Sixth problem set
Math 414
Due Wednesday, April 9, 2008

1. Let V be an inner product space. Prove that if x and y are vectors in V, we have

$$\|x - y\|^2 + \|x + y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

(Geometrically, this says that the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its sides.)

2. Let V be an inner product space.

(a) Let $u_1, \ldots, u_n$ be a finite orthonormal system of vectors in V, let W denote the span of $u_1, \ldots, u_n$, and let v be a vector in V which does not lie in W. Set

$$u_{n+1} = \frac{v - \text{proj}_W(v)}{\|v - \text{proj}_W(v)\|}.$$

Prove that $u_1, \ldots, u_{n+1}$ is an orthonormal system and has the same span as $u_1, \ldots, u_n, v$.

(b) Prove that if $\{v_1, \ldots, v_n\}$ is a finite set of vectors in V and if W denotes its span, then for any vector $x \in V$, the span of $\{v_1, \ldots, v_n, x\}$ is equal to the span of $W \cup \{x\}$. Also prove that if $x \in W$ then the span of $\{v_1, \ldots, v_n, x\}$ is equal to W.

(c) Using the results of (a) and (b), prove by induction on $n \geq 0$ that if a subspace W of V is spanned by a set of n vectors, then W is spanned by an orthonormal system of at most n vectors.

3. Let f and g be continous functions on $\mathbf{R}$ which are periodic in the sense that $f(x + 2\pi) = f(x)$ and $g(x + 2\pi) = g(x)$ for every $x \in \mathbf{R}$. Recall that the convolution $f \star g$ is defined by

$$(f \star g)(x) = \int_{-\pi}^{\pi} f(x - y)g(y)\, dy$$

for every $x \in \mathbf{R}$. Prove that $f \star g$ is continuous and periodic. (For continuity you will need the fact that a continuous function on a closed interval is uniformly continuous. For periodicity you will need the substitution rule for integrals.)