First problem set
Math 414
Due Monday, Jan. 28, 2008

1. Let $\sum_{n=1}^{\infty}$ be any series of real numbers. For each n, set

$$a_n^+ = \begin{cases} a_n \text{ if } a_n \geq 0 \\ 0 \text{ if } a_n < 0 \end{cases}$$

and

$$a_n^- = \begin{cases} -a_n \text{ if } a_n \leq 0 \\ 0 \text{ if } a_n > 0. \end{cases}$$

Note that $a_n^+ \geq 0$, $a_n^- \geq 0$, $a_n = a_n^+ - a_n^-$, and $|a_n| = a_n^+ + a_n^-$.

(a) Prove that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if and only if $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ are convergent. (You will need the comparison test for series of non-negative terms.)

(b) Use the result of part (a) to prove that $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series, then $\sum_{n=1}^{\infty} a_n$ is convergent and

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^-.$$

2. This problem gives an alternative proof that a rearrangement of an absolutely convergent series is absolutely convergent and has the same sum as the given series.

(a) Let $\sum_{n=1}^{\infty} a_n$ be a convergent sequence of non-negative terms. Define a *generalized partial sum* of $\sum_{n=1}^{\infty} a_n$ to be a number of the form $a_{m_1} + \cdots + a_{m_k}$, where $m_1, \ldots, m_k$ are distinct natural numbers. Prove that every generalized partial sum of $\sum_{n=1}^{\infty} a_n$ is less than or equal to some partial sum of $\sum_{n=1}^{\infty} a_n$.

(b) Use the result of (a) to show that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of non-negative terms, the set of all partial sums of $\sum_{n=1}^{\infty} a_n$ and the set of all generalized partial sums of $\sum_{n=1}^{\infty} a_n$ are both bounded above and have the same supremum (least upper bound).

(c) In the theory of sequences it is shown that a (weakly) monotone increasing sequence converges to the least upper bound of its terms. Combine this with the result of (b) to show that the sum of a convergent series of non-negative terms is the supremum of its generalized partial sums.

(d) Use the result of part (c) to show that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of non-negative terms, then any rearrangement of $\sum_{n=1}^{\infty} a_n$ is convergent and has the same sum as $\sum_{n=1}^{\infty} a_n$.

(e) Combine the results of problems 1(b) and 2(d) to prove that if $\sum_{n=1}^{\infty} a_n$ is an abolutely convergent series then any rearrangement of $\sum_{n=1}^{\infty} a_n$ is convergent and has the same sum as $\sum_{n=1}^{\infty} a_n$.

3. A series $\sum_{n=1}^{\infty} a_n$ is said to *diverge to* $+\infty$ if for every real number $A > 0$ there is a natural number N such that for every $K \geq n$, the K-th partial sum $\sum_{n=1}^{K} a_n$ is greater than A. Prove that every conditionally convergent series has a rearrangement which converges to $+\infty$. (This includes part (a) of the theorem that I stated on Wednesday, January 23. You should use the same strategy as in part (b) except that in place of making $b_1 + \cdots + b_{r_1} - c_1 - \cdots - c_{s_1} + \cdots + b_{r_{m-1}+1} + \cdots + b_{r_m}$ greater than S, you should make it greater than the natural number m.)

4. In class we showed that if $\sum_{m=1}^{\infty} a_m$ is conditionally convergent, and if we set $P = \{p \in \mathbf{N} : a_p \geq 0\}$ and $N = \{n \in \mathbf{N} : a_n < 0\}$, then $P = \{p_1 < p_2 < \cdots\}$ and $N = \{n_1 < n_2 < \cdots\}$ are infinite, and that if we set $b_i = a_{p_i}$ and $c_i = -a_{n_i}$ then the series $\sum_{i=1}^{\infty} b_i$ and $\sum_{i=1}^{\infty} c_i$ are divergent. Verify this *directly* for the case where $\sum_{m=1}^{\infty} a_m$ is the series $1 - (1/2) + (1/3) - \cdots$ by writing down the series $\sum_{i=1}^{\infty} b_i$ and $\sum_{i=1}^{\infty} c_i$ and applying tests for convergence that were discussed in class.