Motive Cohomology and Arithmetic Intersection Theory

Henri Gillet

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Goal: To give a purely sheaf theoretic construction of arithmetic Chow groups

- Give a direct construction of the intersection product, without using the moving lemma.

- Use to give nice definition of $\widehat{CH}$, and its product structure, for stacks.

Codimension 1

$$\widehat{\mathrm{CH}}^1(X) \simeq \widehat{\mathrm{Pic}}(X) \simeq \mathbb{H}^2(X, \mathcal{O}_X^* \stackrel{\log|\,|}{\to} \mathcal{A}_X^0)$$

Here $\mathcal{A}_X^0$ is the sheaf of C^∞ real valued functions on $X(\mathbb{C})$.

Codimension 2

If we are over a field, have Bloch's formula:

$$\mathrm{CH}^2(X) \simeq H^2(X, K_2(\mathcal{O}_X))$$

Is there a map $K_2(\mathcal{O}_X) \to \mathcal{A}_X^1$?
Have regulator

$$K_2(X) \to H^2_{\mathcal{D}}(X, \mathbb{R}(1))$$

K_2 is generated by symbols:

$$\mathcal{O}_X^* \otimes \mathcal{O}_X^* \twoheadrightarrow K_2(\mathcal{O}_X)\ ,$$

Symbols and the regulator

$$\begin{aligned}\lambda_2 : \mathcal{O}_X^* \otimes \mathcal{O}_X^* &\longrightarrow \mathcal{A}_X^{1,0} \oplus \mathcal{A}_X^{0,1} \\ f \otimes g &\mapsto \log|f| * \log|g|\end{aligned}$$

where

$$\log|f| * \log|g| = \log|f|\left(\frac{\partial g}{g} - \frac{\overline{\partial} g}{g}\right) - \log|g|\left(\frac{\partial f}{f} - \frac{\overline{\partial} f}{f}\right)$$

But

$$\lambda_2(\{f\} \otimes \{1-f\}) \neq 0$$

Can we write:

$$\lambda_2(\{f\} \otimes \{1-f\}) = d(?)$$

Look for complex which computes (Milnor) K-theory), and map from that complex to ??

Milnor K-theory and Bloch's formula

If F is a field, Milnor K-theory is the graded ring:

$$K_*^M(F) := \bigwedge^*(F^*)/(\{f\} \wedge \{1-f\})$$

Bloch's formula: (Quillen (for K-theory), Kato, Rost for Milnor K-theory)
If X is a regular variety over a field:

$$\mathrm{CH}^p(X) \simeq H^p(X, \mathcal{K}_p^M)$$

Here $\mathcal{K}_p^M$ is defined as a subsheaf of the constant sheaf $K_p^M(k(X))$.

Proof depends on **Gersten's conjecture** for regular local rings – which is a theorem of Quillen for local rings on regular varieties over fields.

Motivic Cohomology

If X is a regular and U a variety over k, $\mathrm{Cor}(X,U) =$ cycles on $X \times U$, finite over X.

- contravariant with respect to X
- covariant with respect to U.

For $i = 1 \ldots k$, have $j_i : \mathbb{G}_m^{k-1} \to \mathbb{G}_m^k$. Set:

$$\mathrm{Cor}(X, \mathbb{G}_m^{\wedge k}) = \mathrm{Cor}(X, \mathbb{G}_m^k) / \sum_{i=1}^{k} j_{i,*}(\mathrm{Cor}(X, \mathbb{G}_m^{k-1})) .$$

Have cosimplicial scheme

$$n \mapsto \mathbb{A}^n = \mathrm{Spec}(\mathbb{Z}[t_0, \ldots, t_n]/(\sum_i t_i = 1)) .$$

Associated simplicial group:

$$n \mapsto \mathrm{Cor}(X \times \mathbb{A}^n, \mathbb{G}_m^{\wedge k})$$

Definition(Voevodsky et. al.):
1.

$$\mathbb{Z}(n)^i(U) := \mathsf{Cor}(U \times \mathbb{A}^{n-i}, \mathbb{G}_m^{\wedge n}) \ .$$

This is a complex of presheaves of abelian groups.
2.

$$H^p(X, \mathbb{Z}(n)) := \mathbb{H}^p(X, \underline{\mathbb{Z}}^*(n))$$

Theorem. *If X is a regular variety over a field,*

$$\mathsf{CH}^p(X) \simeq H^{2p}(X, \mathbb{Z}(p))$$

Again proof depends on Gersten's conjecture.

Real Deligne Cohomology

Recall, that following Burgos, we have a nice description of this:

If X is a smooth variety over $\mathbb{C}$, have the complex

$$E^{*,*}_{\log}(X) := \lim_{\substack{\rightarrow \\ X=\overline{X}-D}} E^{*,*}_{\log D}(\overline{X})$$

The algebra $E^{*,*}_{\log D}(\overline{X})$ of forms with logarithmic singularities along the D.N.C. D is the global sections of the subsheaf of $\mathcal{E}^{*,*}(\overline{X})$-algebras of $\mathcal{E}^{*,*}(X)$ generated locally by:

$$\log z_i\overline{z}_i \ , \ \frac{dz_i}{z_i} \ , \ \frac{\overline{d}z_i}{z_i}$$

where z_i is a local equation of a smooth component of D.

Facts

- This complex computes $H^*(X, \mathbb{C})$
- the subcomplex of forms with real coefficients $E^{*,*}_{\log,\mathbb{R}}(X)$ computes $H^*(X, \mathbb{R})$.
- The bigrading gives the Hodge Filtration.

The real Deligne-Beilinson cohomology of X $H^*_{\mathcal{D}}(X, \mathbb{R}(p))$ may then be computed by the complex (writing $A^{*,*}(p)$ for $(2\pi i)^p E^{*,*}_{\log,\mathbb{R}}(X)$):

$\mathfrak{D}^*(X, p) :=$

$$\ldots \to A^{p-2,p-1}(p-1) \oplus A^{p-1,p-2}(p-1) \to$$

$$A^{p-1,p-1}(p-1) \xrightarrow{-2\partial\overline{\partial}} A^{p,p}(p)$$

$$\to A^{p,p+1}(p) \oplus A^{p+1,p}(p) \to \ldots$$

Note that $A^{p-1,p-1}(p-1)$ is in degree $2p-1$!

If $f \in \mathcal{O}_X^*$, $\log|f| \in \mathfrak{D}^1(X,1)$,

Hence on $\mathbb{G}_m^{\wedge n} = \mathrm{Spec}(\mathbb{Z}[z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}])$ have an $(n-1, n-1)$-form

$\sigma_n \in \log|z_1| * \log|z_2| * \ldots * \log|z_n| \in \mathfrak{D}^n(X \times \mathbb{G}_m^{\wedge k}, n)$

We have the standard topological n-simplex:

$$\Delta^n \subset \mathbb{A}^n(\mathbb{R}) \; .$$

Define

$$\lambda^i(n) : Z^i(X,n) \;\; \to \;\; \tilde{\mathfrak{D}}^i(X,n)$$
$$\zeta \mapsto \int_{\Delta^{n-i} \times \mathbb{G}_m^k} \sigma_n \wedge \delta_\zeta$$

Here $\tilde{\mathfrak{D}}$ denoted forms with mild singularities

Theorem. • *$\lambda^*(n)$ is a map of complexes, and is compatible with the product structures on $Z^*(X,*)$ and $\mathfrak{D}^*(X,*)$.*

• *Let X be a regular variety over $\mathbf{Q}$. There is an isomorphism:*

$$\widehat{\mathsf{CH}}^p(X) \simeq$$

$$\mathbb{H}^{2p}(\mathsf{simple}(Z^*(X,p) \to \tau^{\leq 2p-1}\tilde{\mathfrak{D}}^*(X,p)))$$

• *This isomorphism is compatible with products (at least up to homotopy)– note that the product on RHS is the product on the simple of a map of DGA's.*

Note: Goncharov gives a construction of $\widehat{\mathsf{CH}}^p(X)$ using Bloch's higher Chow groups. This works over $\mathbb{Z}$, but does not have obvious products.

Avoiding Gersten's Conjecture?

Know that product on $\mathrm{CH}^*(X)_{\mathbb{Q}}$ may be defined via

$$\mathrm{CH}^p(X)_{\mathbb{Q}} \simeq \mathrm{Gr}^p_\gamma(X)_{\mathbb{Q}} \,.$$

Grayson – filtration $F^i_{\mathrm{Gr}}K(X)$ via K-theory of commuting automorphisms.

Theorem.

$$\mathrm{Gr}^p_{\mathrm{Gr}}(K(X)) \simeq Z^*_{\mathrm{Gr}}(X,p)$$

Here $Z^i_{\mathrm{Gr}}(X,p)$ is defined using modules on

$$X \times \mathbb{A}^{p-i} \times \mathbb{G}_m^{\wedge p}$$

which are finite and flat over $X \times \mathbb{A}^{p-i}$.

Theorem. (Suslin) *For regular varieties over a field, the natural map*

$$Z^*_{\mathsf{Gr}}(X,p) \to Z^*(X,p)$$

is a quasi-isomorphism.

Conjecture. *This filtration computes the γ-filtration on K-theory.*

Conjecture $\Rightarrow \mathbb{H}^{2p}(X, Z^*_{\mathsf{Gr}}(X,p)) \simeq \mathsf{CH}^p(X)_{\mathbb{Q}}$

This would then give a product structure on $\widehat{\mathsf{CH}}^*(X)_{\mathbb{Q}}$, for $X/\mathbb{Z}$, defined by hypercohomology of sheaves.