Seventh problem set
Math 414
Due Friday, April 18, 2008

1. (a) Recall that the Fourier series of an odd function $f \in \mathcal{C}([-\pi, \pi])$ has the form

$$\sum_{n=1}^{\infty} a_n \sin nx.$$

Find the a_n in the case $f(x) = x^3$. (Note that the answer is consistent with the fact proved in class that if $f(-\pi) \neq f(\pi)$ then the Fourier series of f is not uniformly convergent.)

(b) Recall that for an odd function $f \in \mathcal{C}([-\pi, \pi])$, Parsival's identity is

$$\|f\|_2^2 = \sum_{n=1}^{\infty} a_n^2.$$

Taking $f(x) = x^3$ and applying the result of (a), express the right hand side of Parsival's identity in the form $\alpha\zeta(2) + \beta\zeta(4) + \gamma\zeta(6)$, where α, β and γ are constants. Then evaluate the left hand side of Parsival's identity and use the known values $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$ to find $\zeta(6)$.

2. In this problem, for any $f, g \in P$, I will denote by $\langle f, g\rangle$ the usual inner product on P, defined by

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} fg.$$

(a) Use the fundamental theorem of calculus to show that if f is a 2π-periodic function which is differentiable on $\mathbf{R}$, if f' is continuous on $\mathbf{R}$, and if f takes the value 0 at some point in $[-\pi, \pi]$, then for every $x \subset [-\pi, \pi]$ we have

$$|f(x)| \leq \pi\langle |f'|, 1\rangle,$$

where $|f'|$ simply means the function in P whose value at x is $|f'(x)|$.

(b) Combine the result of (a) with the Cauchy-Schwarz inequality to show that if f is a 2π-periodic function which is differentiable on $\mathbf{R}$, if f' is continuous on $\mathbf{R}$, and if f takes the value 0 at some point in $[-\pi, \pi]$, then for every $x \subset [-\pi, \pi]$ we have

$$|f(x)| \leq \pi\sqrt{2}\|f'\|_2.$$

(c) Observe that for every $f \in P$, the N-th partial sum S_N of the Fourier series for f is equal to $\text{proj}_W(f)$, where $W \subset P$ is the span of the first N terms of the sequence

$$\frac{1}{\sqrt{2}}, \sin x, \cos x, \sin 2x, \ldots.$$

By quoting a general result on inner product spaces, deduce that $f - S_N$ is orthogonal to the constant function 1. From this deduce that if $N \geq 1$ then $f - S_N$ takes the value 0 at some point in $[-\pi, \pi]$.

(d) Suppose that f is a 2π-periodic function which is differentiable on $\mathbf{R}$, and let S_N denote the N-th partial sum of the Fourier series for f. Show that if N is an odd number, so that $N = 2k + 1$ for some integer k and

$$S_N = \frac{b_0}{\sqrt{2}} + \sum_{n=1}^{k} (a_n \sin nx + b_n \cos nx),$$

then S'_N is the N-th partial sum of the Fourier series for f'. Combine this with the results of parts (b) and (c), and the "big theorem" about convergence of Fourier series in $(\mathcal{C}([-\pi, \pi], d_2)$, to show that the Fourier series for f converges pointwise to f.