**Solutions to eighth problem set**
**Math 414**

**1.** We have

$$D_s(\log\zeta(s)) = \frac{\zeta'(s)}{\zeta(s)}$$

and

$$D_s(-\log(1-p_i^{-s})) = \frac{(\log p_i)p_i^{-s}}{1-p_i^{-s}},$$

which suggests the formula

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{i=1}^{\infty}\frac{(\log p_i)p_i^{-s}}{1-p_i^{-s}}.$$

To justify this, we first recall the general result that if a series of functions $\sum_{i=1}^{\infty} f_i(s)$ converges pointwise on an open interval $I$ to a function $F(s)$, if each $f_i$ has a continuous derivative on $I$ and if $\sum_{i=1}^{\infty} f_i'(s)$ converges uniformly on $I$, then $F$ is differentiable on $I$ and $F'(s) = \sum_{i=1}^{\infty} f_i'(s)$. We apply this with $I = (\alpha, \infty)$ where $\alpha$ is a constant greater than 1, and with $f_i(s) = -\log(1-p_i^{-s})$. It follows that if the series

$$\sum_{i=1}^{\infty}\frac{(\log p_i)p_i^{-s}}{1-p_i^{-s}}$$

converges uniformly on $(\alpha,\infty)$ then its sum will be equal to the derivative of $-\log(1-p_i^{-s})$, namely $\zeta'(s)/\zeta(s)$, on the interval $(\alpha,\infty)$. Hence if this is true for every $\alpha > 1$, the formula given above for $\zeta'(s)/\zeta(s)$ will be valid on the entire interval $(1,\infty)$.

It therefore suffices to prove that

$$\sum_{i=1}^{\infty}\frac{(\log p_i)p_i^{-s}}{1-p_i^{-s}}$$

converges uniformly on $[\alpha,\infty)$ for every $\alpha > 1$. For each $i \geq 1$ and each $s \geq \alpha$ we have $1 - p_i^{-s} \geq 1 - 2^{-1} = 1/2$ and $p_i^{-s} \leq p_i^{-\alpha}$. Hence

$$0 \leq \frac{(\log p_i)p_i^{-s}}{1-p_i^{-s}} \leq 2(\log p_i)p_i^{-\alpha}.$$

According to the Weierstrass M-test, it therefore suffices to show that the series of positive terms $\sum_{i=1}^{\infty} 2(\log p_i)p_i^{-\alpha}$ converges for any $\alpha > 1$.

The terms of this series form a subsequence of the terms of the series

$$\sum_{n=1}^{\infty} 2(\log n)n^{-\alpha}.$$

By the comparison test it therefore suffices to show that the latter series converges. For this purpose we recall that for any constant $\beta > 0$ we have $\lim_{x\to\infty}(\log x)/x^\beta = 0$. Since $\alpha > 1$ we may apply this, taking $\beta = (\alpha-1)/2$. We deduce that in particular for large enough $n$, say for $n \geq n_0$, we have $\log n < /n^\beta$, and hence

$$2(\log n)n^{-\alpha} = \frac{2\log n}{n^{1+2\beta}} < 2n^{-(1+\beta)}.$$

Since $\beta > 0$ we know that $\sum_{n=1}^\infty n^{-(1+\beta)}$ converges, and hence $\sum_{n=1}^\infty 2(\log n)n^{-\alpha}$ converges by the comparison test.

**2. (a)** We recall that for any $y \neq -1$ and any $n \geq 1$ we have

$$\frac{1}{1+y} = 1 - y + y^2 - \cdots + (-1)^n y^n + (-1)^{n+1}\frac{y^{n+1}}{1+y}.$$

Substituting $t^2$ for $y$, we deduce for any $t \in \mathbf{R}$ and any $n \geq 1$ we have

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - \cdots + (-1)^n t^{2n} + (-1)^{n+1}\frac{t^{2n+2}}{1+t^2}.$$

Integrating from $t = 0$ to $t = x$, where $x$ is any real number, we get

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n\frac{x^{2n+1}}{2n+1} + \int_0^x \frac{t^{2n+2}}{1+t^2}\,dt.$$

Now suppose that $0 \leq x \leq 1$. In this case we have

$$\begin{aligned} 0 &\leq \int_0^x \frac{t^{2n+2}}{1+t^2}\,dt \\ &\leq \int_0^1 t^{2n+2}, dt \\ &= \frac{1}{2n+3}. \end{aligned}$$

Since $1/(2n+3) \to \infty$ as $n \to \infty$, it follows that

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots.$$

**(b)** We have

$$L(s) = \frac{1}{1^s} - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \cdots$$

for every $s > 0$. Setting $s = 1$ we obtain

$$L(1) = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

By part (a) we have

$$\frac{\pi}{4} = \arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

Hence $L(1) = \pi/4$.