Solutions to first problem set
Math 414

1. (a) If $\sum_{n=1}^\infty b_n$ and $\sum_{n=1}^\infty c_n$ are convergent series of real numbers then $\sum_{n=1}^\infty (b_n+c_n)$ is convergent. Hence if $\sum_{n=1}^\infty a_n^+$ and $\sum_{n=1}^\infty a_n^-$ are convergent then $\sum_{n=1}^\infty (a_n^+ + a_n^-) = \sum_{n=1}^\infty |a_n|$ is convergent, i.e. $\sum_{n=1}^\infty a_n$ is absolutely convergent.
Conversely, suppose that $\sum_{n=1}^\infty a_n$ is absolutely convergent, i.e. that $\sum_{n=1}^\infty |a_n|$ is convergent. Since $0 \le a_n^+ \le |a_n|$ and $0 \le a_n^- \le |a_n|$ for every n, the comparison test guarantees that $\sum_{n=1}^\infty a_n^+$ and $\sum_{n=1}^\infty a_n^-$ are convergent.

(b) If $\sum_{n=1}^\infty b_n$ and $\sum_{n=1}^\infty c_n$ are convergent series of real numbers then $\sum_{n=1}^\infty (b_n-c_n)$ is convergent and

$$\sum_{n=1}^\infty (b_n - c_n) = \sum_{n=1}^\infty b_n - \sum_{n=1}^\infty c_n.$$

If $\sum_{n=1}^\infty a_n$ is an absolutely convergent series, then since $\sum_{n=1}^\infty a_n^+$ and $\sum_{n=1}^\infty a_n^-$ are convergent by part (a), the series $\sum_{n=1}^\infty a_n$ is convergent and $\sum_{n=1}^\infty (a_n^+ - a_n^-)$ is convergent and

$$\sum_{n=1}^\infty (a_n^+ - a_n^-) = \sum_{n=1}^\infty a_n^+ - \sum_{n=1}^\infty a_n^-.$$

But $a_n^+ - a_n^- = a_n$. Hence $\sum_{n=1}^\infty a_n$ is convergent and

$$\sum_{n=1}^\infty a_n = \sum_{n=1}^\infty a_n^+ - \sum_{n=1}^\infty a_n^-.$$

2. (a) Let $\sum_{n=1}^\infty a_n$ be a convergent sequence of non-negative terms. Define a *generalized partial sum* of $\sum_{n=1}^\infty a_n$ to be a number of the form $a_{m_1}+\cdots+a_{m_k}$ be a generalized partial sum. After reordering the finite sum we may assume $m_1 < \cdots < m_k$. Set $N = m_k$. Since all the a_n are non-negative we have

$$a_{m_1} + \cdots + a_{m_k} \le a_1 + \cdots a_N,$$

and the right hand side is a partial sum.

(b) Let $\mathcal{S}$ and $\mathcal{S}'$ denote, respectively, the set of upper sums and the set of generalized upper sums of $\sum_{n=1}^\infty a_n$. Since $\sum_{n=1}^\infty a_n$ is convergent, $\mathcal{S}$ is bounded. Let B denote its supremum. By part (a), every element of $\mathcal{S}'$ is bounded above by an element of $\mathcal{S}$ and hence by B. It follows that $\mathcal{S}'$ is bounded above and that its supremum B' is at most B. On the other hand, it is obvious that $S \subset S'$ and hence that B' is an upper bound for S, so that $B \le B'$. Hence $B' = B$.

(c) Suppose that $\sum_{n=1}^\infty a_n$ is a convergent series of non-negative terms. Let S denote its sum. By definition S is the limit of the sequence of partial sums $(S_N)_{N\ge1}$, where $S_N = \sum_{n=1} N a_N$. By the result quoted from the theory of sequences, S

is the supremum of the S_N. But by part (b), this is the same as the supremum of the generalized partial sums.

(d) Suppose that $\sum_{n=1}^{\infty} a_n$ is a convergent series of non-negative terms, and that $\sum_{n=1}^{\infty} b_n$ is a rearrangement of that $\sum_{n=1}^{\infty} b_n$. Let $\mathcal{S}$ and $\mathcal{T}$ denote, respectively the set of generalized partial sums of $\sum_{n=1}^{\infty} a_n$ and the set of generalized partial sums of $\sum_{n=1}^{\infty} b_n$. According to part (c), the sum S of $\sum_{n=1}^{\infty} a_n$ is the supremum of $\mathcal{S}$, and the sum T of $\sum_{n=1}^{\infty} b_n$ is the supremum of $\mathcal{T}$. But since $\sum_{n=1}^{\infty} b_n$ is a rearrangement of $\sum_{n=1}^{\infty} a_n$, it follows from the definitions that $\mathcal{S} = \mathcal{T}$. Hence $S = T$.

(e) Consider any abolutely convergent series $\sum_{n=1}^{\infty} a_n$. According to problem #1, the series $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ are convergent, and

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^-.$$

Now consider an arbitrary rearrangement $\sum_{n=1}^{\infty} a_{\pi(n)}$ of $\sum_{n=1}^{\infty} a_n$. The series $\sum_{n=1}^{\infty} a_{\pi(n)}^+$ is a rearrangement of $\sum_{n=1}^{\infty} a_n^+$, which is a convergent series of non-negative terms. Hence by part (d), $\sum_{n=1}^{\infty} a_{\pi(n)}^+$ is convergent and has the same sum as $\sum_{n=1}^{\infty} a_n^+$. Similarly, $\sum_{n=1}^{\infty} a_{\pi(n)}^-$ is convergent and has the same sum as $\sum_{n=1}^{\infty} a_n^-$. Since $\sum_{n=1}^{\infty} a_{\pi(n)}^+$ and $\sum_{n=1}^{\infty} a_{\pi(n)}^-$ are convergent, it follows from 1(a) that $\sum_{n=1}^{\infty} a_{\pi(n)}$ is **absolutely** convergent. Furthermore, it follows from 1(b) that we have

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\pi(n)}^+ - \sum_{n=1}^{\infty} a_{\pi(n)}^- = \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^- = \sum_{n=1}^{\infty} a_n.$$

Note: I wrote “absolutely” in bold because the word was inadvertently omitted from the assigned problem.

3. Define the b_i and the c_i as in the proof of part (b) of the theorem. Since $b_1 + b_2 + \cdots$ is a divergent series of non-negative terms, there is a natural number r_1 such that $b_1 + b_2 + \cdots + b_{r_1} > 1$. Since $b_{r_1+1} + b_{r_1+2} + \cdots$ is a divergent series of non-negative terms, there is a natural number $r_2 > r_1$ such that

$$b_1 + b_2 + \cdots + b_{r_1} - c_1 + b_{r_1+1} + b_{r_1+2} + \cdots + b_{r_2} > 2.$$

Similarly, there is a natural number $r_3 > r_1$ such that

$$b_1 + b_2 + \cdots + b_{r_1} - c_1 + b_{r_1+1} + b_{r_1+2} + \cdots + b_{r_2} - c_2 + b_{r_2+1} + b_{r_2+2} + \cdots + b_{r_3} > 3.$$

Continuing in this way we obtain a rearrangement

$$b_1 + b_2 + \cdots + b_{r_1} - c_1 + b_{r_1+1} + b_{r_1+2} + \cdots + b_{r_2} - c_2 + \cdots$$

of the given series which I claim diverges to $+\infty$.
It suffices to show that if K is a natural number, there is a natural number N such that every partial sum for the rearranged series, from the N-th one on, is greater than K. Since we showed in the proof of part (b) that $c_i \to 0$, we may fix an M such that $c_i < 1$ whenever $i \geq M$. Set $K' = \max(K+1, M)$ and $N = K' + r_{K'}$. Then the N-th partial sum is

$$b_1 + \cdots + b_{r_1} - c_1 + \cdots + b_{r_{K'-1}+1} + \cdots + b_{r_{K'}} - c_{K'}.$$

Any partial sum from the N-th one on is bounded below by a partial sum of the form

$$b_1 + \cdots + b_{r_1} - c_1 + \cdots + b_{r_{L-1}+1} + \cdots + b_{r_L} - c_L$$

for some $L \geq K'$. By construction we have

$$b_1 + \cdots + b_{r_1} - c_1 + \cdots + b_{r_{L-1}+1} + \cdots + b_{r_L} > L \geq K' \geq K + 1.$$

Since $L \geq K' \geq M$ we have $c_L < 1$. Hence

$$b_1 + \cdots + b_{r_1} - c_1 + \cdots + b_{r_{L-1}+1} + \cdots + b_{r_L} - c_L > K + 1 - 1 = K.$$

4. The series in question are

$$1 + \frac{1}{3} + \frac{1}{5} + \cdots$$

and

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots.$$

The second series is obtained from the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

by multiplying each term by $1/2$, and therefore diverges. The terms of the first series are bounded below by those of the second series, and the comparison test therefore shows that the first series is divergent.