Algebraic Topology - Spring 2013 - Assignment Number 3.

1. Given cell complexes X and Y with assigned boundary orientations so that the chain complexes are defined, you can build a chain complex for X x Y by taking the cell complex structure of cells A x B as a k+s cell when A is a k-cell of X an B is an s-cell of Y. You can obtain a chain complex by defining the boundary of A x B by the formula

$$d(A \times B) = d(A) \times B + (-1)^k A \times d(B)$$

where d denotes the differential in the chain complexes for X and also for Y and we extend the product linearly to the chain complexes so that (ra + b) x c = r(a x c) + (b x c) where r is in the coefficient ring and a, b,c are any chains (similarly on the other side of the product). In other words, we identify the chain complex for X x Y as the tensor product of the chain complexes for X and Y respectively, with the differential as defined above on the tensor product of chains in X and Y.

Compute the homology and cohomology over the integers and over Z/2Z for the following spaces. In each case, describe the cell decomposition of the space and describe the corresponding chain and cochain complexes.

(i) P x P x ... x P (n factors) where P is the projective plane. Keep track of differences when n is odd and when n is even.

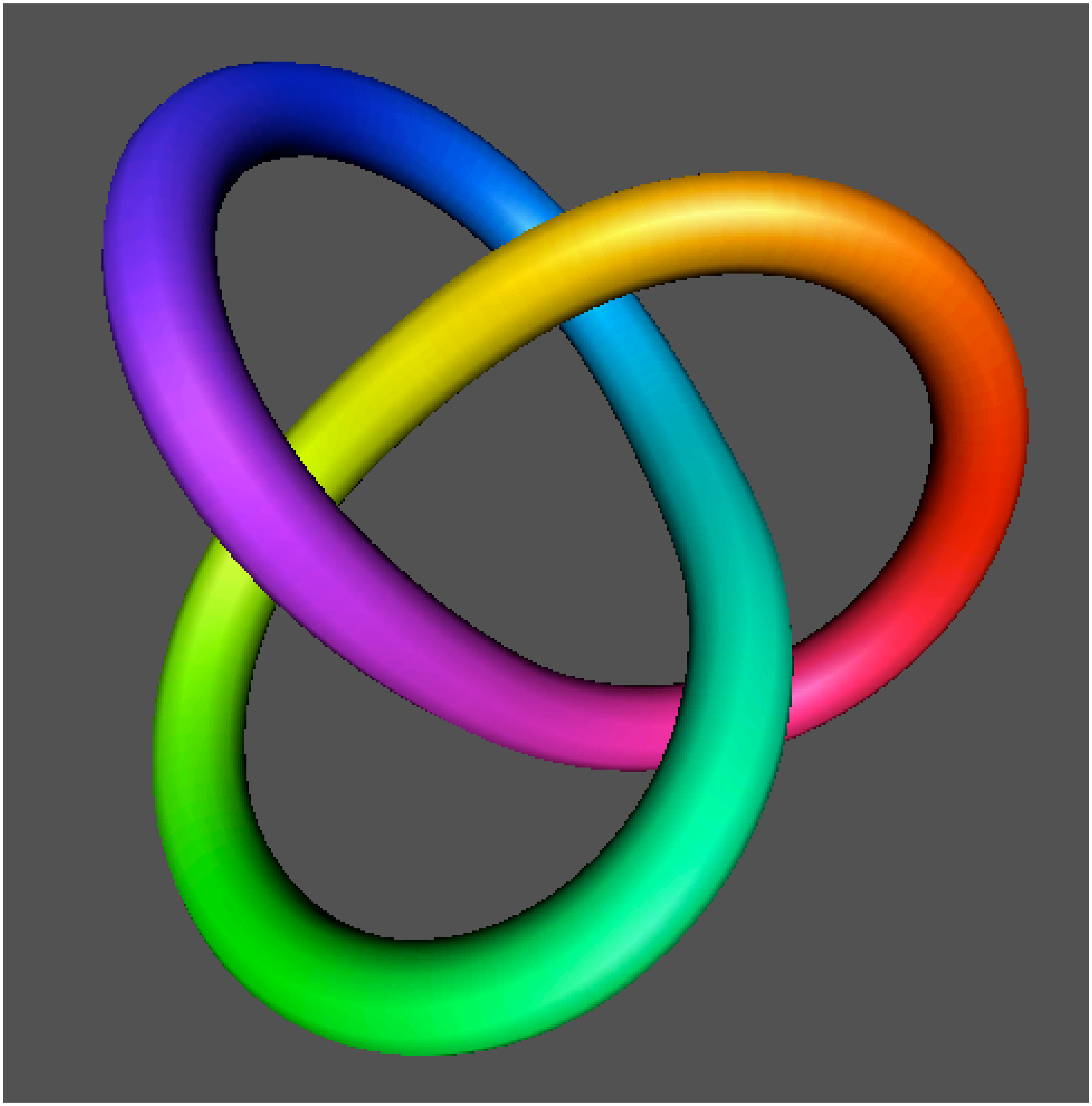
(ii) P^n x P^m where P^k is the projective space obtained as the quotient of the k-dimensional sphere by the antipodal map.

(iii) S^1 x S^1 x K where K is the Klein bottle and S^1 is the circle.

(iv) M x M x ... x M where this denotes a product of n copies of the Mobius band.

2. (i) Read the proof in the Stillwell notes describing the proof that the Wirtinger presentation of the fundamental group of the complement of a knot in three space is presented as described by Stillwell, with one relation for each crossing in a knot diagram. Write an account of that proof in your own words and illlustrations. The aim here is to produce a concise (about one page) proof that satisfies you that this result is true. You will assume the van Kampen Theorem for this proof.

(ii) Show that the Wirtinger presentation of the fundamental group of the knot complement of a trefoil knot



gives a group isomorphic to the following presentation $(a,b \mid a^2 = b^3)$. Give an algebraic proof, transforming the Wirtinger presentation into this presentation.

(iii) Given that the abelianization of the fundamental group is the first homology group, reformulate the van Kampen Theorem to give a Theorem about the first homology group of a space that is the union of two spaces.

3. Hatcher page 132. Problems 18, 19, 22, 26.

4. Hatcher page 176. Problem 6
(You need to read at least pages 170 - 172).

5. Let [012...n] denote an abstract n-simplex, so that (e.g.)
d([012]) = [12] - [02] + [01]
is the boundary of [012] in the chain complex generated by
[012] and all its faces. Let C denote this chain complex and
CxC denote the tensor product of C with itself, generated by
elements a x b where a and b are faces of [012...n] (including the
entire simplex). Define Delta: C ------> CxC by
Delta[01...k] =
[0][012...k] + [01][12...k] + [012][23...k] + ...+ [01...k][k].
Here [a][b] =[a] x [b] in C x C.
Show that Delta is a chain map. That is, show that
Delta d = d Delta where juxtaposition denotes composition of maps.
Here the boundary operator on the tensor product of complexes is
as given in problem 1.