Solutions to sixth problem set
Math 414

1. We have

$$\begin{aligned}\|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \\ &= \|x\|^2 + 2\langle x,y\rangle + \|y\|^2.\end{aligned}$$

As the identity

$$\|x+y\|^2 = \|x\|^2 + 2\langle x,y\rangle + \|y\|^2$$

is now seen to hold for all x and y, we may substitute $-y$ for y to obtain

$$\|x-y\|^2 = \|x\|^2 - 2\langle x,y\rangle + \|y\|^2.$$

Adding the last two identities we obtain

$$\|x-y\|^2 + \|x+y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

2. (a) We are given that the set $\{u_1, \dots, u_n\}$ is orthonormal. Hence to show that $\{u_1, \dots, u_{n+1}\}$ is orthonormal, it suffices to show that $\|u_{n+1}\| = 1$ and that $\langle u_i, , u_{n+1}\rangle = 0$ for $i = 1, \dots, n$.

We recall the definition of $\mathrm{proj}_W(v)$. There are unique vectors $x \in W$ and $y \in W^\perp$ such that $v = x + y$, and by definition we have $\mathrm{proj}_W(v) = x$. Hence $y = v - \mathrm{proj}_W(v)$. Since $v \notin W$ we have $y \neq 0$, so that $u_{n+1} = y/\|y\|$ is defined and

$$\|u_{n+1}\| = \frac{\|y}{\|y\|} = 1.$$

For $i = 1, \dots, n$, since $y \in W^\perp$ and $u_i \in W$, we have $\langle u_i, y\rangle = 0$, and hence

$$\langle u_i, u_{n+1}\rangle = \frac{\langle u_i, y\rangle}{\|y\|} = 0.$$

This shows that $\{u_1, \dots, u_{n+1}\}$ is orthonormal.

It remains to show that $\{u_1, \dots, u_{n+1}\}$ has the same span as $\{u_1, \dots, u_n, v\}$. For this purpose, we first note that since x lies in W, which is the span of $\{u_1, \dots, u_n\}$, there are scalars $\beta_1, \dots, \beta_n$ such that $x = \beta_1 u_1 + \cdots \beta_n u_n$. We have

$$\begin{aligned}v &= x + y \\ &= x + \|y\| u_{n+1} \\ &= \beta_1 u_1 + \cdots \beta_n u_n + \|y\| u_{n+1}.\end{aligned}$$

If z is any vector in the span of $\{u_1, \ldots, u_n, v\}$, there are scalars $\alpha_1, \ldots, \alpha_n, \alpha_{n+1}$ such that $z = \alpha_1 u_1 + \ldots, \alpha_n u_n + \alpha_{n+1} v$. We now have

$$\begin{aligned} z &= \alpha_1 u_1 + \ldots + \alpha_n u_n + \alpha_{n+1}(\beta_1 u_1 + \cdots + \beta_n u_n + \|y\| u_{n+1}) \\ &= (\alpha_1 + \alpha_{n+1}\beta_1)u_1 + \cdots + (\alpha_n + \alpha_{n+1}\beta_n)u_n + \alpha_{n+1}\|y\| u_{n+1}, \end{aligned}$$

which shows that z lies in the span of $\{u_1, \ldots, u_n, v\}$.

Conversely, suppose that z is a vector in the span of $\{u_1, \ldots, u_{n+1}\}$. Then there are scalars $\gamma_1, \ldots, \gamma_n, \gamma_{n+1}$ such that $z = \gamma_1 u_1 + \ldots, \gamma_n u_n + \gamma_{n+1} u_{n+1}$. We now have

$$\begin{aligned} z &= \gamma_1 u_1 + \ldots + \gamma_n u_n + \frac{\gamma_{n+1}}{\|y\|}(v - \beta_1 u_1 - \cdots - \beta_n u_n) \\ &= (\gamma_1 - \frac{\gamma_{n+1}\beta_1}{\|y\|})u_1 + \cdots + (\gamma_n - \frac{\gamma_{n+1}\beta_n}{\|y\|})u_n + \frac{\gamma_{n+1}}{\|y\|}v, \end{aligned}$$

which shows that z lies in the span of $\{u_1, \ldots, u_n, v\}$.

(b) First suppose that z is a vector in the span of $\{v_1, \ldots, v_n, x\}$. Then there are scalars $\alpha_1, \ldots, \alpha_n, \alpha_{n+1}$ such that $z = \alpha_1 v_1 + \ldots, \alpha_n v_n + \alpha_{n+1} x$. If we set $w = \alpha_1 v_1 + \ldots, \alpha_n v_n$, then $w \in W$ and $z = 1w + \alpha_{n+1} x$, so that z is in the span of $W \cup \{x\}$.

Conversely, suppose that z is a vector in the span of $W \cup \{x\}$. By definition this means that there are scalars $\beta_1, \ldots, \beta_m$ and vectors $y_1, \ldots, y_m \in W \cup \{x\}$ such that $z = \beta_1 y_1 + \ldots \beta_m y_m$. After re-indexing we may assume that $y_i \in W$ for $1 \le i \le k$ and $y_i = x$ for $k < i \le m$, where k is some integer with $0 \le k \le m$. Setting $w = \beta_1 y_1 + \ldots \beta_k y_k$ and $\gamma = \beta_{k+1} \ldots + \beta_m$, we find that $w \in W$ and that $x = w + \gamma x$. We may write $w = \delta_1 v_1 + \cdots \delta_n v_n$ for some scalars $\delta_1, \ldots, \delta_n$. We then have $x = \delta_1 v_1 + \cdots \delta_n v_n + \gamma x$, so that z is in the span of $\{v_1, \ldots, v_n, x\}$.

In the special case where $x \in W$ we have $W \cup \{x\} = W$, so the result proved above shows that the span of $\{v_1, \ldots, v_n, x\}$ is equal to the span of W. But W is its own span since it is a subspace.

(c) For $n = 0$ this asserts that if W is spanned by an empty set of vectors then it is spanned by an empty orthonormal set. This is true since an empty set is orthogonal by default.

Assume the assertion is true for a given n, and that a subspace W_{n+1} of V is spanned by vectors $z_1, \ldots, z_{n+1}$. Let W_n denote the span of $z_1, \ldots, z_n$. By the induction hypothesis, W_n is spanned by an orthonormal set $\{u_1, \ldots, u_m\}$ for some $m \le n$. In the case where $z_{n+1} \in W_n$, it follows from part (b) that $W_{n+1} = W_n$, so that W_{n+1} is spanned by the orthonormal set $\{u_1, \ldots, u_m\}$. Now suppose that $z_{n+1} \notin W_n$. According to part (b), W_{n+1} is the span of $W_n \cup \{z_{n+1}\}$. Applying part (b) again we deduce that part (b), W_{n+1} is the span of $\{u_1, \ldots, u_m, z_{n+1}\}$. Now set

$$u_{m+1} = \frac{z_{n+1} - \mathrm{proj}_W(z_{n+1})}{\|z_{n+1} - \mathrm{proj}_W(z_{n+1})\|}.$$

Then according to part (a), $u_1, \ldots, u_{m+1}$ is an orthonormal system and has the same span as $u_1, \ldots, u_m, z_{n+1}$, namely W_{n+1}. Since $m+1 \leq n+1$, this completes the induction.

3. Set $M = \|g\|$. Given any $\epsilon > 0$, the uniform continuity of f on $[-2\pi, 2\pi]$ gives a positive δ such that for all $x, x' \in [-2\pi, 2\pi]$ with $|x - x'| < \delta$ we have $|f(x) - f(x')| < \epsilon/(2\pi M)$. We may take $\delta < \pi$. Then for any $x, x' \in \mathbf{R}$ with $|x - x'| < \delta$, there is an integer n such that $x_0 = x + 2\pi_n$ and $x'_0 = x' + 2\pi_n$ lie in $[-2\pi, 2\pi]$. We have $|x_0 - x'_0| = |x - x'| < \delta$, and therefore $|f(x) - f(x')| = |f(x_0) - f(x'_0)| < \epsilon/(2\pi M)$.

Now let x and x' be real numbers with $|x - x'| < \delta$. We have

$$\begin{aligned}|(f \star g)(x) - (f \star g)(x')| &= |\int_{-\pi}^{\pi} f(x-y)g(y)\,dy - \int_{-\pi}^{\pi} f(x'-y)g(y)\,dy| \\ &= |\int_{-\pi}^{\pi} (f(x-y) - f(x'-y))g(y)\,dy| \\ &\leq \int_{-\pi}^{\pi} |f(x-y) - f(x'-y))||g(y)|\,dy|.\end{aligned}$$

For every $y \in [-pi, \pi]$ we have $|(x-y) - (x'-y)| = |x - x'| < \delta$ and hence $|f(x - y) - f(x' - y))| < \epsilon/(2\pi M)$. We also have $g(y) \leq M$. Hence

$$|(f \star g)(x) - (f \star g)(x')| < \int_{-\pi}^{\pi} \frac{\epsilon}{2\pi M} M = \epsilon,$$

which shows that $f \star g$ is uniformly continuous, and in particular continuous.

For periodicity we need only notice that

$$\begin{aligned}f \star g(x + 2\pi) &= \int_{-\pi}^{\pi} f((x+2\pi) - y)g(y), dy \\ &= \int_{-\pi}^{\pi} f((x-y) + 2\pi)g(y), dy \\ &= \int_{-\pi}^{\pi} f(x-y)g(y), dy \\ &= f \star g(x).\end{aligned}$$

(The substitution rule is not needed.)