Eighth problem set
Math 414
Due Friday, May 2, 2008

1. From the identity

$$\zeta(s) = \prod_{i=1}^{\infty}(1 - p_i^{-s})^{-1},$$

which is valid for $s > 1$, it follows that

$$\log \zeta(s) = \sum_{i=1}^{\infty} - \log(1 - p_i^{-s}).$$

By differentiating the latter identity, obtain a series expression for $\zeta'(s)/\zeta(s)$ which is valid for $s > 1$. Justify the resulting identity analytically. (You will have to quote results about differentiation of series proved in class, and this will require showing that your series expression converges uniformly on each closed sub-interval of $(1, \infty)$.)

2. **(a)** In the first lecture of the semester, starting with the identity

$$\log(1 + x) = \int_0^x \frac{dt}{1 + t}$$

which is valid for $x > 0$, I obtained the power series expression for $\log(1 + x)$ for $0 \leq x \leq 1$. Use the same method, starting with the identity

$$\arctan x = \int_0^x \frac{dt}{1 + t^2},$$

which is valid for every real x, to obtain the power series expression for $\arctan x$ for $0 \leq x \leq 1$. Of course you will need to do an explicit estimates of a certain integral.

(b) In class I defined $\chi(n)$, where n is an integer, by $\chi(2k) = 0$, $\chi(4k + 1) = 1$, and $\chi(4k - 1) = -1$. I also is the defined a function $L(s)$ for $s > 1$, by

$$L(s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

Use the result of part (a) to find an elementary expression for $L(1)$.