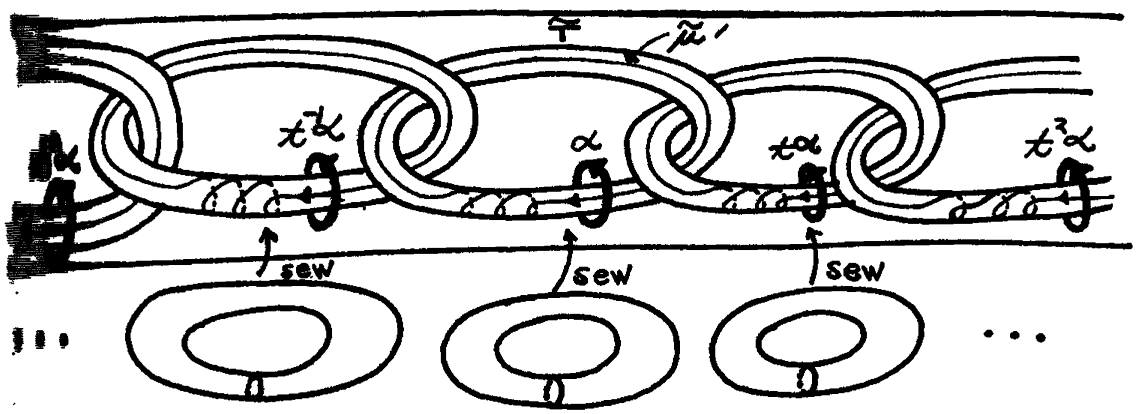
The infinite cyclic cover of $S^3-h(K)$ is

$$p : \mathbb{R}^1 \times \mathbb{R}^2 \longrightarrow S^1 \times \mathbb{R}^2 \cong S^3-h(K).$$



X_∞ is obtained by removing each t^nT and replacing it with the meridian along $t^n\mu'$. <u>Sewing in T kills $\mu' = t\alpha-3\alpha+t^{-1}\alpha$</u>. $H_1(X_\infty) = (\alpha \,|\, (t-3+t^{-1})\alpha = 0)$ as a $Z[t,t^{-1}]$ module. $\Delta(t) = t-3+t^{-1} \doteq t^2 - 3t+1$, the Alexander polynomial of the figure-eight knot.

ALEXANDER'S METHOD

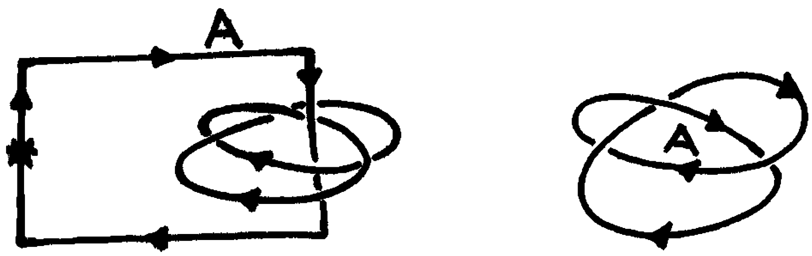
Example 9.11. It was by way of the geometry of coverings and particularly the infinite cyclic covering that people realized that the Alexander polynomial could be extracted from the fundamental group of the knot complement. This comes about as follows: By construction, $\pi_1(X_\infty) \cong G'$ where G' denotes the commutator subgroup of $G = \pi_1(S^3-K)$. Thus $H_1(X_\infty) \cong G'/G''$, and the action of $Z[t,t^{-1}]$ on

$H_1(X_\infty)$ corresponds algebraically to $t \cdot g = sgs^{-1}$ where $s \in \pi_1(S^3-K)$ is a chosen element having linking number with K. By definition, $\Delta_K(t)$ is the balance class (i.e., defined up to $\pm t^n$) of the ideal of elements $\rho \in Z[t,t^{-1}]$ such that $\rho g = 0$ for all $g \in G'/G''$. Thi can be computed purely group theoretically by using a standard presentation of $\pi_1(S^3-K)$. [One says that $\Delta_K(t) \doteq$ <u>the order of</u> <u>G'/G''</u> <u>over</u> $Z[t,t^{-1}]$.]

There are many algorithms of this sort. Here we wil sketch one that yields the computational method given by Alexander in his original paper [A1]: If $K \subset S^3$ is a knot and $G = \pi_1(S^3-K)$, then $G/G' \cong Z$ and the map $f : G \longrightarrow Z$ is given by $f(\alpha) = \ell k(\alpha,K)$. [In the case of oriented links, everything works in similar fashion to gi a map $G \longrightarrow Z$ even though this is not the abelianizatio Thus the action of $R = Z[t,t^{-1}]$ can be obtained by fin ing $s \in G$ with $f(s) = 1$.

Choose a presentation for the fundamental group $G = (g_0,\cdots,g_n | r_1,\cdots,r_n)$ with one more generator than there are relations. (For example, the Dehn presentation—we will use it in the next paragraph.) We can cho $s = g_0$ and <u>rewrite generators and relations</u> so that $f(g_1) = \cdots = f(g_n) = 0$ (replacing g_k by $s^{i_k} g_k$ when necessary). Then $g_1,\cdots,g_n \in G'$ and one can show that $G' = (g_1,\cdots,g_n | r_i = 1,\ i = 1,\cdots,n)$ is a presentation o G' as an R-module. Now abelianize G', and look for the relations.

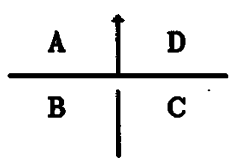
To make this concrete we use the Dehn presentation of $\pi_1(S^3-K) = G$.



$\ell k(A,K) = 2$

In the Dehn presentation, generators of $\pi_1(S^3-K)$ are in 1-1 correspondence with all-but-one of the regions of the knot diagram. We let the base-point, $*$, live in the unbounded region. Each of the other regions becomes an element of $\pi_1(S^3-K)$ by taking a path through it as illustrated above.

Each crossing gives rise to a relation:



$AB^{-1}CD^{-1} = 1.$

__Exercise__. Draw a picture of this relation.

The linking number of a generator with the knot is computed by a method of indexing the regions of the diagram with integers:

(1) Index (outer unbounded region) = 0.

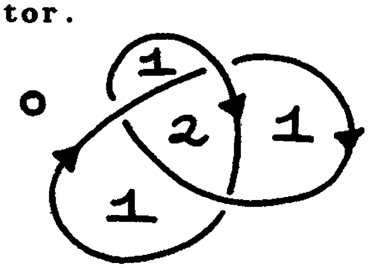
(2) Relative indices across an oriented edge form

this pattern.

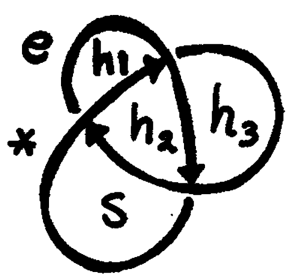
$$\left\{ p \uparrow p+1 \right\}$$

$$\left\{ p \downarrow p-1 \right\}$$

The index of a region is the linking number of the corresponding generator.



Let's use this format to get a presentation of G'/G''. First $G = (s, h_1, \cdots, h_n \mid r_1, \cdots, r_n)$ where s correspon to a region adjacent to the unbounded region.



$$\frac{e \uparrow h_1}{s \uparrow h_2}\ r_1 \qquad \frac{e \uparrow h_3}{h_1 \uparrow h_2}\ r_2 \qquad \frac{e \uparrow s}{h_3 \uparrow h_2}$$

$$r_1 = es^{-1}h_2h_1^{-1} = 1$$

<u>$e = 1$</u>

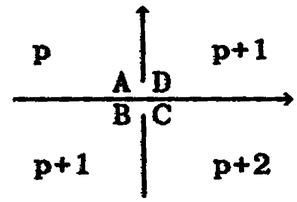
$$r_2 = eh_1^{-1}h_2h_3^{-1} = 1$$

$$r_3 = eh_3^{-1}h_2s^{-1} = 1$$

$$G = (s, h_1, h_2, h_3 \mid r_1 = 1,\ r_2 = 1,\ r_3 = 1)$$

But now we want to let $g_k = s^{-i_k}(h_k)$ so that $\ell k(g_k, K) = 0$. Thus $i_k = \text{Index}(h_k)$. And we have to rewrite the relations in terms of this new basis:

Here is one case. We leave the other as an exercis



With these orientations, the indices are p, $p+1$, $p+2$ as indicated.

$$r = A\bar{B}C\bar{D} \quad \text{(using bar } (\bar{\ }) \text{ for } (\ )^{-1}).$$

Let
$$a = s^{-p}A \qquad c = s^{-p-2}C$$
$$b = s^{-p-1}B \qquad d = s^{-p-1}D.$$

Then
$$A = s^{p}a \qquad C = s^{p+2}c$$
$$B = s^{p+1}b \qquad D = s^{p+1}d$$

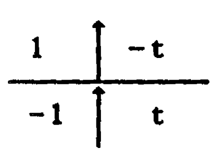
$$r = (s^{p}a)(\overline{s^{p+1}b})(s^{p+2}c)(\overline{s^{p+1}d})$$
$$= (s^{p}a)(\bar{b}s^{-p-1})(s^{p+2}c)(\bar{d}s^{-p-1})$$
$$r = (s^{p}as^{-p})(s^{p}\bar{b}s^{-p})(s^{p+1}cs^{-p-1})(s^{p+1}\bar{d}s^{-p-1}).$$

This is now written correctly as an element of G' as an module, where $ta = sas^{-1}$. In $H = G'/G''$ (the abelianization), this relation becomes:

$$t^{p}a-t^{p}b+t^{p+1}c-t^{p+1}d = 0$$

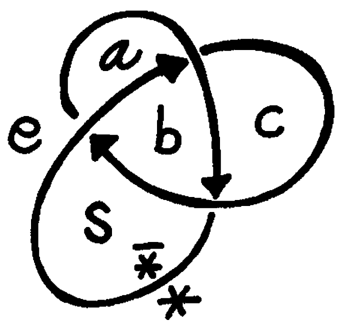
or equivalently

$$\boxed{a-b+tc-td = 0}$$



This "code" may help remember how to write the relation.

Thus, for the trefoil, we get



e ↑ a	e ↑ c	e ↑ s
s | b	a | b	c | b
e−s+tb−ta	e−a+tb−tc	e−c+tb−t
‖	‖	‖
0	0	0

But $e \equiv s \equiv 0$ in H.

Thus the relations become $tb-ta = 0$, $-a+tb-tc = 0$, $-c+tb = 0$,

or
$$\begin{cases} b-a = 0 \\ -a+tb-tc = 0 \\ -c+tb = 0 \end{cases}$$

or
$$\begin{cases} -a+ta-tc = 0 \\ c = tb = ta \end{cases}$$

or
$$\begin{cases} -a+ta-t^2a = 0 \\ (t^2-t+1)a = 0 \end{cases}$$

And so $\Delta_K(t) \doteq t^2-t+1$. Not a surprise by now.

Using Alexander's formalism, you can find $\Delta_K(t)$ directly by taking a determinant of the $n\times n$ relation matrix. Thus here we have:

	e	s	a	b	c	
1st crossing	1	−1	−t	t	0	$e-s+tb-ta = 0$
2nd crossing	1	0	−1	t	−t	$e-a+tb-tc = 0$
3rd crossing	1	−t	0	t	−1	$e-c+tb-ts = 0$

Let $M = \begin{bmatrix} -t & t & 0 \\ -1 & t & -t \\ 0 & t & -1 \end{bmatrix}$ (deleting the columns corresponding to a and s). Then M is a relation matrix for $H = G'/G''$ as a $Z[t,t^{-1}]$-module, and $\Delta_K(t) \doteq D(M)$.

Remark: Alexander begins his paper [A1] by giving this formula as the definition! If you read diligently, there are hints about fundamental group and covering spaces in the last two pages of the paper. I had the pleasure of discovering that there is a whole world of combinatorics related to this version of $\Delta_K(t)$. And it yields another model of the Conway axioms. For more, read [K1]. The discovery of the first generalized polynomial by Jones, Ocneanu, Lickorish, Millet, Hoste, Freyd, Yetter, Przytycki and Traczyk (!) may be regarded as a remarkable confirmation of Alexander's intuition in formulating a combinatorial approach. (See these notes, Chapter VI, sections 18, 19, 20, and the Appendix for more about generalized polynomials.)