

**Solutions to second problem set
Math 414**

1. (a) For every $n \geq 1$ and every $x \in (0, \infty)$, we have

$$0 < \frac{1}{(x+n)^2} < \frac{1}{n^2},$$

and the series of positive terms

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent by the integral test. Hence the Weierstrass M -test shows that

$$\sum_{n=1}^{\infty} \frac{1}{(x+n)^2}$$

is uniformly convergent on $(0, \infty)$.

(b) For every $n \geq 1$ we have

$$D_x \frac{1}{(x+n)^2} = \frac{-2}{(x+n)^3}$$

for $x > 0$. Hence if

$$\sum_{n=1}^{\infty} \frac{1}{(x+n)^2}$$

is pointwise convergent on $(0, \infty)$, and

$$\sum_{n=1}^{\infty} \frac{-2}{(x+n)^3}$$

is uniformly convergent on $(0, \infty)$, it will follow that

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{(x+n)^2}$$

is differentiable on $(0, \infty)$ and that

$$f(x) = \sum_{n=1}^{\infty} \frac{-2}{(x+n)^3}.$$

Certainly

$$\sum_{n=1}^{\infty} \frac{1}{(x+n)^2}$$

is pointwise convergent on $(0, \infty)$, since it is uniformly convergent by part (a). On the other hand, for every $n \geq 1$ and every $x \in (0, \infty)$, we have

$$\left| \frac{-2}{(x+n)^3} \right| < \frac{2}{n^3},$$

and the series of positive terms

$$\sum_{n=1}^{\infty} \frac{2}{n^3}$$

is convergent by the integral test. Hence the Weierstrass M -test shows that

$$\sum_{n=1}^{\infty} \frac{-2}{(x+n)^3}$$

is uniformly convergent on $(0, \infty)$.

2. (a) For every $n \geq 1$ and every $x \in (-\infty, \infty)$, we have

$$|f_n(x)| = \frac{1}{n^2} |\sin nx| \leq \frac{1}{n^2},$$

and the series of positive terms

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent by the integral test. Hence the Weierstrass M -test shows that

$$\sum_{n=1}^{\infty} f_n(x)$$

is uniformly convergent on $(-\infty, \infty)$.

(b) For every $n \geq 1$ we have

$$f'_n(0) = \frac{1}{n} \cos(n \cdot 0) = \frac{1}{n}.$$

By the integral test, the series

$$\sum_{n=1}^{\infty} f'_n(0) = \sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent. In particular, the series

$$\sum_{n=1}^{\infty} f'_n(x)$$

is not pointwise convergent on $(-\infty, \infty)$, and hence not uniformly convergent.

- 3.** Define a sequence of functions $(g_n(x))$ on $[0, 1]$, for $n \geq 1$, by $g_1(x) = x$ and $g_n(x) = x^n - x^{n-1}$ for $n \geq 2$. Then we have

$$\sum_{n=1}^N g_n(x) = x^N$$

for every $N \geq 1$. I pointed out in class that the sequence of functions $(x^N)_{N \geq 1}$ converges pointwise but not uniformly on $[0, 1]$. Hence the series

$$\sum_{n=1}^{\infty} g_n(x)$$

converges pointwise but not uniformly on $[0, 1]$.

- 4. (a)** The series converges pointwise because

$$\sum_{i=0}^N (-1)^i x^i = \frac{1 - (-x)^{N+1}}{1 + x},$$

so that

$$\sum_{i=0}^{\infty} (-1)^i x^i = \frac{1}{1 + x}$$

for $0 \leq x < 1$. Uniform convergence would mean that

$$\sup_{0 \leq x < 1} \left| \frac{1 - (-x)^{N+1}}{1 + x} - \frac{1}{1 + x} \right| \rightarrow 0$$

as $N \rightarrow \infty$. But for every N we have

$$\sup_{0 \leq x < 1} \left| \frac{1 - (-x)^{N+1}}{1 + x} - \frac{1}{1 + x} \right| = \sup_{0 \leq x < 1} \left| \frac{x^{N+1}}{1 + x} \right| = \frac{1}{2}$$

since $x^N/(1+x)$ tends to $1/2$ as x approaches 1 from the left. Hence the convergence is not uniform.

- (b)** For $x = 1$ the series does not converge because the terms all have absolute value 1. Hence it does not converge pointwise on $[0, 1]$, and in particular it does not converge uniformly.

- (c)** Starting from the identity

$$\sum_{i=0}^N (-1)^i t^i = \frac{1 + (-t)^{N+1}}{1 + t}$$

and integrating from 0 to x , where $0 \leq x \leq 1$, we obtain

$$\sum_{i=0}^N (-1)^i \frac{x^{i+1}}{i+1} = \log(1+x) + \int_0^x \frac{(-t)^{N+1}}{1+t} dt.$$

Hence

$$\begin{aligned} \left| \sum_{i=0}^N (-1)^i \frac{x^{i+1}}{i+1} - \log(1+x) \right| &= \int_0^x \frac{t^{N+1}}{1+t} dt \\ &\leq \int_0^x t^{N+1} dt \\ &= \frac{x^{N+2}}{N+2} \leq \frac{1}{N+2}. \end{aligned}$$

It follows that

$$\sup_{0 \leq x \leq 1} \left| \sum_{i=0}^N (-1)^i \frac{x^{i+1}}{i+1} - \log(1+x) \right| \leq \frac{1}{N+2}$$

and hence that

$$\sup_{0 \leq x \leq 1} \left| \sum_{i=0}^N (-1)^i \frac{x^{i+1}}{i+1} - \log(1+x) \right| \rightarrow 0$$

as $N \rightarrow \infty$. This is uniform convergence.