Lecture 16

Theorem. Let R be an affine k-algebra (quotient of a polynomial ring). Then

$$\dim R = \text{tr deg}_k\, R = \text{tr deg}_k\, Q(R)\ )$$

Proof. Let r be the transcendence degree over k of R. We will prove $r \geq \dim R$. By the Going-Up Theorem, $R = k[x_1, ..., x_n]/\mathfrak{p}$. If $r = 0$, then that implies R is a field, so that dim $R = 0$. Let $S = k[x_1, ..., x_n]$. Then it suffices to show if $P \subset Q \subset S$ with $P \neq Q$ then $S/P \to S/Q$ surjectively.

We claim that tr $\deg_k S/Q <$ tr $\deg_k S/P$. By surjection, the inequality $\leq$ is apparent. So assume we have equality. Write $S/Q = k[\beta_1, ..., \beta_n]$ and $S/P = k[\alpha_1, ..., \alpha_n]$, where β_i and α_i are the appropriate images of $x_1, ..., x_n$. Let $m =$ tr $\deg_k S/Q$. Then $\beta_1, ..., \beta_m$ form a transcendence basis over k for S/Q an that implies $\alpha_1, ..., \alpha_m$ form a transcendence basis over k for S/P. Now pick the multiplicative system $T = k[x_1, ..., x_n] - \{0\} \subset S$. We woul dlike to localize. Notice $T \cap P = \emptyset$ and $T \cap Q = \emptyset$; otherwise, the $\alpha_1, ..., \alpha_m$ and $\beta_1, ..., \beta_m$ wouldn't be algebraically independent. Then $T^{-1}S = k(x_1, .., x_m)[x_{m+1}, ..., x_n]$. Then

$$T^{-1}S/P(T^{-1}S) = k(\alpha_1, ..., \alpha_m)[\alpha_{m+1}, ..., \alpha_n]$$

and it

$$\text{ht}\, p + \text{coht}\, p = \dim\, R.$$

Proof. By Noether normalization,

$$k \subseteq k[Z_1, ..., Z_r] \subseteq R,$$

with $r = \text{tr deg}_k\, R = \dim R$. Let $\text{ht}\, p = h$. By homework exercise, $R \subset S \subseteq Q$ with $P = Q \cap R$ and $R \subset S$ an itnegral extension, dim $R =$ dim S, $\text{ht}\, p = \text{ht}\, Q$, and coht $P =$ coht Q. We can assume $R = k[Z_1, ..., Z_r]$.

Hint: The previous argument shows that $\exists y_1, ..., y_r$ such that R is integral over $k[y_1, ..., y_r]$ having the property $p \cap k[y_1, ..., y_r] = (y_1, ..., y_h)$ (improved version of Noether normalization). Then ht$(y_1, ..., y_r) = h$, coht$(y_1, ..., y_r) = r - h$ so the sum is r.

Lecture 17

Graded rings and modules

If A^N is a graded ring, S a collection of groups, $(S_d)_{d \in \mathbb{N}}$ such that $S = \bigoplus_{d > 0} S_d$ homogeneous of degree d, and $S_d S_e \subseteq S_{d+e}$. In part, S_0 is a ring, S is an S_0-algebra.

Example. If $S = R[x_1, ..., x_n]$ is graded, deg $R = 0$ and deg $x_i = 1$ with

$$S = \bigoplus_{d \geq 0} R[x_1, ..., x_n]_d,$$

where each term is the ring of homogeneous polynomials of degree d. There exist many other gradings on polynomial rings, by assigning deg $x_i = e_i \in \mathbb{N}$.

Example. Look at $S = k[x_1, ..., x_n]/\mathcal{I} = \bigoplus_{d \geq 0} k[x_1, ..., x_n]_d \,/\, \mathcal{I}_d$ where $\mathcal{I}$ is a homogeneous ideal (generated by homogeneous elements).

Fix S graded. Then a graded S-module M is a collection of Abelian groups $\{M_e\}_{e \in \mathbb{N}}$ such that $M = \bigoplus_{e \geq 0} M_e$. The operation $S_d M_e \subseteq M_{d+e}$. In part, each M_e is an S_0-module.

Example. $M = k[x_1, ..., x_n]/\mathcal{I}$ is a graded module over $k[x_1, ..., x_n]$.

We will now introduce the Hilbert polynomial and function.

Definition. The function $f : \mathbb{N} \to \mathbb{Q}$ is called <u>polynomial-like</u> if there exists a polynomial $P \in \mathbb{Q}[x]$ such that $f(n) = P(n)$ for $n \gg 0$. Furthermore, deg $f =$ deg P.

Lemma. For $f : \mathbb{N} \to \mathbb{Q}$ a function, define $\Delta f : \mathbb{N} \to \mathbb{Q}$ to be $\Delta f(n) = f(n+1) - f(n)$. Then f is polynomial-like of degree r if and only if Δf is polynomial-like of degree $r - 1$. (deg $0 =$ -1)

Proof. First, for all $p \geq q \in \mathbb{N}$, $f(p) - f(q) = \sum_{k=q}^{p-1} \Delta f(k)$. Furthermore, for every $r \in \mathbb{N} - \{0\}$, $\Delta\left(\frac{n!}{r!}\right) = \frac{r!n!}{(r-1)!}$. We can use these two facts to obtain the lemma. $\square$

Note: For a finitely-generated graded module M decomposable into submodules, we can always assume the generators of M are homogeneous.

Theorem. Let $S = \bigoplus_{d \geq 0} S_d$ be a graded ring such that $S_0 = k$ a field, and S is finitely generated over k (as an algebra) by $a_1, \ldots, a_r \in S_1$. Then for every finitely generated graded module $M = \bigoplus_{n \geq 0} M_n$ over S, the function $h_M(n) := \dim_k M_n$ is polynomial-like of degree less than r.

Proof. We can use induction on r. If $r = 0$, then $S = S_0 = k$ is a field. Take M to be a finitely generated module, then say by $x_1, \ldots, x_k$, deg $d_1 \leq \ldots \leq d_k$. That implies $M_n = 0$ for all $n > d_k$ so $h_M(n) = 0$ (degree -1).

Now assume $r > 0$. Consider $\varphi_r : M \to M$ given by multiplication by a_r. Then $a_r \in S_1$ (has degree 1), so $\varphi_r(M_n) \subseteq M_{n+1}$. Then for all n, we have an exact sequence

$$0 \to K_n = \ker(\varphi_r) \to M_n \xrightarrow{\varphi_r} M_{n+1} \to C_n = \operatorname{co\,ker}(\varphi_r) \to 0.$$

Then $K := \bigoplus_{n \geq 0} K_n$ and $C := \bigoplus_{n \geq 0} C_n$ are graded modules over S. Then $C \subseteq M \twoheadrightarrow K$ so that both C and K are finitely generated algebras over $R \rightsquigarrow h_C(n), h_K(n)$ are well-defined, so that $\dim_k K_n - \dim_k M_n + \dim_k M_{n+1} - \dim_k C_n = 0$. Hence,

$$\Delta h_M(n) = h_M(n+1) - h_M(n) = h_C(n) - h_K(n).$$

Then by construction, $a_r \cdot K = 0$ and $a_r \cdot C = 0$. So in fact, K and C are graded modules over $S' = k[a_1, \ldots, a_{r-1}] \subsetneq S$. Then by induction, h_C and h_K are polynomial-like of degree $\leq r - 2$ so that Δh_M is as well and hence h_M is polynomial-like of degree less than r by our lemma. $\square$

Definition. The function h_M given in the previous theorem is the <u>Hilbert function</u> of M. If $h_M(n) = P_M(n)$ for $n \gg 0$, P_M is the <u>Hilbert polynomial of M</u>.

Example. If $S = k[x_1, \ldots, x_n] = \bigoplus_{m \geq 0} S_m$, then $S_m = k[x_1, \ldots, x_n]_m = \{$space of homogeneous polynomials of degree m $\}$ and

$$h_S(m) = \binom{n-1+m}{m} = \binom{n-1+m}{n-1} = \frac{(m+n-1)\cdot\ldots\cdot(m+1)}{(n-1)!} = \frac{1}{(n-1)!}m^{n-1} + \underbrace{\mathcal{O}(m^{n-2})}_{\text{remainder}}.$$

Remark: Notice $\dim S = \deg h_S + 1$.

Lecture 18

Artinian Rings

Definition. A ring R is Artinian if it satisfies the descending chain condition (DCC) on ideals, i.e., there exists a decreasing chain of ideals $I_1 \supseteq ... \supseteq I_m \supseteq ...$ so that there exists an $n \in \mathbb{Z}^+$ such that the chain stabilizes after n, that is, $I_n = I_{n+1} = ...$ holds. The same definition holds for modules with respect to inclusions of submodules.

Examples. (1) $\mathbb{Z}$ is not Artinian.

(2) $\mathbb{Z}/d\mathbb{Z}$ is Artinian.

(3) $k[x_1, ..., x_n]/(x_1, ..., x_n)^m$ with $m \geq 1$ is Artinian.

(4) Product of fields $k_1 \times ... \times k_r$ for $r \geq 2$ and k_i fields.

Lemma 1. If R is Artinian and a domain, then R is a field.

Proof. Pick $a \in R$. Then we have a chain $(a) \supseteq (a^2) \supseteq ... \supseteq (a^m) \supseteq ...$. By the DCC, there exists an n such that $(a^n) = (a^{n+1})$ which implies there is a $b \in R$ so that $a^n = ba^{n+1}$, which means $a^n(1 - ba) = 0$, so that a has an inverse b. □

Lemma 2. If R is Artinian, then every prime ideal in R is maximal, and there are only finitely many.

Proof. If $\underline{p} \subseteq R$ is a prime, then $R/\underline{p}$ is Artinian and a domain, so by the previous lemma, it is a field, and hence $\underline{p}$ is maximal. To show there are finitely many, notice the family

$$\{\underline{m}_1 \cap ... \cap \underline{m}_k \,|\, \underline{m}_i \text{ maximal in } R\}$$

has a minimal element with respect to inclusion. Now say $I = \underline{m}_1 \cap ... \cap \underline{m}_k$ is minimal. Then take $\underline{m} \subseteq R$ to be maximal. Then $m \cap I = \underline{m} \cap \underline{m}_1 \cap ... \cap \underline{m}_k \in \mathcal{F}$. But $m \cap I \subseteq I$ is minimal so that $m \cap I = I$. But then $\underline{m} \subseteq \underline{m}_1 \cap ... \cap \underline{m}_k$ where $\underline{m}$ and each $\underline{m}_i$ are prime. Hence, $\exists i$ such that $\underline{m} = \underline{m}_i$. □

Remark: We can use this lemma to show that all Artinian rings are a finite product of local Artinian rings. (i.e., Chinese Remainder Theorem).

Definition. If R is a ring and $M \neq 0$ is an R-module, then M is <u>simple</u> if it has no submodules different from 0 and itself. Then $Rx \subseteq M$ for M simple implies $Rx \cong M$, and hence $Rx \cong R/\text{Ann}(x)$. Hence M is simple if and only if $\text{Ann}(x)$ is maximal. Hence, M simple implies $M \cong R/m$ for some maximal ideal m.

Definition. A <u>composition series</u> of M is a finite filtration:

$$M = M_0 \supseteq M_1 \supseteq ... \supseteq M_n = 0$$

such that M_i/M_{i+1} is simple for all $i = 0, ..., n - 1$.

Jordan-Hölder Theory

If the composition series exists, then the length of any two is the same:

$$\ell_R(M) = \text{length}(M) = \begin{cases} \text{length of any such series} & \text{if a composition series exists} \\ \infty & \text{otherwise} \end{cases}.$$

Furthermore $\ell_R(M) < \infty$ if and only if M is Artinian and Noetherian. Also,

$$0 \to M \to N \to P \to 0 \text{ implies } \ell_R(N) = \ell_R(M) + \ell_R(P)$$

for an exact sequence of R-modules. If M is a k-vector space, then $\ell(M) = \dim_k M$.

Example. For $(R, \underline{m})$,

$$R \supseteq \underline{m} \supseteq \underline{m}^2 \supseteq \ldots$$

$$R/\underline{m} \oplus \underline{m}/\underline{m}^2 \oplus \underline{m}^2/\underline{m}^3 \oplus \ldots$$

$$\underline{m} \supseteq I \supseteq \underline{m}^k \;\Rightarrow R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \ldots$$

$\underline{m}^k/\underline{m}^{k+1}$ has finite length ($\dim_{R/\underline{m}} \underline{m}^k/\underline{m}^{k+1} < \infty$). Then $\underline{m}^n = \underline{m}^{n+1}$ implies that $\underline{m}^n = 0$ by Nakayama's Lemma. Then

$$\ell(R) = \ell(R/\underline{m}) + \ell(\underline{m}/\underline{m}^2) + \ldots + \ell(\underline{m}^{n-1}/\underline{m}^n).$$

Then $\underline{m} = (x_1, \ldots, x_n)$ (a system of parameters), and $\underline{m}^k/\underline{m}^{k+1} = \{$ homogeneous polynomials of degree k in n variables $\}$. Then $\dim_{R/\underline{m}} \underline{m}^k/\underline{m}^{k+1} = \binom{n-1+k}{k}$.

Proposition. For M a finitely-generated module and R a Noetherian ring, the following are equivalent:

(1) $\ell_R(M) < \infty$

(2) All primes in $\text{Ass}(M)$ are maximal.

(3) All primes in $\text{Supp}(M)$ are maximal.

Remark: Notice this implies $\text{Ass}(M) = \text{Supp}(M)$

Proof. [(1) $\Rightarrow$ (2)] By our earlier lemma, there is a filtration $M = M_0 \supseteq \ldots \supseteq M_n = 0$ such that $M_{i-1}/M_i \cong R/\underline{p}_i$ for $\underline{p}_i$ prime, with $\text{Ass}(M) \subseteq \{p_1, \ldots, p_n\}$, and

$$\infty > \ell_R(M) = \sum \ell_R(M_{i-1}/M_i) = \sum \ell_R(R/\underline{p}_i).$$

But then $\infty > \ell_R(R/\underline{p}_i)$ so that $R/\underline{p}_i$ is an Artinian R-module, and it must also be a domain. Hence $\underline{p}_i$ is maximal by the earlier lemma.

[(2) $\Rightarrow$ (3)] We know $\text{Ass}(M) \subseteq \text{Supp}(M)$, and they have the same minimal primes. Pick a prime $Q \in \text{Supp}(M)$. Whether or not it is minimal, $\exists P \subseteq Q$ that is minimal, so this means that $P \in \text{Ass}(M)$ meaning it is maximal, and hence Q is maximal.

[(3) $\Rightarrow$ (1)] Exercise: $\forall \underline{p}_i$ they are contained in $\text{Supp}(M)$. If $\underline{p}_i$ are all maximal, then $R/\underline{p}_i$ is all fields, so $\ell_R(R/\underline{p}_i) = 1$ and hence we have a composition series, and $\ell_R(M) = n < \infty$. ☐

Lecture 19

Theorem A. Let R be a Noetherian ring. The following are equivalent:

(i) R is Artinian.

(ii) Every prime is maximal.

(iii) Every associated prime is maximal.

Proof. We know (i) implies (ii) from lemma 2 last time; (ii) implies (iii) is obvious; and (iii) implies (i) is true by (2) implies (1) in the proposition from last time. □

Theorem B. A ring R is Artinian if and only if $\ell_R(R) < \infty$.

Proof. Let $\ell_R(R) < \infty$. Then obviously R is Artinian and Noetherian. Now we claim there exist maximal ideals $\underline{m}_1, ..., \underline{m}_k$ such that $\underline{m}_1 \cdot ... \cdot \underline{m}_k = 0$ (since then $\underline{m}_1 ... \underline{m}_k \supseteq \underline{m}_1 ... \underline{m}_k \underline{m}_{k+1}$ has to stop by the descending chain condition, so apply Nakayama's Lemma). We have $R \supseteq \underline{m}_1 \supseteq \underline{m}_1 \underline{m}_2 \supseteq ... \supseteq \underline{m}_1 ... \underline{m}_k = 0$. Then each $N_i = \underline{m}_1 ... \underline{m}_{i-1} / \underline{m}_1 ... \underline{m}_i \rightsquigarrow R/\underline{m}_i$-moduli (vector space). Notice $IM = 0 \Longrightarrow M$ is an R/I-module. Also, $\ell_{R/\underline{m}_i}(N_i) < \infty$ implies $\ell_R(N_i) < \infty$ (because R is Artinian), and then the fact ℓ_r is additive in filtrations implies $\ell_R(R) < \infty$. □

Theorem C. A ring R is Artinian if and only if R is Noetherian and every prime ideal is maximal.

Proof. We proved the adverse in theorem A. By theorem B, $\ell_R(R) < \infty$ so that R is Noetherian, and then by Theorem A we know each prime ideal is maximal. □

Hilbert function and dimension

We can now look at graded rings of the form $S = \bigoplus_{d \geq 0} S_d$ with S_0 Artinian. Then there exists a Hilbert polynomial of positive degree such that S is generated by S_1/S_0.

Definition. If $(R, \underline{m})$ is a local ring, then an <u>ideal of definition</u> for R is $I \subseteq R$ such that there exists a $k \geq 1$ with $\underline{m}^k \subseteq I \subseteq \underline{m}$.

Lemma. An ideal I is of definition if and only if R/I is Artinian.

Proof. (Sketch) I is an ideal of definition if and only if $\text{rad}(I) = \underline{m}$ (so there does not exist non-maximal primes containg I). □

Definition. If $I \subseteq (R, \underline{m})$ is an ideal of definition with M a finitely-generated R-module, then the <u>associated graded ring</u> $\text{gr}_I(R) = \bigoplus_{n \geq 0} I^n/I^{n+1}$. The <u>associated graded module</u> $\text{gr}_I(M) = \bigoplus_{n \geq 0} I^n M / I^{n+1} M$.

Remark. If $a_1, ..., a_r$ are generators for I, then $\overline{a}_1, ..., \overline{a}_r$ generate I^m/I^2.

$$\text{gr}_I(R) \text{ over } qr_0 = R/I.$$

- R/I is Artinian, as before.
- If M/IM is finitely generated over R/I then it is Artinian, which implies for all $k \geq 1$, $\ell_R(R/I^k) < \infty$, $\ell_R(M/IM) < \infty$ and so • $\ell_R(I^{k-1}M/I^kM) < \infty$ (I^k is also an ideal of definition).
- $h_{gr_I(M)}(n) = \ell_R(I^nM/I^{n+1}M)$. By the Hilbert polynomial theorem, this is polynomial-like of degree $\leq r-1$ (for $I = (a_1, ..., a_r)$).

Definition. The Hilbert-Samuel function of M (with respect to I) is

$$S_M^I(n) = \ell_R(M/I^nM) < \infty.$$

Proposition. The Hilbert-Samuel function is polynomial-like of degree $\leq r$.

Proof. There exists an exact sequence

$$0 \to I^nM/I^{n+1}M \to M/I^{n+1}M \to M/I^nM \to 0.$$

So that for all n, $\Delta S_M^I(n) = S_M^I(n+1) - S_M^I(n) = h_{\mathrm{gr}_I(M)}(n)$ and so by the earlier bullet point statement, S_M^I is polynomial-like of degree $\leq r$. (where ΔS_M^I is as defined in the lemma in Lecture 17) ☐

Proposition. The degree of $S_M^I(n)$ does not depend on I (call it $d(M)$).

Proof. Start with the fact I is an ideal of definition, i.e., there is a k such that $\underline{m}^k \subseteq I \subseteq \underline{m}$. Then we can look at S_M^I and $S_M^{\underline{m}}$, and if we can prove they are equal we're done since the latter is ideal invariant. For each $p \geq 1$, we get $\underline{m}^{kp} \subseteq I^p \subseteq \underline{m}^p$. Then $S_M^{\underline{m}}(kp) \geq S_M^I(p) \geq S_m^{\underline{m}}(p)$ for every p, so deg $S_M^I =$ deg $S_M^{\underline{m}}$. ☐

Lecture 20

Proposition. Setting as above [last time], for any exact sequence of finitely generated R-modules, $0 \to M' \to M \to M'' \to 0$, we have $S_{M'}^I(n) + S_M^I(n) = S_M^I(n) + r(n)$ where $r(n)$ is polynomial like of degree $< d(M)$, with non-negative leading coefficients.

Proof. We have an exact sequence

$$0 \to M'/(M' \cap I^nM) \to M/I^nM \to M''/I^nM'' > 0.$$

Let's say $M'_n := M' \cap I^nM$. From the above sequence, we get by the additivity of the Hilbert function that $\ell_R(M'/M'_n)$ (implies $\ell_R(M'/M'_n)$ is polynomial-like). Now notice for all m, $I^{n+m}M' \subseteq I^{n+m}M \cap M' = M'_{n+m}$ (since $M' \subset M$). The Artin-Reese lemma states there exists an m such that for each $n \geq m$, $IM'_n = M'_{n+1}$ with $(I^k(M' \cap I^nM)) = M' \cap I^{n+k}M$. Hence, we get $I^{n+m}M' \subseteq M'_{n+m} = I^nMm'$ [Artin-Reese Lemma] $\subseteq I^nM'$. Therefore, $\ell_R(M'/I^{n+m}M') \geq \ell_R(M'/M'_{n+m}) \geq \ell_R(M'/I^nM')$. Notice the first term in this inequality equals $S_{M'}^I(n+m)$ and the latter $S_{M'}^I(n)$. Then make

$n \to \infty$ and we get that $S^I_{M'}(n)$ and $\ell_R(M'/M'_n)$ have the same degree and same leading coefficient. Then define $r(n) \coloneqq \ell_R(M'/M'_n) - S^I_{M'}(n)$. This is a polynomial-like term of degree $< d(M') \leq d(M)$ with a non-negative leading coefficient. ☐

Let M be a finitely generated module over R. Then

$$\text{dim R} = \begin{cases} \text{dim}(R/\text{Ann}(M)) & \text{if } M \neq 0 \\ -1 & M = 0 \end{cases}.$$

Lemma. The following are equivalent:

(1) dim $M = 0$ (2) $\ell_R(M) < \infty$ (3) All primes $\underline{p} \in$ Supp(M) are maximal.

(4) All associated primes $\underline{p} \in$ Ass(M) are too.

Definition. If $(R, \underline{m})$ is a Noetherian local ring with M finitely generated over R, the <u>Chevalley dimension</u> of M is

$$\delta(M) \coloneqq \min\{r \in \mathbb{N} \mid \exists a_1, ..., a_r \in \underline{m} \text{ s.t. } \ell_R(M/(a_1, ..., a_r)M) < \infty\}.$$

This definition makes sense because $\ell_R(M/\underline{m}M) < \infty$.

Theorem. (*Dimension Theorem*) If M is finitely generated over $(R, \underline{m})$ a Noetherian local ring, then dim $M = d(M) = \delta(M)$.

Corollary 1. The dim $M < \infty$ for any M a finitely generated module over R. In particular, dim $R < \infty$.

Corollary 2. Each $\underline{p} \subseteq R$ prime has finite height, so the set of primes in R satisfy the descending chain condition.

Proof. dim $R_{\underline{p}} =$ ht $\underline{p}$. ☐

Corollary 3. dim $R \leq$ dim$_k$ $\underline{m}/\underline{m}^2$ where $k = R/\underline{m}$ (embedding dim of R).

Proof. If $\overline{a_1}, ..., \overline{a_r}$ is a basis of $\underline{m}/\underline{m}^2$, then $a_1, ..., a_r$ generate $\underline{m}$ so by corollary 1, dim $R \leq r$. ☐

Corollary 4. The dim $k[[x_1, ..., x_n]] = n$ for k a field. Then $(x_1, ..., x_n) = \underline{m}$ implies by corollary 1 that dim $R \leq m$. Furthermore, $(0) \subseteq (x_1, x_2) \subseteq ... \subseteq (x_1, ..., x_n)$ implies dim $R \geq n$.

Lecture 22

Theorem. (*Generalized Krull principal ideal theorem*) If R is a Noetherian local ring and $\underline{p} \subseteq R$ is a prime, the following are equivalent:

(1) ht $\underline{p} \leq n$ (# of generators).

(2) $\exists$ ideals $I \subset R$ generated by n elements such that $\underline{p}$ is minimal over I.

Proof. [(1) => (2)] We have dim $R_{\underline{p}} =$ ht $\underline{p} \leq n$. Then there exists $J \subseteq R_{\underline{p}}$ generated by $\left(\frac{a_1}{s}, ..., \frac{a_n}{s}\right)$, $a_i \in R$ such that J is an ideal of definition for $R_{\underline{p}}$. But then

$$(\underline{p}\, R_{\underline{p}})^k \subseteq J \subseteq \underline{p}\, R_{\underline{p}} \Leftrightarrow J \text{ is } \underline{p}\, R_{\underline{p}}\text{-primary},$$

so that $I = (a_1, ..., a_n) \subseteq \underline{p}$ a minimal prime. So then in $R_{\underline{p}}$, $IR_{\underline{p}}$ is $\underline{p}R_{\underline{p}}$-primary which means $IR_{\underline{p}}$ is an ideal of definition so that dim $R_{\underline{p}} \leq n$.

Theorem. (*Krull principal ideal theorem*) If R is Noetherian with $x \notin Z(R)$ and $x \notin R^*$, then for every minimal prime $\underline{p}$ over (x), ht $\underline{p} = 1$.

Proof. Since $x \notin R^*$, by the previous theorem ht $\underline{p} \leq 1$. Assume ht $\underline{p} = 0$. But we know that $R\underline{p} \neq 0$. Notice if $\frac{x}{1} = 0 \in R_{\underline{p}}$ then $\exists s \notin p$ such that $sx = 0$, but this is impossible since $x \notin Z(R)$. Since $Z(R) = \bigcup_{p \in \text{Ass}(R)} \underline{p}$, we have $x \in \underline{p} \subseteq Z(R)$, our desired contradiction. ☐

Definition. Let $(R, \underline{m})$ be a Noetherian local ring with M a finitely-generated R-module and dim $M = n$. Then a <u>system of parameters</u> for M is a set $\{a_1, ..., a_n\} \subseteq \underline{m}$ such that $\ell_R(M/(a_1, ..., a_n)M) < \infty$. (exists because dim $M = \delta(M)$)

Examples. (1) Let $I = (a_1, ..., a_n)$ be an ideal of definition. Then $\{a_1, ..., a_n\}$ is a system of parameters.

(2) $\{x_1, ..., x_n\} \subseteq k[[x_1, ..., x_n]]$ is a system of parameters.

Theorem. Take M to be a finitely generated module over a Noetherian local ring. Take $a_1, ..., a_t \in \underline{m}$. Then dim $M/(a_1, ..., a_t)M \geq$ dim $M - t$. In addition, we have equality if and only if $\{a_1, ..., a_t\}$ is part of a system of parameters.

Proof. Let $a \in M$ and define $N := M/aM$. Let $r =$ dim $N = \delta(N)$. Then $\exists$ $b_1, ..., b_r \in R$ such that $\ell_R(N/(b_1, ..., b_r)) < \infty$. But $N/(b_1, ..., b_r)N \cong M/(a, b_1, ..., b_r)$. So then $\delta(M) \leq r + 1 = \delta(M/aM) + 1$.

Now use induction on t. Start with $P = M/(a_2, ..., a_t)M$. By induction, dim $P \geq$ dim $M + (t - 1)$. For equality, [...see proof in book]

Examples. (1) $\{a\}$ is an M-sequence if and only if $a \notin \mathfrak{J}(M)$.

(2) In $k[x_1, ..., x_n]$ or $k[[x_1, ..., x_n]]$, $\{x_1, .., \}$

Lecture 23

Theorem. If M is a finitely generated module over $(R, \underline{m})$ a Noetherian local ring, and if $a_1, ..., a_t$ is an M-regular sequence, then $\{a_1, ..., a_t\}$ is part of a system of parameters.

Proof. By induction on t, for $t = 1$ we have dim $M/a_1M =$ dim $M - 1$. So by one of the theorem from earlier, $\{a_i\}$ is part of a system of parameters. If $t > 1$, then assume $\{a_1, ..., a_{t-1}\}$ is an M-regular sequence which is part of a system of parameters. Then dim $M/(a_1, ..., a_t)M =$ dim $M - (t-1)$. Hence, dim $M/(a_1, ..., a_t)M =$ dim $M/(a_1, ..., a_{t-1})M - 1 =$ dim $M - t + 1 - 1 =$ dim $M - t$. Again by the theorem from last time, this means $\{a_1, ..., a_t\}$ is part of a system of parameters. ☐

Depth. Let M be a finitely generated module over $(R, \underline{m})$. The *depth* of M in R (or $\underline{m}$) is the supremum over the length of all M-regular sequences, i.e., sup $\{t \mid \{a_1, ..., a_t\}$ an M-regular sequence$\}$.

Note: Later, we will see the depth equals the length of any maximal M-regular sequence.

Proposition. depth $M \leq$ dim M.

Proof. Every M-regular sequence extends to a system of parameters.

Definition. A module M as above is *Cohen-Macaulay* (CM) if depth $M =$ dim M.

A Noetherian local ring $(R, \underline{m})$ is CM if it is CM over itself.

Proposition. If M is a finitely generated module over Noetherian R, then if $\{a_1, ..., a_t\}$ is such that a^k is M-regular, then the sequence contained in $\mathfrak{J}(R) = \bigcup_{\underline{m} \subset R} \underline{m}$, and then any permutation is again an M-regular sequence. In part, if $(R, \underline{m})$ is local, then any permutation of any M-regular sequence is an M-regular sequence.

Proof. It is enough to prove that $\{a_2, a_1, ..., a_t\}$ is an M-regular sequence. We need to prove that $a_2 \notin Z(M)$, and $a_1 \notin Z(M/a_2M)$. Then say there exists an $x \in M$ such that $a_1\overline{x} = 0$ if and only if $a_1x \in a_2M$ meaning $\exists y \in M$ such that $a_1x = a_2y$. Then $y \in a_1M$ so $\exists z$ such that $y = a_1z$. But then $a_1 = a_1a_2z$ so that $a_1(x - a_2z) = 0$, but $a_1 \notin Z(M)$ so that $x = a_2z \in a_2M$ so $\overline{x} = 0$. ☐

Definition. A Noetherian local ring $(R, \underline{m})$ is *regular* if the maximal ideal $\underline{m}$ can be generated by $a_1, ..., a_r$, where $r =$ dim R.

Examples. (1) If dim $R = 0$, then R is regular if and only if R is a field.

(2) If dim $R = 1$, then R is regular if and only R is a discrete valuation ring.

(3) If $R = k[[x_1, ..., x_n]]$ is regular local then $x_1, ..., x_n$ must be a regular system of parameters.

(4) For X an algebraic variety, $x \in X$ is smooth if and only if $\mathcal{O}_{X,x}$ is a regular local ring.

(5) If $R = k[X,Y](Y^2 - X^3)$ is a cusp, then dim $R =$ dim $k[X,Y] - 1 = 1$.

Lecture 24

Theorem 1. If R is a regular local ring then R is a domain.

Theorem 2. If $(R, \underline{m})$ is a regular local ring of dim r with $a_1, ..., a_t \in \underline{m}$ for $1 \le t \le r$, then the following are equivalent:

(1) $a_1, ..., a_t$ can be extended to a regular system of parameters.

(2) $\overline{a_1}, ..., \overline{a_t}$ are linearly independent over k in $\underline{m}/\underline{m}^2$.

(3) $R/(a_1, ..., a_t)$ is a regular local ring.

Proof. [(1) $\Longleftrightarrow$ (2)] By Nakayam's Lemma, $a_1, ..., a_t, b_{t+1}, ..., b_r$ is a regular system of parameters if and only if $\overline{a_1}, ..., \overline{a_t}, \overline{b_{t+1}}, ..., \overline{b_r}$ is a basis for $\underline{m}/\underline{m}^2$.

[(1) $\Longrightarrow$ (3)] Say $\{a_1, ..., a_t, b_{t+1}, ..., b_r\}$ is a regular system of parameters. Then for any system of parameters, by an older theorem, dim $R/(a_1, ..., a_t) = r - t$. So then $\{\overline{b_{t+1}}, ..., \overline{b_r}\}$ generate a maximal ideal in $R/(a_1, ..., a_t)$ so that $R/(a_1, ..., a_t)$ is regular.

[(3) $\Longrightarrow$ (1)] We have $R/(a_1, ..., a_t)$ regular so that $\{\overline{b_{t+1}}, ..., \overline{b_r}\}$ is a regular system of parameters. So then pick any $x \in \underline{m}$, so that $\overline{x} = \sum_{j=t+1}^{r} c_j \overline{b_j}$ for some c_j, so that $x - \sum c_j b_j \in (a_1, ..., a_r)$. Hence, $x = \sum c_j b_j + \sum c_i a_i$ so $x \in (a_1, ..., a_t, b_{t+1}, ..., b_r) = \underline{m}$. □

Proof. (of Theorem 1) We will prove by induction on $r =$ dim R. If $r = 0$, then R is a field and if $r = 1$ then R is a discrete value ring. If $r > 1$, $\exists x \in \underline{m}/\underline{m}^2$. Let the minimal primes of R be $\underline{p}_1, ..., \underline{p}_t$ (want all $\underline{p}_i = 0$). Then we can also assume $x \notin \underline{p}_i$ $\forall i$. If $\underline{m} \subseteq \underline{m}^2 \cup \underline{p}_1 \cup ... \cup \underline{p}_t$, then $\underline{m} \subseteq \underline{m}^2$ or $\underline{m}_i \subseteq \underline{p}_i$ for some i. Now look at $R/(x)$. Then $0 \ne \overline{x} \in \underline{m}/\underline{m}^2$. By Theorem 2, $R/(x)$ is regular, but dim $R/(x) = r - 1$, so inductively, this is a domain. Then since (x) is prime, $\exists i$ s.t. $\underline{p}_i \subseteq (x)$ so we claim $\underline{p}_i = x p_i$ for $x \in \underline{m}$, and by Nakayama's Lemma, $\underline{p}_i = 0$. Then we claim $y \in \underline{p}_i$ implies $\exists z$ such that $y = zx$ with $x \notin \underline{p}_i$ so that $z \in \underline{p}_i$. □

Theorem. Let $(R, \underline{m})$ be a Noetherian local ring. Then R is regular if and only if $\underline{m}$ can be generated by a regular sequence. In addition, the length of any such regular sequence is equal to dim R.

Proof. If R is regular, take $\{a_1, ..., a_r\}$ to be regular for any system of parameters. Then for all t, by Theorem 2 we have $R/(a_1, ..., a_t)$ is regular, so by Theorem 1, $R/(a_1, ..., a)$ is a domain. So hence $a_{t+1} \notin \mathcal{Z}(R/(a_1, ..., a_t))$. On the other hand, let $\underline{m} = (a_1, ..., a_s)$. Then by the previous theorem $\{a_1, ..., a_s\}$ is part of a system of parameters. So then $0 =$ dim $R/\underline{m} =$ dim $R - s = r - s$. Then $s = r$ implies R is regular.

The reason for this theorem is that it gives the following important corollary:

Corollary. A regular local ring is Cohen-Macaulay.

Proof. We always know depth $R \le \dim R$. On the other hand, by the theorem depth $R \ge \dim R$. ☐

Homological algebra

Now we start over, and learn some homological algebra in order to prove some more important theorems later on.

Fix a ring A. Then a chain complex C is a sequence of R-modules C_n with $n \in \mathbb{Z}$ so that

$$\ldots \to C_{n+1} \to C_n \to C_{n-1} \to \ldots$$

with $d_i : C_i \to C_{i-1}$ for R-modules hom s.t. $d_n \circ d_{n+1} = 0 \ \forall n$.

We call $\mathcal{Z}_n(C_\bullet) := \ker \ d_n$ n-*cycles*, and $B_n(C_\bullet) := \text{Im} \ d_{n+1}$ an n-*boundary*. Then $d_n d_{n+1} = 0$ implies $B_n \subseteq Z_n$.

We can define the *n-th homology* R-*module* of $C_\bullet$ by $H_n(C_\bullet) := \mathcal{Z}_n(C_\bullet)/B_n(C_\bullet)$. Furthermore, a *homology of complexes* is a collection of R-module homomorphisms,

$$f : C_\bullet \to D_\bullet, f_n : C_n \to D_n$$

$$\ldots \to C_{n+1} \to C_n \to C_{n-1} \to \ldots$$

$$\downarrow \qquad \downarrow f_n \quad \downarrow$$

$$D_{n+1} \to D_n \to D_{n-1}$$

Then we can easily check $f_n(Z_n) \subseteq Z_n$ and $f_n(B_n) \subseteq B_n$. So then f induces homomorphisms $H_n(f_\bullet) : H_n(C_\bullet) \to H_n(D_\bullet)$ (on homologies).

Lecture 25

Definition. Let $f, g : C. \to D.$ be modules of complex. A *homotopy* between f and g is a collection of $h_n : C_n \to D_{n+1}$ s.t. $f_n - g_n = h_{n-1} \circ d_n + d_{n+1} \circ h_n$.

$$C_n \to C_{n-1}$$

$$\swarrow \quad \searrow f_n - g_n \searrow h_{n-1}$$

$$D_{n+1} \overset{d_{n+1}}{\to} D_n \longleftarrow \ /$$

Lemma. If f and g are homotopic, then $H_n(f) = H_n(g)$.

Proof. (Homework)

Theorem. (Snake lemma) Assume we have two exact sequences with commutative diagrams.

$$A \to B \to C \to 0$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$0 \to D \to E \to F$$

OKAY forget trying to type up this diagram just look up the lemma.

Theorem. A short exact sequence of complexes

$$0 \to C_\bullet \xrightarrow{f} D_\bullet \xrightarrow{g} E_\bullet \to 0$$

means there exists a long exact sequence of homology modules

$$... \to H_{n+1}(E_\bullet) \xrightarrow{\partial} H_n(C_\bullet) \xrightarrow{H_n(f)} H_n(D_\bullet) \xrightarrow{H_n(g)} H_n(E_\bullet) \xrightarrow{\gamma} H_{n-1}(C_\bullet) \to ...$$

Proof. Steal from someone else's lecture notes.

Lemma. Every commutative diagram of short exact sequences

$$0 \to C \to D \to \varepsilon \to 0$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$0 \to C' \to D' \to E' \to 0$$

induces a commutative diagram of long exact sequences of homology groups

$$... \to H_{n+1}(E_\bullet) \to H_n(C_\bullet) \longrightarrow H_n(D_\bullet) \longrightarrow H_n(E_\bullet) \to H_{n-1}(C_\bullet) \to ...$$

$$\downarrow$$

$$... + H_n(C'_\bullet) \to ....$$

Definition. An R-module P is *projective* if for all surjective homomorphisms of R-modules, for all homomorphisms $f : P \to N'$, there exists a homomorphism $h : P \to M$ making the ofllowing diagram commutative:

$$P$$

$$h \swarrow \ \downarrow f$$

$$M \to N \to 0$$

where the h is called a lift.

Proposition. Every free module is projective.

Proof. He proved it in class, but see Dummit and Foote.

See also the Dummit and Foote theorem about equivalent conditions for projective modules!

Lecture 27

Theorem. (*Baer's Criterion*) E is an injective R-module if and only if $\forall I \subseteq R$ ideal and $\forall\ f : I \to E$, $\exists h : R \to E$ extending f.

Proof. ($\Longrightarrow$) By definition, $M \subseteq M_0 \subseteq N$.

($\Longleftarrow$) If $0 \to M \xrightarrow{f'} N$ with $M \xrightarrow{g'} E$ and $N \overset{h'}{\rightsquigarrow} E$ (lifts to). Then $\exists$ a maximal extension $h_0 : M_0 \to E$ with $h_0 : M_0 \to E$ and $h_0 |_M = g'$ (by Zorn's Lemma).

If $M_0 = N$, we are done. Assume it's not, then $\exists x \subseteq N \backslash M_0$. If $I := \{r \in R \,|\, rx \in M_0\}$, then define $f: I \to E$ by $f(r) = h_0(rx)$. This can be extended to $h: R \to E$. Define $h_0': M_0 + Rx \to E$ (with the former a proper subset of M_0) with $h_0'\ (x_0 + rx) = h_0(x_0) + rh(1)$ (with $x \in M_0$). This is well-defined and extens h_0 so we have a contradiction. ☐

Theorem. Every R-module can be embedded in an injective R-module.

Proposition 1. Every abelian group can be embedded in a divisible group (iff injective).

Proof. If $0 \to K \to F \to M \to 0$ with $\mathbb{Z} \subseteq \mathbb{Q}$ and $\mathbb{Z}^I \subseteq \mathbb{Q}^I$. Then $M \cong F/K \subseteq \mathbb{Q}^I/\mathbb{K}$ divisible. ☐

Proposition 2. If D is a divisible abelian group and R is a commutative ring, then $E := \text{Hom}_{\mathbb{Z}}(R, D)$ is an injective R-module.

Proof. Note that $\text{Hom}_{\mathbb{Z}}(R, D)$ is an R-module (we can always do $rf(s) = f(rs)$). We want $0 \to M \to N$ with $M \to E$ and N lifting to E. Then $\text{Hom}(N, E) \to \text{Hom}(M, E) \to 0$. We want $\text{Hom}_R(N, \text{Hom}_{\mathbb{Z}}(R, D)) \to \text{Hom}_R(M, \text{Hom}_{\mathbb{Z}}(R, D))$. The former is isomorphic to $\text{Hom}_{\mathbb{Z}}(N \otimes_R R, D)$ and the latter to $\text{Hom}_{\mathbb{Z}}(M \otimes_R R, D)$ and again $N \otimes_R R \cong N$ and $M \otimes_R R \cong M$ for D divisible implying we have injective over $\mathbb{Z}$ for $\text{Hom}_{\mathbb{Z}}(N, D) \to \text{Hom}_{\mathbb{Z}}(M, D) \to 0$. ☐

Proof (of theorem). We have $M \hookrightarrow$ injective module, so M an R-module implies M an abelean group implies (by Proposition 1) that $\exists M \hookrightarrow D$ a divisible group. Let $E = \text{Hom}_{\mathbb{Z}}(R, D)$ injective over R as by Proposition 2. We then claim that there is an injective R-module homomorphism $\varphi: M \to E$ with $m \mapsto f_m$ for $f_m(r) = rm \in M \subseteq D$. Then if φ is injective, $f_{m_1} = f_{m_2}$ implies $f_{m_1}(1) = f_{m_2}(1)$ so that $m_1 = m_2$. If φ is an R-module homomorphism, $s \in R$ means $f_{sm}(r) = rsm = (rs)m = f_m(sr) = (sf_m)(r)$. ☐

Resolutions

Definition. The *left resolution* of a module M is an exact sequence

$$... \to P_1 \to P_0 \to M \to 0.$$

It is a *projective* (free) resolution if all the P_i's are projective (free). A deleted resolution is one of the form $P_2 \to P_1 \to P_0 \to 0$ (i.e., P_0 is not exact anymore).

Similar definition for a right resolution.

Lemma. Every module M admits projective (in fact free) and injective resolutions.

Proof. We have $0 \to K_0 \to F_0 \to M$ and $0 \to K_1 \to F_1 \to K_0 \to 0$, and continue like this. ☐

Definition. An R-module M is *flat* if the functor $M \otimes_R \_\_$ is exact, i.e., for all short exact sequences $0 \to A \to B \to C \to 0$ of R-modules, then $0 \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$ is exact.

Examples. (1) R is flat over R because R $\otimes_R$ A $\cong$ A for any A.

(2) (Exercise) $\bigoplus M_i$ is flat $\Leftrightarrow \forall M_i$ is flat.

(3) Every projective R-module is flat.

Lecture 28

Flat modules

$M \otimes_R \_\_$ is right exact $\forall M$ R-modules.

Proof. Start with a short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$. We want to show

$$M \otimes_R A \xrightarrow{1 \otimes f} M \otimes_R B \xrightarrow{1 \otimes g} M \otimes_R C \to 0$$

is everywhere exact.

First, notice $1 \otimes g$ is surjective: $1 \otimes g\left(\sum m_i \otimes b_i\right) = \sum m_i \otimes g(b_i) = \sum m_i \otimes c_i$. Second, (2) Im$(1 \otimes f) \subseteq$ ker$(1 \otimes g)$: $(1 \otimes g) \circ (1 \otimes f) = 1 \otimes g \circ f = 0$.

Furthermore, ker$(1 \otimes g) \subseteq$ im$(1 \otimes f) := D$. Then by (2) we have $D \subseteq$ ker$(1 \otimes g)$ implies $\exists$ induced map $\overline{g} : M \otimes_R B - \gg M \otimes_R$. Then

$$\overline{g} \circ \pi(m \otimes b) = \overline{g}(m \otimes b) = m \otimes g(b) \text{ and ker } (\overline{g} \circ \pi) = \text{ker}(1 \otimes g).$$

We claim that it is enough to show $\overline{g}$ is an isomorphism. Construct the inverse

$$\overline{h} : M \otimes_R C \to M \otimes_R D \Big/ D \text{ with } M \times C \text{ lift to } M \otimes_R C \text{ and } M \times C \xrightarrow{h} M \otimes_R D \Big/ D.$$

Then

$$h : M \times C \to M \otimes_R B \Big/ D \text{ with } h((m,c)) = m \otimes b \text{ for any } b \text{ s.t. } g(b) = c.$$

This is well-defined and R-bilinear. ☐

Examples (of flatness)

(1) R flat over R

(2) $\forall$ projective module is flat over R

(3) $\mathbb{Z}$-module (abelian group) is flat if and only if it is torsion-free.
(remember for these every torsion-free abelian group is free so it is projective and flat).

We say it is not torsion free if $\exists n \in \mathbb{Z}$ such that $nx = 0$..

(4) $\mathbb{Q}$ is flat, but not projective over $\mathbb{Z}$. It is torsion free so it is flat. But it is not free so it is not projective.

(5) (*Homework*) (a) For $R \subset S$, if S is flat over R and M is flat over an S-module, then M is flat over S. (b) If $R \subset S$, M is a flat R-module implies $S \otimes_R M$ is flat over S. (c) If M is flat over R, then $S \subseteq R$ is a multiplication system then $S^{-1}M$ is flat over S^{-1}R.

Derived functions

(1) *Right exact functors*: If F is right exact on R modules, then if we take A to be an R-module, we can construct a projective resolution

$$... \to P_2 \to P_1 \to P_0 \to A \to 0$$

$$.... \to P_2 \to P_1 \to P_0 \to 0$$

Apply F to P_0 and we get $F(P_0) \to$ commplex. Then if $F: \ A \to B \to C$ with $0 \to A_1 \to B$ and $A \to A_1$ and $B \to B_1 \to 0$ and $0 \to B \to C$, then $F(A_1) \to F(B_1) \to F(C_1)$ is an exact sequence. We know $F(f) = F(B_1)$. Then $F(g) = F(v) \circ F(u)$ and Ker$(F(u)) \cong K$ ($\subset$ ker$(F(g))$ because $F(v)$ is not injective.

Definition. The *left derived functors* of F are the functors $(L_n F)(A) = H_n(F(P_0))$.

This doesn't depend on the resolution.

We also get an induced long exact sequence:

$$... \to (L_{m+1}F)(C) \to (L_n F)(A) \to (L_n F)(B) \to (L_n F)(C) \to (L_{n-1}F)(A) \to ...$$

Our "favorite gadget" will be:

Definition. $\text{Tor}_i^R(M, \_\_) := L_i M \otimes_R \_\_$

Lemma. If P is projective, then $\text{Tor}_i^R(M, P) = 0 \, \forall i > 0$ and $\text{Tor}_0^R(M, P) \cong M \otimes_R P$.

Proposition. The following are equivalent:

(1) M is flat over R (2) $\text{Tor}_n^R(M, N) = 0 \, \forall n \geq 1, \, \forall N$ (3) $\text{Tor}_i^R(M, N) = 0 \, \forall N$

Proof. (1) $\implies$ (2) $P_0 \to N$ is a projective resolution with $1 \otimes_R M$ flat. Then

$$P_0 \otimes_R M \to N \otimes_R M \to 0 \text{ is exact.}$$

So then $\text{Tor}_n^R(M, N) = 0 \, \forall n \geq 1$.

(2) $\implies$ (3) Obvious.

(3) $\implies$ (1) $0 \to A \to B \to C \to 0$ gives

$$... \to \text{Tor}_1^R(M, B) \to \text{Tor}_1^R(M, C) \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0. \ \square$$

Lecture 30

Left exact functors

Start with $(\text{Hom}_R(M, \_\_))$ and $(\text{Hom}_R(\_\_, M))$. Then for F left exact on an R-module, let A be an R-module and take injection resoultion:

$$0 \to A \to E_0 \to E_1 \to ...$$

with E_0 a deleted resolution.Then $F(E_0)$ is a complex (like in the last lecture). Then

$$(R^n F)(A) = H^n(F(E_0)) \text{ (independent of } E_0\text{)}.$$

For an exact sequence $0 \to A \to B \to C \to 0$, we get a long exact sequence

$$\ldots \to (R^{n-1}F)(C) \to (R^nF)(A) \to R^nF(B) \to (R^nF)(C) \to (R^{n+1}F)(A) \to \ldots$$

Definition. $\operatorname{Ext}^n_R(M, \_\_) := R^n \operatorname{Hom}_R(M, \_\_)$

Lemma. $\forall\, F$ left or right exact and $\forall A$ R-modules,

$$(L_0F)(A) \cong F(A) \cong (R^0F)(A).$$

Proof. We have a right exact sequence $P_0 \to A \to 0$ with $F(P_1) \to F(P_0) \to 0$ and $F(P_0) \to F(A)$. By definition, $L_0F(A)$ is a homology at $F(P_0)$, specifically, $F(P_0)/\text{Im}(F(P_1) \to F(P_0))$. Since $P_1 \to P_0 \to A \to 0$, applying F to a right exact sequence, $F(P_1) \to F(P_0) \to F(A) \to 0$ is exact. □

Lemma. F is left exact and E is injective implies $(R^nF)(E) = 0\, \forall n > 0$. In particular

$$\operatorname{Ext}^n_R(M, E) = 0\ \forall n > 0,\ \forall M.$$

Remark: Can also look at $\operatorname{Ext}^n_R(\_\_, N) = R^n\operatorname{Hom}_R(\_\_, N)$. Can do it by picking projective resolutions.

Proposition. The following are equivalent:

(1) M is projective. (2) $\operatorname{Ext}^n_R(M, N) = 0\ \forall n \geq 1, \forall N$.

(3) $\operatorname{Ext}^n_R(M, N) = 0\ \ \forall N$.

Proof. (1) $\implies$ (2) We have a projective resolution $0 \to M \to M \to 0$ so then

$$\operatorname{Ext}^n_R(M, M) = R^n \operatorname{Hom}_R(M, N) = 0.$$

(2) $\implies$ (3) Clear.

(3) $\implies$ (1) Take the exact sequence $0 \to G \to F \to M \to 0$ $(*)$. Then apply $\operatorname{Hom}_R(\_\_, N)$:

$$0 \to \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_R(F, N) \to \operatorname{Hom}_R(G, N) \to \operatorname{Ext}^n_R(M, N) \to \operatorname{Ext}^1_R(F, N) \to \_\_.$$

Take $N = G$. Then

$$0 \to \operatorname{Hom}_R(M, G) \to \operatorname{Hom}_R(F, G) \to \operatorname{Hom}_R(G, G) \to 0.$$

Then $(*)$ splits, so M is a direct summand of F, so M is projective. □

Proposition. The following are equivalent:

(1) N is injective. (2) $\operatorname{Ext}^n_R(M, N) = 0\ \forall n \geq 1\, \forall M$. (3) $\operatorname{Ext}^1_R(M, N) = 0\, \forall M$.

Proof. Homework exercise.

Examples. (1) $\operatorname{Ext}^i_R(R, M) = 0\, \forall i > 0\, \forall M$ (from proposition). Then by definition $\operatorname{Ext}^0_R(R, M) = \operatorname{Hom}_R(R, M) \cong M$.

(2) $x \in R$ is neither a unit nor a zero-divisor. We want to compute $\operatorname{Ext}^i_R(R/xR, M)$ for any M. We have

$$0 \to R \to R \to R/xR \to 0.$$

We get the long exact sequence

$$0 \to \text{Hom}_R(R/xR, M) \to \text{Hom}_R(R, M) \to \text{Hom}_R(R, M) \to \text{Ext}^1(R/xR, M) \to \text{Ext}^1(R, M)$$

and

$$\ldots \to \text{Ext}^{i-1}(R, M) \to \text{Ext}^i(R/xR, M) \to \text{Ext}^i(R, M) \to \ldots$$

for $i \geq 2$. Then $\text{Ext}^1(R/xR, M) \cong M/xM$, and $\text{Hom}_R(R/x, M) = \{m \in M \mid xm = 0\} =$ *socle of* x.

(3) $\text{Tor}_i^R(M, R) = 0 \quad \forall i > 0$, and $\text{Tor}_0^R(M, R) \cong M \otimes_R R \cong M$. As in (2), compute $\text{Tor}_i^R(R/xR, M)\, \forall i$.

(4) For any $I \subseteq R$, what is $\text{Tor}_i(R/I, M)$?

Lecture 31

(4) We want to compute $\text{Tor}_i^R(R/xR, M)$ where x is not a unit or zero divisor.

$$0 \to R \xrightarrow{x} R \to R/xR \to 0$$

So we need $1 \otimes M$. So we get

$$\text{Tor}_1(R, M) \to \text{Tor}_1(R/xR, M) \to R \otimes_R M \to R \otimes_R M \to R/xR \otimes_R M \to 0.$$

But by isomorphisms,

$$0 \to \{m \mid x \cdot m = 0\} \to M \to M \to M/xM \to 0.$$

So $\{m \mid x \cdot m = 0\} \cong \text{Tor}_1(R/xR, M)$, and $M/xM = \text{Tor}_0(R/xR, M)$. So

$$\text{Tor}_i(R, M) \to \text{Tor}_i(R/xR, M) \to \text{Tor}_i(R, M)$$

for $i \geq 2$.

(5) Take $I \subseteq R$ any ideal. Then $\text{Tor}_i(R/I, M) =$? $\forall i$. Then

$$0 \to I \to R \to R/I \to 0$$

and we tensor with M.

$$0 \to \text{Tor}_1(R/I, M) \to I \otimes_R M \to M \to M/IM \to 0.$$

$$\text{Tor}_i(R, M) \to \text{Tor}_i(R/I, M) \to \text{Tor}_{i-1}(I, M) \to \text{Tor}_{i-1}(R, M).$$

Then for $i \geq 2$, $\text{Tor}_i(R/I, M) \cong \text{Tor}_{i-1}(I, M)$. Notice we know

$$\text{Tor}_1(R/I, M) = \ker(I \otimes_R M \to IM) \ \ (a \otimes m \mapsto am).$$

Homological Dimension

Definition. If M is an R-module, take a projective resolution

$$P_\bullet := 0 \to P_n \to \ldots \to P_1 \to P_0 \to M \to 0$$

of *length* n. The *projective (homological) dimension* of M, denoted $\text{pd}_R(M)$ is the infimum (minimum) over the length of all such resolutions (could be ∞).

Notice $\text{pd}_R(M) = 0 \Leftrightarrow M$ is projective.

Lemma. Let R be a principal ideal domain, and M an R-module. Then $\text{pd}_R(M) \leq 1$. Equality holds if and only if the torsion part of M is non-trivial.

Proof. Notice there is an exact sequence $0 \to F_1 \to F_0 \to M \to 0$ where F_0 is free and F_1 is the kernel. Since F_0 is free, F_1 must be torsion-free as it is on a principal ideal domain. Hence it must be free. Thus, we have found a resolution of length 1, so that $\text{pd}_R(M) \leq 1$. Indeed $\text{pd}_R(M) = 0$ if and only if M is projective if and only if M is free if and only if (since we're on a PID) the torsion part is trivial. □

Definition. The *global homological dimension of* R is $\text{gd}(R) = \sup_M \text{pd}_R(M)$ (it could be infinite).

Examples. (1) If R is a field, then $\text{gd}(R) = 0$.

(2) If R is a PID, then $\text{gd}(R) = 1$.

Theorem. The following are equivalent for a given R-module M:

(1) $\text{pd}_R(M) \leq n$. (2) $\text{Ext}^i_R(M, N) = 0 \, \forall i > n \, \forall N$ R-modules.

(3) $\text{Ext}^{n+1}(M, N) = 0 \, \forall N$ R-modules.

(4) If there is an exact sequence $0 \to Q_n \to P_{n-1} \to \ldots \to P_1 \to P_0 \to M \to 0$ where P_i are all projective, then Q_n is also projective.

Proof. (4) $\implies$ (1) and (2) $\implies$ (3) are true by definition and inspection.

(1) $\implies$ (2) Take a proj. resolution of M: $0 \to P_0 \to M \to 0$ such that length$(P.) \leq n$. Then $\text{Ext}^i_R(M, N) = R^i \, \text{Hom}(P., \, N) = 0$ for $i > n$ by basic notion of homology.

(3) $\implies$ (4) $0 \to Q_n \to P_{n-1} \to P_{n-2} \to \ldots \to P_2 \to P_1 \to P_0 \to M$ where we have $P_{n-1} \to K_{n-1} \to 0$ and $0 \to K_{n-1} \to P_{n-2}$, ..., and $P_2 \to K_1 \to 0$, $0 \to K_1 \to P_1$, $0 \to K_1 \to P_0$, and $P_1 \to K_0 \to 0$, where K_i is the so-called ith syzygy module. This gives

$$0 \to K_0 \to P_0 \to M \to 0$$

$$0 \to K_1 \to P_1 \to K_0 \to 0$$

We know that $\text{Ext}^{n+1}_R(M, N) = 0 \; \forall N$. From last time, Q_n is projective if and only if $\text{Ext}^1_R(Q_n, N) = 0 \, \forall N$. Then

$$\text{Ext}^n(M, N) \to \text{Ext}^n(P_0, N) \to \text{Ext}^n(K_0, N) \to \text{Ext}^{n+1}(M, N) = 0.$$

Since P_0 is projective, and Ext of anything projective is 0, we have $\text{Ext}^n(P_0, N)$. So we've shifted the index, so that $\text{Ext}^n(K_0, N) = 0 \, \forall N$. Then

$$0 = \text{Ext}^n(P_1, N) \to \text{Ext}^{n-1}(K_1, N) \to \text{Ext}^n(K_0, N) = 0$$

so this implies $\text{Ext}^{n-1}(K_1, N) = 0 \; \forall N$. Continue this way. Then eventually $Q_m \cong K_{n-1}$. Then $\text{Ext}^1(Q_n, N) = 0 \; \forall N$. □

Corollary. $\text{gl}(R) = \inf \{n \,|\, \text{Ext}^n_R(M, N) = 0 \, \forall M \, \forall N\}$.

Definition. Similar definition for *injective resolution* and *injective dimension* (id_R).

Theorem. For R-module N, $n \geq 0$, the following are equivalent:

(1) $\mathrm{id}_R(N) \leq n$. (2) $\mathrm{Ext}^i_R(M,N) = 0\ \forall i > n\, \forall M$ (3) $\mathrm{Ext}^{n+1}_R(M,N) = 0\ \forall M$

(4) $\forall$ exact sequences $0 \to N \to E_0 \to \ldots \to E_{n-1} \to Q_n \to 0$ with E_i injective, Q_n is also injective.

Proof. Homework.

Corollary. $\mathrm{gd}(R) = \sup_N \{\mathrm{id}_R(N)\} = \inf_N \left\{n \,|\, \mathrm{Ext}^{n+1}_R(M,N) = 0\ \forall M\right\} =$ $\inf \{n \,|\, \mathrm{Ext}^{n+1}(M,N) = 0\ \forall M\}$.

Lecture 32

This lecture we will apply homological methods to obtain some results.

Proposition. Start with $(R, \underline{m})$ a Noetherian local ring, with $k = R/\underline{m}$ the residue field. Let M be a finitely generated R-module. Then M is free if and only if $\mathrm{Tor}^R_1(M,k) = 0$.

Proof. $(\implies)$ If M is free, then it is projective, so that $\mathrm{Tor}_i(M,N) = 0\, \forall i > 0$, $\forall N$.

$(\impliedby)$ Take a minimal set of generators for M, say $x_1, \ldots, x_n$. Take a free module F of rank n, with basis $e_1, \ldots, e_n$. We have

$$0 \to K \to F \to M \to 0 \text{ with } e_i \mapsto x_i.$$

We then tensor with $k = R/\underline{m}$, so we get

$$\mathrm{Tor}_1(M,k) \to K \otimes_R k \to F \otimes_R k \to M \otimes_R k \to 0.$$

Notice $\mathrm{Tor}_1(M,k) = 0$, $F \otimes_R k \cong F/\underline{m}F \cong M/\underline{m}M$ [Nakayama] $\cong M \otimes_R k$. But then $K = \underline{m}K$ so by Nakayama's Lemma, $K = 0$ which implies $M \cong F$.

Corollary. If $(R, \underline{m})$ is a Noetherian local ring, with M a finitely generated R-module, then M is free if and only M is projective if and only if M is flat.

Proof. Since M is free, it is projective, and so flat, and so $\mathrm{Tor}_1(M,k) = 0$ (since all $\mathrm{Tor}_1(M,N) = 0$ for $n > 0$). By the proposition this in turn implies M is free. ☐

Theorem. If $(R, \underline{m})$ is a Noetherian local ring, and M is a finitely generated R-module, then the following are equivalent:

(1) $\mathrm{pd}_R(M) \leq n$. (2) $\mathrm{Tor}^R_i(M,N) = 0\, \forall i > n\, \forall N$ R-modules.

(3) $\mathrm{Tor}^R_{n+1}(M,N) = 0\, \forall N$ R-modules.

(4) $\mathrm{Tor}_{n+1}(M,R) = 0$.

Proof. [(1) $\implies$ (2)] Take a projective resolution $0 \to P_\bullet \to M \to 0$ of length $\leq n$.

$$\mathrm{Tor}^R_i(M,N) = H_i(P_\bullet \otimes N) = 0\, \forall i > n.$$

[(2) $\implies$ (3) $\implies$ (4)] Obvious.

[(4) $\Longrightarrow$ (1)] It's enough to show (as was seen in the previous lecture) that if we have an exact sequence

$$0 \to Q_n \to P_{n-1} \to ... \to P_1 \to P_0 \to M \to 0$$

with P_i projective, then Q_n is projective. So this is what we need to show. By the earlier proposition, we only need to show that $\text{Tor}_1(Q_n, k) = 0$ (which is much more manageable).

$$\text{Tor}_1(Q_n, k) \cong \text{Tor}_2(K_{n-2}, k) \cong \text{Tor}_3(K_{n-3}, k) \cong ... \cong \text{Tor}_n(K_0, k) \cong \text{Tor}_{n+1}(M, k) = 0$$

by (4). □

Corollary. If $(R, \underline{m})$ is a Noetherian local ring, $k = R/\underline{m}$, and $n > 0$, then the following are equivalent:

(1) $\text{gd}(R) \le n$. (2) $\text{Tor}^R_{n+1}(M, N) = 0 \, \forall M, N$ finitely generated modules.

(3) $\text{Tor}_{n+1}(k, k) = 0$

Proof. $[(1) \Leftrightarrow$ (2)] Notice $\text{pd}_R(M) \le n$ is true if and only if $\text{Tor}_{n+1}(M, N) = 0 \, \forall N$ which is true if and only if $\text{Tor}_{n+1}(M, k) = 0$.

(2) is true if and only if $\text{Tor}_{n+1}(M, k) \cong \text{Tor}_{n+1}(k, M) = 0 \, \forall M$, so by the previous theorem, this is true if $\text{Tor}_{n+1}(k, k) = 0$. □

First application of homological methods

We will discuss the lenght of M-regular sequences.

Definition. If R is a Noetherian local ring, $I \subseteq R$, M is a finitely generated R-module, $IM \ne M$, then the $\text{grade}_I(M) = \max_n \{x_1, ..., x_n$ M-regular sequence $| \, x_i \in I \, \forall i\}$.

Example. If $(R, \underline{m})$ is a Noetherian local ring, then depth $M = \text{grade}_{\underline{m}}(M)$.

Theorem. If R is a Noetherian local ring, $I \subseteq R$, M is a finitely generated R-module, then any two maximal M-regular sequences in I have the same length. This length is equal to $\min\{n \, | \, \text{Ext}^n(R/I, M) \ne 0\}$.

We will prove this shortly.

Proposition. Let M and N be R-modules, $x_1, ..., x_n$ an M-regular sequence. Assume that $(x_1, .., x_n) \cdot N = 0$. Then $\text{Ext}^n(N, M) \cong \text{Hom}(N, M/(x_1, ..., x_n)M)$.

Proof. Consider $0 \to M \xrightarrow{x_1} M \subseteq M/x_1 M \to 0$. Then this implies there is

$$... \to \text{Ext}^{n-1}(N, M) \to E^{n-1}(N, M/x_1 M) \to \text{Ext}^n(N, M) \xrightarrow{x_1} \text{Ext}^n(N, M) \to ...$$

which means $x_1 \, \text{Ext}^n(N, M) = 0 \, \forall n$ (exercise). Then for $n = 1$,

$$0 \to \text{Hom}(N, M) \xrightarrow{x_1} \text{Hom}(N, M) \to \text{Hom}(N, M/x_1 M) \to \text{Ext}^1(N, M) \to 0.$$

But notice $\text{Hom}(N, M) = 0$. This says $\text{Ext}^1(N, M) \cong \text{Hom}(N, M/x_1 M)$. We then claim that $\varphi \in \text{Hom}(N, M/(x_1, ..., x_{k-1})M) = 0$. Then $x_k \varphi(n) = \varphi(x_k n) = \varphi(0) = 0$ with $x_i N = 0$ with $x_k \notin Z(M/(x_1, ..., x_{k-1})M)$. This implies $\varphi(n) = 0$. So then

$$0 \to \text{Hom}(N, M/(x_1, ..., x_{n-1})) \to \text{Hom}_R(N, M/(x_1, ..., x_n)M)$$

which induces

$$... \to \text{Ext}^{n-1}(N, M) \to \text{Ext}^{n-1}(N, M/x_1 M) \to \text{Ext}^n(N, M) \xrightarrow{x_1} \text{Ext}^n(N, M). \ \square$$

Lecture 34

Theorem. If $(R, \underline{m})$ is a Noetherian local ring, then a complex $F.$ of free modules over R is minimal if and only if $d_n \otimes 1_R : F_n \otimes_R \underline{k} \to F_{n-1} \otimes_R \underline{k}$ if and only if the matrices representing d_n have all entries in the maximal ideal $\underline{m}$.

Minimal free resolutions of a given module M are unique up to isomorphism.

Theorem. (*Auslander-Büchsbaum*) If $(R, \underline{m})$ is Noetherian local and M is a finitely generated R-module such that $\text{pd}_R(M) < \infty$, then $\text{pd}_R(M) + \text{depth}(M) = \text{depth}(R)$.

Example of application

We want to detect when a ring is Cohen-Macaulay. We can do this with the following corollary.

Corollary. (a) If there is a finitely generated module M with $\text{pd}_R(M) = \dim(R)$, then the ring R is Cohen-Macaulay.

(b) If R is Cohen-Macaulay and M is a finitely generated R-module with $\text{pd}_R(M) = \dim(R)$, then $\underline{m} \in \text{Ass}(M)$.

Proof. In general, $\text{depth}(R) \leq \dim R$ with equality if and only R is Cohen-Macaulay. But then $\dim\ R \leq \text{pd}_R(M) + \text{depth}\ M = \text{depth}\ R \leq \dim\ R$ holds if and only if $\text{depth}\ R = \dim\ R$ (gives Cohen-Macaulayness) and $\text{depth}\ M = 0$ (if and only if $\underline{m} \in \text{Ass}(M)$). $\square$

Proof. (of theorem) We will use induction on the projective dimension $p = \text{pd}_R(M)$. If $p = 0$, this is equivalent to saying M is projective, but the ring is local so this is equivalent to M being free. This implies $\text{depth}(M) = \text{depth}(R/\text{Ann}(M)) = \text{depth}(R)$.

Now consider $p = 1$. We pick a minimal free resolution,

$$0 \to R^m \xrightarrow{f} R^n \to M \to 0$$

where f has entries in $\underline{m}$. Recall $\text{depth}(M) = \inf\{i \mid R/\underline{m} = \text{Ext}^i(k, M) \neq 0\}$ (theorem from last time). This gives

$$... \to \text{Ext}^i(k, R^m) \to \text{Ext}^i(k, R^n) \to \text{Ext}^i(k, M) \to \text{Ext}^{i+1}(k, R^m) \to ...$$

But notice $\text{Ext}^i(k, R^\xi) \cong \bigoplus_{\xi \text{ times}} \text{Ext}^i(k, R)$ for $\xi \in \{m, n\}$. But then the map

$$\bigoplus_m \text{Ext}^i(k,R) \xrightarrow{\widetilde{f}} \bigoplus_n \text{Ext}^i(k,R)$$

is the same matrix as f. So then from earlier $x\,\text{Ext}^i(N,M)=0$, so the map

$$\bigoplus_m \text{Ext}^i(k,R) \xrightarrow{\widetilde{f}} \bigoplus_n \text{Ext}^i(k,R) \to \text{Ext}^{i+1}(k,R^m) \to \ldots$$

is in fact 0. Furthermore,

$$0 \to \bigoplus_n \text{Ext}^i(k,R) \to \text{Ext}^i(k,M) \to \bigoplus_m \text{Ext}^{i+1}(k,R) \to 0.$$

Then depth$(M) = \min\{i \mid \text{Ext}^i(k,M) \neq 0\}$, and depth$(R) = \min\{i \mid \text{Ext}^i(k,R) \neq 0\}$. Notice $\text{Ext}^i(k,M)=0$ implies $\text{Ext}^{i+1}(k,R)=0$. On the other hand, $\text{Ext}^i(k,M)\neq 0$ implies $\text{Ext}^i(k,R)\neq 0$ or $\text{Ext}^{i+1}(k,R)\neq 0$ so that depth R = depth $M+1 = \text{pd}_R(M)$.

Finally, consider $p>1$. Take the presentation $0\to K\to R^n \to M\to 0$. Then $\text{pd}_R(M)=p$ implies $\text{pd}_R(k)=p-1$. By induction, $p-1+$ depth K = depth R. Now we only need to show depth K = depth $M+1$. We have

$$\ldots \to \text{Ext}^{i-1}(k,M) \to \text{Ext}^i(k,K) \to \text{Ext}^i(k,R)^n \to \text{Ext}^i(k,M) \to \ldots$$

So then depth $R >$ depth K. Then if we let d = depth K,

$$\text{Ext}^{d-1}(k,R) = \text{Ext}^d(k,K) = 0,$$

so that $\text{Ext}^d(K,k) \cong \text{Ext}^{d-1}(k,M)$. Then the earlier long sequence has to be minimal, so depth M = depth $K-1$. ☐

Proposition. Let $(R,\underline{m})$ be a Noetherian local ring, and M a finitely generated R-module. Take

$$0 \to F_n \to F_{n-1} \to \ldots \to F_1 \to F_0 \to M \to 0$$

to be a minimal free resolution. Then

(1) rank$(F_i) = \dim_k \text{Tor}_i(M,k)$. (where the rank is the so-called Betti # of M)

(2) $\text{pd}_R(M) = n = \sup\{i \mid \text{Tor}_i(M,k)\neq 0\}$.

(3) $\text{gd}(R) = \text{pd}_R(k)$.

Furthermore,

(1) $$\ldots \xrightarrow{0} F_i\otimes k \xrightarrow{0} F_{i-1}\otimes k \to \ldots$$

where the homology here is $\text{Tor}_i(M,k)$. Then $\text{Tor}_i(M,k)\cong F_i \otimes_R k$.

(2) We know from the previous theorem that $\text{pd}_R(M) = \sup\{i \mid \text{Tor}_i(M,k)\neq 0\} = n$.

(3) We can compute $\text{Tor}_i(k,M)$ by taking the minimum free resolution for k. So

$$\text{pd}_R(M) \leq \text{pd}_R(k).$$

Lecture 35

Koszul complex

This is the most important example of a complex. Let R be a ring with $E \triangleq R^n$ with basis $e_1, ..., e$ and $\lambda : E \to R$ a linear form (in E^*). Construct $K_\bullet(\lambda)$ sas follows:

$$K_i = \bigwedge^i E \cong R^{\binom{n}{i}}$$

with $d_i : K_i \to K_{i-1}$ given by $\bigwedge^i E \xrightarrow{d_i} \bigwedge^{i-1} E$. Then

$$d_i(v_1 \wedge ... \wedge v_i) = \sum_{j=0}^{i} (-1)^{j-1} \lambda(v_j) \; v_1 \wedge ... \wedge \widehat{v}_j \wedge ... \wedge v_i$$

where $\widehat{v}_j$ means we are excluding v_i from the $\wedge$'s.

Exercise. (1) If you have two differential forms with $\omega \in \bigwedge^p E,$ and $\eta \in \bigwedge^q E$, then

$$\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^p \, \omega \wedge \mathrm{d}\eta.$$

(2) Use (1) to show $d_i \circ d_{i+1} = 0 \; \forall i$.

We get a complex

$$0 \to \bigwedge_R^n E \to \bigwedge^{n-1} E \to ... \to \bigwedge^2 E \xrightarrow{d} E^{d=\lambda} \to R \to R/\mathrm{Im}(\lambda) \to 0.$$