Fourth problem set
Math 414
Due Monday, March 3, 2008

1. Prove that a sequence in $\mathbf{R}^m$ is bounded if and only if each of its coordinate sequences is bounded.

2. This problem gives an alternative proof of the theorem known as the Weierstrass M-test. The theorem states that a series $\sum_{n=1}^{\infty} f_n$ of real-valued functions on a domain D is uniformly convergent on D if it satisfies the following hypothesis:

($\star$) There is a convergent series $\sum_{n=1}^{\infty} M_n$ of non-negative constants such that

$$|f_n(x)| \leq M_n$$

for each $n \in \mathbf{N}$ and each $x \in D$.

(a) Prove that the hypothesis ($\star$) holds if and only if $f_n \in \mathcal{B}(D)$ for each $n \in \mathbf{N}$ and $\sum_{n=1}^{\infty} \|f_n\|_{\sup}$ is convergent.

(b) Using the result of part (a) together with the completeness of $\mathcal{B}(D)$, adapt the alternative proof given in class that an absolutely convergent series is convergent to give an alternative proof of the M-test.

3. **(a)** Let X, Y and Z be metric spaces, and suppose that $f : X \to Y$ and $g : Y \to Z$ are continuous maps. Prove that $g \circ f : X \to Z$ is continuous.

(b) Let X be a metric space and let $f : X \to \mathbf{R}^m$ be continuous. Write $f(x) = (f^{(1)}(x), \ldots, f^{(m)}(x))$ for each $x \in X$, so that $f^{(1)}, \ldots, f^{(m)}$ are real-valued functions on X. Prove that f is continuous if and only if the functions $f^{(1)}, \ldots, f^{(m)}$ are all continuous.

(c) Prove that the map $\sigma : \mathbf{R}^2 \to \mathbf{R}$ defined by $\sigma(x, y) = x + y$ is continuous. (This requires an "$\epsilon/2$ argument.")

(d) Combine the results of parts (a)–(c) to give a **short** proof (**without** computations) of the following fact: If X is a metric space, and f and g are continous real-valued functions on X, then $f + g$ is continuous on X.

4. Let (X, d) be a metric space. A subset A of X is said to be *dense* if for every $\epsilon > 0$ and every point $x \in X$ there is a point $a \in A$ such that $d(a, x) < \epsilon$. For example, the set $\mathbf{Q}$ of rational numbers is dense in $\mathbf{R}$.

(a) Prove that A is dense in X if and only if for every point $x \in X$ there is a sequence of points in A converging to x.

(b) Let X and Y be metric spaces, let A be a dense subset of X, and let $f, g : X \to Y$ be continuous maps such that $f(x) = g(x)$ for every $x \in A$. Prove that $f = g$. (One approach is to use the result of part (a).)