F_i)$$
$$\{f : \textstyle\bigcup_{i \in I} A_i \to L\}.$$

The problem now is how to extend this from free modules to arbitrary modules. Start with any R-module M. Certainly, there is an F_0, F_1 such that

$$F_1 \to F_0 \xrightarrow{\partial} M \to 0.$$

Define $F_0 \xrightarrow{\partial} M$ so that ∂ is onto. Then $\ker \partial \subset F_0$ is a submodule, and hence free (for R a PID). [Hint: to prove this, use transfinite induction.] Then

$$0 \to F_1 \xrightarrow{\partial} F_0 \to 0$$

and we want (by construction) $H_0(F) = M$ and $H_i(F) = 0$ (for $i > 0$) (i.e., a resolution). By definition, $t_i(M) = H_i(F)$ for $i = 0, 1$. Then

$$t_0(M) = M \otimes L \text{ and } t_1(M) = M * L,$$

the torsion product. We want to be able to compute these things, now.

Proposition. Two short exact resolutions of M are chain homotopy equivalent.

Proof. We have

$$\begin{array}{c} 0 \to F_1 \to F_0 \to M \to 0 \\ f_1 \downarrow \quad f_2 \downarrow \quad \downarrow \mathbb{I}_M \\ 0 \to G_1 \to G_0 \to M \to 0, \end{array}$$

where f_1 and f_2 are natural, with $f_2 = f_1 | F_1$.

Lecture 28 (March 20, 2009) -

Remember problem 20: Take the torus $T^2 = S^1 \times S^1$. Use $U = I \times S^1$ and $V = J \times S^1$. Then $U \cong S^1$ and $V \cong S^1$. For this problem, the MV maps are "obvious".

For problem 21: Take $S^1 \vee ... \vee S^1$ (a disc minus smaller discs). Take

$$U = D^2 - (D_1^2 \cup ... \cup D_n^2) \text{ and } V = D^2 - (D_0^2).$$

Unnamed problem: Show $f, g : S^n \to S^n$ with $\forall x,\ f(x) \neq -g(x)$, are homotopic. Just connect them with a straight line (it can't go through the origin), then project.

Problem 25: We don't have to know anything about the map, just think of groups.

For $\prod_{i=1}^k S^{n_i}$, the rank is $b_q = \text{rank } H_q(X; \mathbb{Z})$ (Betti number), then $\sum_q b_q t^q$.

$$(1 + t^n)(... + b_j t^j + ...) = ... + (b_q + b_{q-n})t^q + ...$$

We know $b_{q-n} = 0$ if $q - n < 0$.

Example. $U \subset I$ for cohomology, C is a free chain complex.

$$0 \to \text{Ext}(H_{q-1}C, L) \to H^q(C; L) \to \text{Hom}(H_q C, L) \to 0$$

is split exact. We would have $f : C_q \to L \in H^q(C; L)$, and $z : H_q C \to L$ in the latter. Suppose we changed z by taking $z + \partial c$. Then $f(z + \partial c) = f(z) + f(\partial c) = f(z)$ because $f(\partial c) = (\partial f)(c)$ but the coboundary is 0.

Recall

$$H_q(\mathbb{R}P^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } q = 0 \\ \mathbb{Z}/2 & \text{if } q \text{ is odd and } 1 \leq q \leq n-1 \\ \mathbb{Z} & \text{if } q = n \text{ and odd} \\ 0 & \text{otherwise.} \end{cases}$$

Similarly,

$$H^q(\mathbb{R}P^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } q = 0 \\ \mathbb{Z}/2 & \text{if } q \text{ is even and } 2 \leq q \leq n \\ \mathbb{Z} & \text{if } q = n \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

Recall

$$H_q(\mathbb{R}P^n, \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & 0 \le q \le n \\ 0 & \text{otherwise.} \end{cases}$$

Take $R = \mathbb{Z}/2$, which is a field! A module over a field is a vector space, so no torsion! Hence,

$$H^q(\mathbb{R}P^n, \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & 0 \le q \le n \\ 0 & \text{otherwise.} \end{cases}$$

We say

$$H^*(\mathbb{R}P^n; \mathbb{Z}/2) = \mathbb{Z}/2\,[\alpha]\,/\,\alpha^{n+1}.$$

Lecture 29 (March 30, 2009) -

Let Λ be a P.I.D. Let $\Delta_q(X, \Lambda)$ be the free Λ-module on the set $\Delta^q \xrightarrow{\sigma} X$. Then the boundary map $\Delta_q \xrightarrow{\partial} \Delta_{q-1}$, and face maps are $\Delta^{q-1} \to \Delta^q$, with $\partial \circ \partial = 0$. Then we form

$$\Delta^p(X; \Lambda) = \text{Hom}_\Lambda(\Delta_q(X; \Lambda), \Lambda)$$
$$c \mapsto (\gamma \mapsto c(\gamma))$$

and we write $c(\gamma) = \langle c, \gamma \rangle$. Define $\Delta^q \xrightarrow{\Gamma} \Delta^{q+1}$ by $\langle \partial c, \alpha \rangle = \langle c, \partial \alpha \rangle$. Define a map $k : H^n X \to \text{Hom}_\Lambda(H_n X, \Lambda)$. Given $x \in H^n X$ and $\xi \in H_n X$, let $z \in Z^n X$ represent x in the co-cycles and $\zeta \in Z_n X$ represent ξ in the cycles. Then $\langle x, \xi \rangle = \langle z, \zeta \rangle$ (**Exercise**), and set $k(x)\xi = \langle x, \xi \rangle$. This is well-defined with $\partial z = \partial \zeta = 0$. We just need to check well-definedness:

$$\langle z + \partial y, \zeta + \partial \eta \rangle = \langle z, \zeta \rangle + \langle z, \partial \eta \rangle + \langle \partial y, \zeta \rangle + \langle \partial y, \partial \eta \rangle =$$
$$\langle z, \zeta \rangle + \langle \partial z, \eta \rangle + \langle y, \partial \zeta \rangle + \langle y, \partial \circ \partial \eta \rangle = \langle z, \zeta \rangle,$$

because z is a co-cycle, ζ is a cycle, and $\partial \circ \partial = 0$.

Theorem A.1. (Special case of Universal Coefficient Theorem) *Let $H_{n-1}X$ be free. Then k is a natural equivalence.*

Proof. Since $\Delta_n X / Z_n \cong B_{n-1} \subset \Delta_{n-1} X$, B_{n-1} is free. Hence, Z_n is a direct summand of $\Delta_n X$, that is, there is a splitting $Z_n \leftrightarrow \Delta_n X$ (with $\leftarrow$ given by r). Hence, for any $f \in \text{Hom}_\Lambda(H_n X, \Lambda)$, $\exists f$ such that $Z_n \to H_n \xrightarrow{f} \Lambda$ with $Z_n \subset \Delta_n X \xrightarrow{F} \Lambda$. ☐

Now, restrict $F \mid B_{n-1} = 0$ to the boundaries, so that $\partial F = 0$ as $\langle \partial F, \alpha \rangle = \langle F, \partial \alpha \rangle$. Hence, $F \in Z^n X$ is a co-cycle. Let F represent $x \in H^n X$. If $\xi \in H_n X$ is represented by $\zeta \in Z_n$, then

$$k(x)\xi = \langle x, \xi \rangle = \langle F, \zeta \rangle = f(\xi).$$

We claim ker $k = 0$. Let $z \in Z^n X$ represent x and assume $k(x) = 0$. We want to show $x = 0$. Then $\forall \zeta \in Z_n$, $\langle z, \zeta \rangle = 0$. **We claim $z \in B^n X$.** For all $\beta \in B_{n-1}$, $\exists \gamma \in \Delta_n X$ such that $\beta = \partial \gamma$. If β is also the boundary of γ_1, that is, $\beta = \partial \gamma_1$, then obviously $\partial(\gamma - \gamma_1) = 0$ so that $\gamma - \gamma_1 \in Z_n X$ (is a cycle) and by assumption that $\langle z, \zeta \rangle = 0$,

$$\langle z, \gamma - \gamma_1 \rangle = 0,$$

so $\langle z, \gamma\rangle = \langle z, \gamma_1\rangle$. Then $z \circ \partial^{-1} : B_{n-1} \to \Lambda$ and $\beta \mapsto \langle z, \gamma\rangle$ is well-defined.

Since $H_{n-1}X$ is free, B_{n-1} is a summand of Z_{n-1}, and hence B_{n-1} is a summand of $\Delta_{n-1}X$. Let $f : \Delta_{n-1}X \to \Lambda$ extend to $z \circ \partial^{-1}$. Then $f \in \Delta^{n-1}X$, and $\forall \sigma$,

$$\langle \partial f, \sigma\rangle = \langle f, \partial\sigma\rangle = \langle z \circ \partial^{-1}, \partial\sigma\rangle = \langle z, \sigma\rangle.$$

Then $\partial f = z \in B^n X$, which is exactly **what we claimed**! (see bold part above) Hence, $x = 0$ in $H^n(X)$. ☐

We want to know $H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\})$. We can't use excission on this, and it turns out

$$H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) = \begin{cases} \Lambda & q = n \\ 0 & q \neq n. \end{cases}$$

We know this already about $H_q(D^n, S^{n-1})$. Then

$$\begin{array}{ccccccccc} \ldots \to & H_q(S^{n-1}) & \to & H_q(D^n) & \to & H_q(D^n, S^{n-1}) & \to & H_{q-1}(S^{n-1}) & \to \ldots \\ & \cong \downarrow & & \downarrow \cong & & \downarrow & & \downarrow \cong & \\ \ldots \to & H_q(\mathbb{R}^n - \{0\}) & \to & H_q(\mathbb{R}^n) & \to & H_q(\mathbb{R}^n, \mathbb{R}^n - \{0\}) & \to & H_{q-1}(\mathbb{R}^n - \{0\}) & \to \ldots \end{array}$$

so by the Five Lemma the penultimate $\downarrow$ is also an isomorphism.

Lecture 30 (April 1, 2009) -

CW spaces (JHC Whitehead).

A CW space is a Hausdorff space partitioned into a collection of $\{e_\alpha\}$ of disjoint subsets such that

(1) $\exists F_\alpha : D^{n(\alpha)} \to X$ with $F_\alpha \mid \overset{\circ}{D}{}^{n(\alpha)}$ a homeomorphism onto e_α.

We can now define the n-skeleton $X^n = \bigcup \{e_\alpha : n(\alpha) \leq n\}$.

(2) If we let $f_\alpha = F_\alpha \mid S^{n(\alpha)-1}$, then this maps into $X^{n(\alpha)-1}$. We say X is finite if there are only finitely many cells. A subset $A \subset X$ is a (finite) CW-subspace if it is closed, and a union of (finitely many) e_α's.

(3) (Closure-Finiteness) Each point in X is contained in a finite subcomplex.

(4) (Weak topology) X has the topology of the direct limit of the finite subcomplexes (which means A is closed $\leftrightarrow$ each finite subcomplex is closed).`

Theorem A2. $\coprod_{\alpha, n(\alpha) = n} H_q(D^n, S^{n-1}) \overset{\amalg F_\alpha}{\longrightarrow} H_q(X^n, X^{n-1})$ is an isomorphism, where

$$H_q(D^n, S^{n-1}) = \begin{cases} 0 & q \neq n \\ \text{free } \Gamma\text{-module} & q = n. \end{cases}$$

Proof. See appendix.

Lecture 31 (April 3, 2009) -

Theorem. $H_n(X) = H_n(C_*)$, where $C_n = H_n(X^n, X^{n-1})$.

$C^n = H^n(X^n, X^{n-1}) = \text{Hom}_\Lambda(C_n, \Lambda)$, the dual of homology (this is cohomology).

Excision

Let $U \subset A \subset X$ with $\overline{U} \subset \overset{\circ}{A}$. We have

$$\text{H}_q(X - U, A - U) \to H_q(X, A)$$

is an isomorphism. By the UCT,

$$\begin{array}{ccccccccc} 0 \to \text{Ext}(H_{q-1}(X; A), L) & \to & H^q(X, A; L) & \to & \text{Hom}(H_q(X, A), L) \to 0 \\ = \downarrow i^*| & & = \downarrow\ i^* & & = \downarrow\ (i_*)^* \\ 0 \to \text{Ext}(H_{q-1}(X - U, A - U), L) & \to & H^q(X - U, A - U; L) & \to & \text{Hom}(H_q(X - U, A - U), L) \to 0. \end{array}$$

By the Five Lemma, the middle is also an isomorphism.

Cup product

Consider $H^p(X, A) \otimes H^q(X, B) \to H^{p+q}(X, A \cup B)$ given by $u \otimes v \mapsto u \cup v$.

Properties. (1) First, we have a map $f : X \to Y$. We get $f^*(u \cup v) = f^*u \cup f^*v$.

(2) It is bilinear and associative. This covers a host of identities, like

$$r(u \cup v) = (ru) \cup v - u \cup (rv)$$
$$(u_1 + u_2) \cup v = u_1 \cup v + u_2 \cup v$$
$$(u \cup v) \cup w = u \cup (v \cup w).$$

(3) We say $u \cup v = (-1)^{pq}\, u \cup v$.

(4) Think of $i : H \hookrightarrow X$, and let $u \in H^p(A)$. Then $\partial u = H^{p+1}(X, A)$. Let $v \in H^q(X)$. Then $\partial u \cup v = \partial(u \cup i^*v)$. Further, $i^*v \in H^q(A)$.

(5) We specify $u_0 \in H^0(X) = \text{Hom}(H_0(X), \Lambda)$ where $u_0 = (1 \mapsto 1)$. Hence, $u_0 \cup v = v$ and it acts as the identity.