# Math 435 – Notes and Problems Weeks 5 + 6



LK.

First we list the homework problems due for week 6, Tuesday, February 25, 2014.

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Please read the notes on permutations ① + ② just after the problems. The further notes on matrices will be discussed in class.

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## Problems

I. page 10: 1.3.1, 1.3.2, 1.3.3
page 15: 1.4.1
page 23: 1.5.3, 1.5.4, 1.5.5, 1.5.6, 1.5.8
page 73: 1.10.2, 1.10.3, 1.10.10

II. Let $C_3 = \{I, R, R^2 \mid R^3 = I\}$ be the cylic group of order 3.
Examine its multiplication table.

| $\cdot$ | $I$ | $R$ | $R^2$ |
|---|---|---|---|
| $I$ | $I$ | $R$ | $R^2$ |
| $R$ | $R$ | $R^2$ | $I$ |
| $R^2$ | $R^2$ | $I$ | $R$ |

Note that each row of the table is a permutation of $I, R, R^2$.

Thus: $I \leftrightarrow \begin{pmatrix} I & R & R^2 \\ I & R & R^2 \end{pmatrix}$

$R \leftrightarrow \begin{pmatrix} I & R & R^2 \\ R & R^2 & I \end{pmatrix}$

$R^2 \leftrightarrow \begin{pmatrix} I & R & R^2 \\ R^2 & I & R \end{pmatrix}$

In fact <u>this must always be so</u>.
For suppose you have a finite group $G = \{g_1, g_2, \ldots, g_n\}$ where this is a list of all the distinct elements of $G$. Let $g \in G$ be some given element of $g$. Then

$$\begin{aligned} g g_1 &= g_{i_1} \\ g g_2 &= g_{i_2} \\ &\cdots \\ g g_n &= g_{i_n} \end{aligned}$$

since $G$ is closed under multiplication.

<u>Claim.</u> $g_{i_k} = g_{i_l} \Leftrightarrow i = l$

<u>Proof.</u>
$$\begin{aligned} g_{i_k} = g_{i_l} &\Leftrightarrow g g_k = g g_l \\ &\Leftrightarrow g^{-1}(g g_k) = g^{-1}(g g_l) \\ &\Leftrightarrow (g^{-1} g) g_k = (g^{-1} g) g_l \\ &\Leftrightarrow I g_k = I g_l \\ &\Leftrightarrow g_k = g_l \\ &\Leftrightarrow k = l \text{ (since the list is a list of distinct elements).} // \end{aligned}$$

This means that $\{g_{i_1}, g_{i_2}, \ldots, g_{i_n}\}$ is a permutation of $\{g_1, g_2, \ldots, g_n\}$.

This problem asks you to work out all the permutations associated with the six rows for the multiplication table for $S_3$. Where we take

$$G = S_3 = (R, F \mid R^3 = I, F^2 = I, RF = FR^2)$$
$$= \{\underset{\overset{\parallel}{g_1}}{I}, \underset{\overset{\parallel}{g_2}}{R}, \underset{\overset{\parallel}{g_3}}{R^2}, \underset{\overset{\parallel}{g_4}}{F}, \underset{\overset{\parallel}{g_5}}{FR}, \underset{\overset{\parallel}{g_6}}{FR^2}\}.$$

Here is an example of the calculation for one row: Let $g = g_4 = F$

Then
$$gg_1 = FI = F = g_4$$
$$gg_2 = FR = g_5$$
$$gg_3 = FR^2 = g_6$$
$$gg_4 = FF = g_1$$
$$gg_5 = FFR = R = g_2$$
$$gg_6 = FFR^2 = R^2 = g_3.$$

Let $[g] = \begin{pmatrix} g_1 & g_2 & g_3 & g_4 & g_5 & g_6 \\ g_4 & g_5 & g_6 & g_1 & g_2 & g_3 \end{pmatrix}$

We can abbreviate:

$$[g_4] = [g] = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{pmatrix}$$

Work out the permutations for $[g_1], [g_2], [g_3], [g_4], [g_5], [g_6]$.

✓ already done

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Since you know how to multiply (compose) permutations you should find that

$$[gh] = [g] \circ [h]$$

when $g$ and $h$ are elements of the group. Try this out on your permutations. Can you write a proof of the formula $[gh] = [g] \circ [h]$ ?

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This exercise has you working and proving Cayley's Theorem that states that every finite group $G$ is isomorphic to a subgroup of the permutation group $S_n$ when $n$ is the number of elements of $G$.

# Redoing Diagrammatic Permutations

Regard $S_n = \{\sigma : \{1,2,\dots,n\} \circlearrowleft \mid \sigma \text{ is } \underbrace{\underline{1\text{-}1} \text{ and } \underline{\text{onto}}}_{1\text{-}1 \text{ correspondence}}\}$

Given $\sigma \in S_n$, let $\sigma(k)$ denote the image of $k \in \{1,2,\dots,n\}$ under $\sigma$. Given $\tau, \sigma \in S_n$ define $\tau\sigma \underset{\text{def}}{=} \tau \circ \sigma$, the composition of these functions. Thus

$$(\tau\sigma)(k) = \tau(\sigma(k)) \text{ for } k = 1,2,\dots,n.$$

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When we write $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \in S_4$

we mean: $\sigma(1) = 2,\ \sigma(2) = 3,\ \sigma(3) = 4,\ \sigma(4) = 1.$

If $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$: $\tau(1) = 4,\ \tau(2) = 1,\ \tau(3) = 3,\ \tau(4) = 2.$

$$\left.\begin{aligned} \tau\sigma(1) &= \tau(\sigma(1)) = \tau(2) = 1 \\ \tau\sigma(2) &= \tau(\sigma(2)) = \tau(3) = 3 \\ \tau\sigma(3) &= \tau(\sigma(3)) = \tau(4) = 2 \\ \tau\sigma(4) &= \tau(\sigma(4)) = \tau(1) = 4 \end{aligned}\right\} \Rightarrow \tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$$

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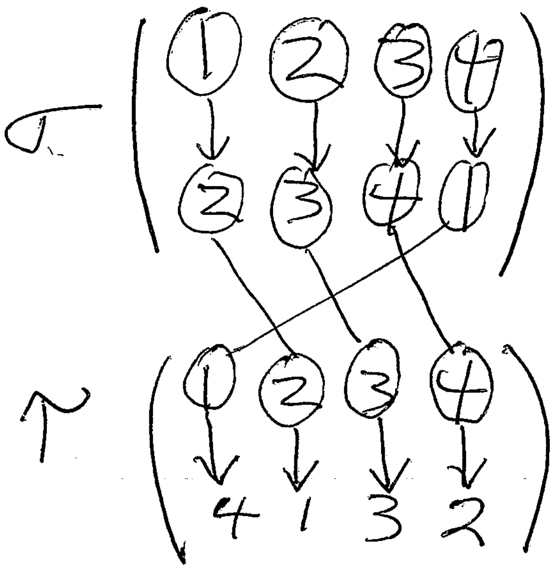
$$\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$$

You can do the multiplication of $\tau\sigma$ by placing $\tau$ next to $\sigma$ and tracing what happens to each number.




$\longrightarrow \begin{pmatrix}1&2&3&4\\1&3&2&4\end{pmatrix}$

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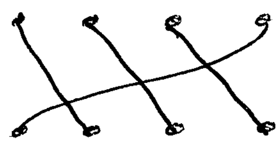
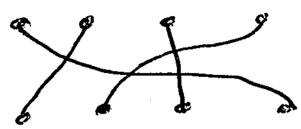
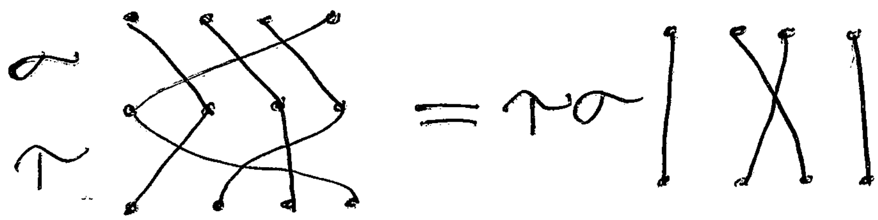
In our diagramming we do it this way:

$\sigma \begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}$ :

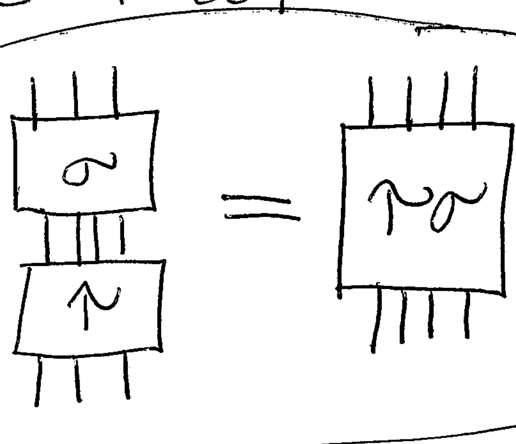


$\tau \begin{pmatrix}1&2&3&4\\4&1&3&2\end{pmatrix}$ :






But note that




We will use this ordering convention from now on.

# Math 435 Notes – Week 6 – LHK

## Linear Algebra and Matrices

### I. More Matrices

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = A = (a_{ij}), \text{ here } 3\times 3.$$

$$\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ a_{21} & \cdots & a_{2m} \\ & \cdots & \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} = A = (a_{ij}), \quad n\times m$$

$n$ rows and $m$ columns.

$$\text{row}_i(A) = r_i(A) = (a_{i1}, a_{i2}, \ldots, a_{im})$$

$i^{th}$ row of $A$

$$\text{col}_j(A) = c_j(A) = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix} \; j^{th} \text{ col of } A.$$

Given $A$, $n\times m$ and $B$, $m\times p$

note =

we get $C = AB$, $n\times p$ via

$$C_{ij} = \text{row}_i(A) \cdot \text{col}_j(B)$$

dot product of vectors

$$C_{ij} = r_i(\underset{n\times m}{A}) \cdot c_j(\underset{m\times p}{B}) \qquad 1 \le i \le n,\ 1 \le j \le p$$

$$= (a_{i1}, \ldots, a_{im}) \cdot \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{pmatrix}$$

$$C_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj}$$

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$$(AB)_{ij} = \sum_{K=1}^{m} a_{iK} b_{Kj}$$

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Sometimes we write

$$(AB)_{ij} = \sum_{K=1}^{m} A_{iK} B_{Kj}$$

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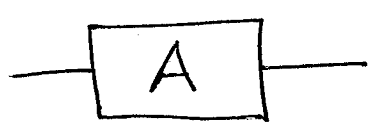
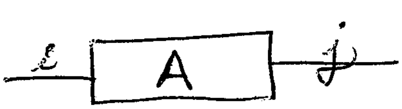
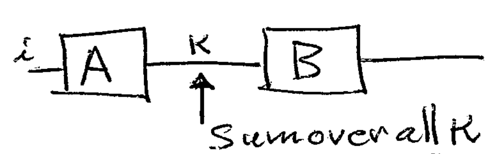
Diagram



$= A_{ij}$

Product 

: 

Examples:

$$\begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} d \\ e \\ f \end{pmatrix} = ad+be+cf.$$

$$\underset{2\times 2,\ 2\times 1}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix}} = \underset{2\times 1}{\begin{pmatrix} ae+bf \\ ce+df \end{pmatrix}}$$

$$\underset{2\times 3,\ 3\times 3}{\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}} = \begin{pmatrix} a+2b+3c & 4a+5b+6c & 7a+8b+9c \\ d+2e+3f & 4d+5e+6f & 7d+8e+9f \end{pmatrix}$$

$$\underset{3\times 1,\ 1\times 3}{\begin{pmatrix} d \\ e \\ f \end{pmatrix}\begin{pmatrix} a & b & c \end{pmatrix}} = \underset{3\times 3}{\begin{pmatrix} da & db & dc \\ ea & eb & ec \\ fa & fb & fc \end{pmatrix}}$$

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Notation: $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$, $\vec{a}^{\,t} = (a_1, a_2, \ldots, a_n)$

Thus $\vec{a}^{\,t}\vec{b} = (a_1 \cdots a_n)\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = a_1b_1 + \cdots + a_nb_n$

$$\| $$

$$\vec{a}\cdot\vec{b}$$

But $\vec{a}\,\vec{b}^{\,t} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}(b_1 \cdots b_n) = \begin{pmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ & \cdots & & \\ a_nb_1 & \cdots & & a_nb_n \end{pmatrix}$

is a matrix.

## Dirac Notation

$$|\vec{a}\rangle = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

$$\langle \vec{a}| = (a_1 a_2 \cdots a_n)$$

Thus $\langle \vec{a}||\vec{b}\rangle \underset{\text{def}}{=} \langle \vec{a}|\vec{b}\rangle = \vec{a}\cdot\vec{b}$

But $|\vec{a}\rangle\langle\vec{b}| = \vec{a}\,\vec{b}^t$ is an $n\times n$ matrix.

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## Permutation Matrices

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = T$

Note $I\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$, $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}$.

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$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$, $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = R$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = F$

$$\left.\begin{array}{l} R\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ c \\ a \end{pmatrix} \\ F\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ c \\ d \end{pmatrix} \end{array}\right\}$$

You see that by permuting rows of $I$ you can obtain all 6 permutations and get a set of 6 matrices that represent $S_3$.

# Determinants

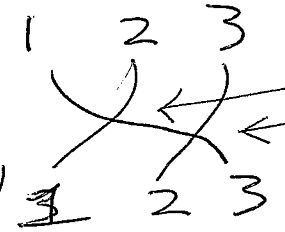
## (a) signs of permutations

$$\mathrm{sgn}(\sigma) \overset{\text{def}}{=} (-1)^{\left[\begin{array}{l}\#\text{ of transpositions} \\ \quad (i\ldots j) \rightsquigarrow (j\ldots i) \\ \text{needed to take the} \\ \text{permutation back} \\ \text{to standard order.}\end{array}\right]}$$

e.g. $3\,1\,2 \rightsquigarrow 1\,3\,2 \rightsquigarrow 1\,2\,3$

$$\Rightarrow \mathrm{sgn}(312) = +1 = (-1)^2$$

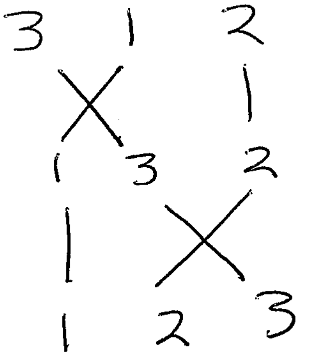
$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$:



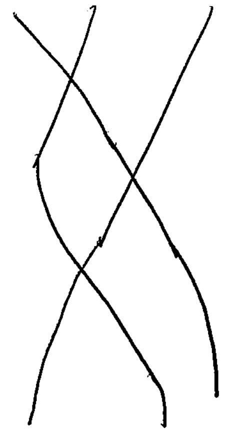

} 2 transpositions

Here we factorized the mapping into transpositions.

or



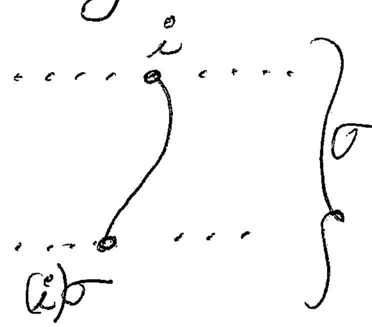

} Here we indicated the actual switching.



$$\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix} = R$$

$$\mathrm{sgn}(R) = (-1)^3 = -1.$$

You can see that if you represent a permutation in $S_n$ by a line diagram




then $\mathrm{sgn}(\sigma) = (-1)^{\#\left(\substack{\text{crossings}\\ \text{in Diagram}(\sigma)}\right)}$.

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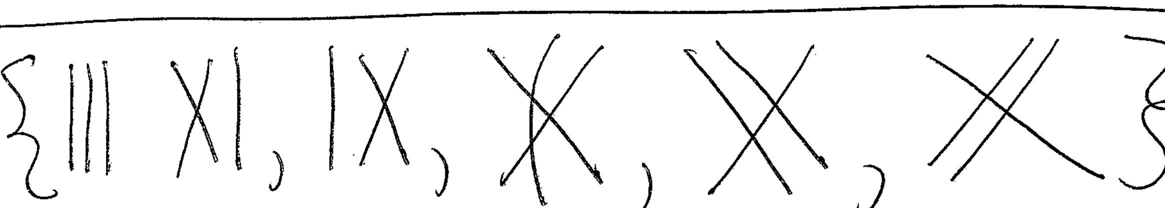

$= S_3$

$+ \quad - \quad - \quad - \quad + \quad +$ <u>sgns</u>

Note that half the elts of $S_3$ have $+$ sgns and half have $-$ sgns.

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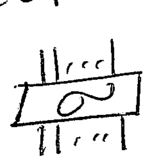
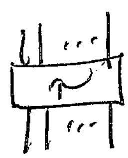
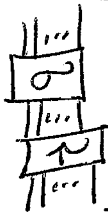
<u>Theorem</u>. $\mathrm{sgn} : \underbrace{S_n}_{\substack{\text{order } n!\\ \text{symmetric}\\ \text{group.}}} \longrightarrow \underbrace{\{+1, -1\}}_{\substack{\text{order } 2\\ \text{group}}} = C_2$

Then $\mathrm{sgn}(1) = 1$ and for $\sigma, \tau \in S_n$, $\mathrm{sgn}(\sigma\tau) = \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\tau)$.

<u>Proof</u>. Represent $\sigma$ and $\tau$ by string diagrams ,  so that

$$\operatorname{sgn}(\sigma) = (-1)^{c(\sigma)}, \quad \operatorname{sgn}(\tau) = (-1)^{c(\tau)}$$

where $c(\sigma)$ = # of crossings in the $\sigma$-diagram & $c(\tau)$ = # crossingings in the $\tau$-diagram.

Then clearly $c\Big[$  $\Big] = c(\sigma) + c(\tau)$.

$$\therefore \operatorname{sgn}(\sigma\tau) = (-1)^{c(\sigma)+c(\tau)}$$
$$= (-1)^{c(\sigma)}(-1)^{c(\tau)}$$
$$= \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) \quad //$$

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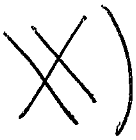
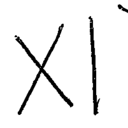
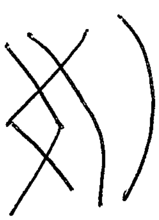
<u>Example</u>. $\operatorname{sgn}\Big($  $\Big) = (-1)^2 = +1$

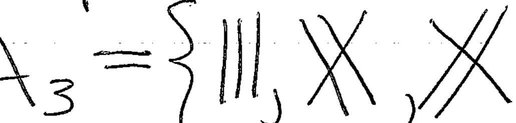
$\operatorname{sgn}\Big($  $\Big) = (-1)^1 = -1$

$\operatorname{sgn}\Big($  $\Big) = (-1)^{2+1} = (-1)^3 = -1$

$\operatorname{sgn}\Big($  $\Big) = -1$ ✓

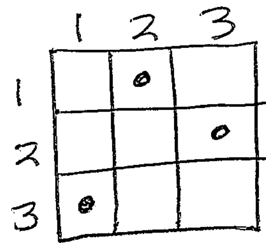
Let $A_n = \{\sigma \in S_n \mid sgn(\sigma) = +1\}$.
You can easily check that $A_n$ is a subgroup of $S_n$.

For example

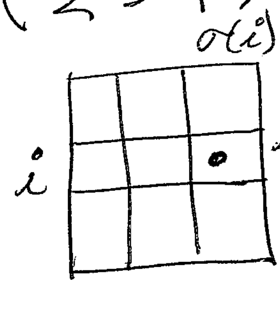
$A_3 = \{$  $\} \subset S_3$.

In general $A_n$ will have $(n!)/2$ elements. Thus $A_4$ should have $(4!)/2 = 24/2 = 12$ elements.
<u>Make a list of the elements of $A_4$ and experiment with their products.</u>

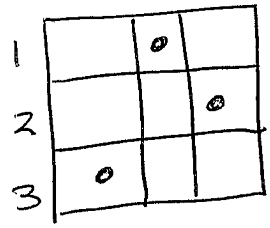
## (b) <u>Permutations and Determinants</u>



You can regard this pattern as the permutation
$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \sigma$$

where  the dot is in column $\sigma(i)$.

Think of this as



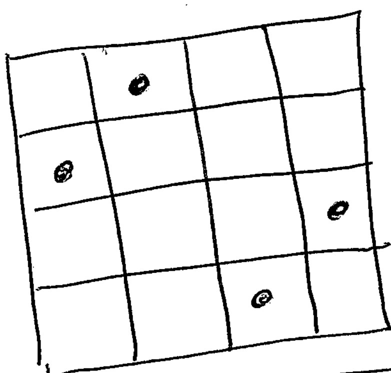

$\leftarrow$ in by 2
$\leftarrow$ in by 3
$\leftarrow$ in by 1

$\longleftrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

<u>and</u>

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}.$$

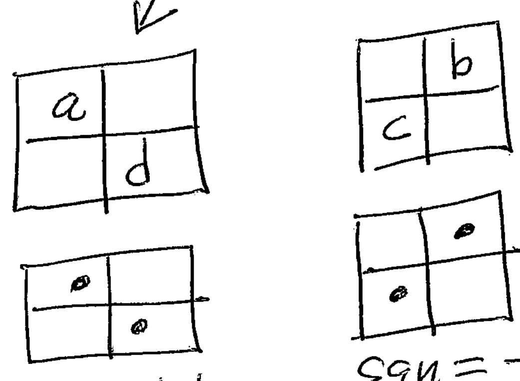
<u>Thus you can find the corres permutation by matrix multiplication.</u>



Work out the corres permutation and matrix mult.

<u>Now Note:</u> $\text{Det}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

We will use this pattern to generalize to Det(A) when A is $n \times n$.

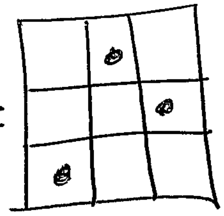



sgn = +1

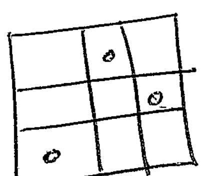
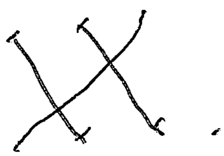
sgn = −1

On an $n \times n$ grid, a <u>permutation pattern</u> $P$ is a placement of dots in the grid so that each dot occupies a unique row and a unique column.

e.g. $n=3$

$P=$ 

As we have seen, there is a 1-1 corres between permutation patterns $P$ on an $n\times n$ grid and elements $\sigma \in S_n$.

 $\longleftrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \Leftrightarrow$ .

Let $\mathcal{P}_n = \left\{ P \,\middle|\, \begin{array}{l} P \text{ is a permutation} \\ \text{pattern on an } n\times n \\ \text{grid.} \end{array} \right\}$

Let $\sigma : \mathcal{P}_n \longrightarrow S_n$

$\sigma(P) =$ elt of $S_n$ corres to $P$.

Define $\operatorname{sgn}(P) = \operatorname{sgn}(\sigma(P))$.

Thus $\operatorname{sgn}\begin{array}{|c|c|}\hline \circ & \\ \hline & \circ \\ \hline\end{array} = +1$

$\operatorname{sgn}\begin{array}{|c|c|}\hline & \circ \\ \hline \circ & \\ \hline\end{array} = -1.$

Define for an $n \times n$ matrix $A$

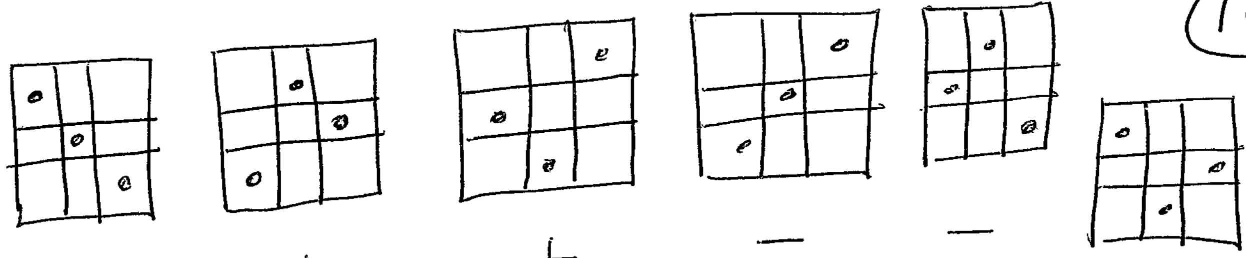
$\langle A | P \rangle =$ product of the entries of $A$ that correspond to dots in $P$.

$$\left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{array}{|c|c|}\hline \circ & \\ \hline & \circ \\ \hline\end{array} \right\rangle = ad$$

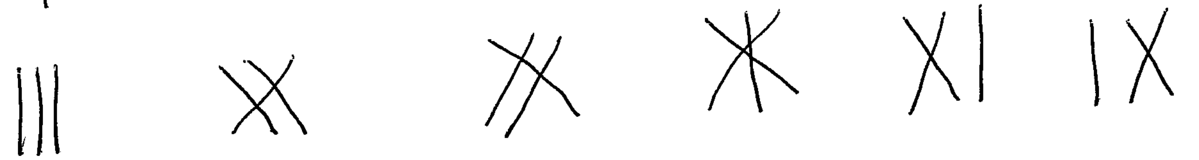
$$\left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{array}{|c|c|}\hline & \circ \\ \hline \circ & \\ \hline\end{array} \right\rangle = bc.$$

Now we can define the determinant of an $n \times n$ matrix $A$:

$$\boxed{\operatorname{Det}(A) = \sum_{P \in \mathcal{P}_n} \operatorname{sgn}(P) \langle A | P \rangle.}$$



$+ \quad + \quad + \quad - \quad - \quad -$



$$\mathrm{Det}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} = aek + bfg + cdh - ceg - bdk - afh.$$

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You can verify that

1. $\displaystyle \mathrm{Det}(A) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\, A_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{n\sigma(n)}.$

2. Regard $\mathrm{Det}(A) = D\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix}$ where $r_i = i$th row of $A$.

(*) Then $D(A') = -D(A)$ if $A'$ obtained from $A$ by interchanging two rows.

(**) $D\begin{pmatrix} \vdots \\ a r_i + b r_i' \\ \vdots \end{pmatrix} = a D\begin{pmatrix} \vdots \\ r_i \\ \vdots \end{pmatrix} + b D\begin{pmatrix} \vdots \\ r_i' \\ \vdots \end{pmatrix}.$

That is, $D$ is a linear function of each row.

One can show that an $n \times n$ matrix $A$ is invertible ($\exists B$ s.t. $AB = I$) if and only if $\mathrm{Det}(A) \neq 0$. In fact there is an explicit formula for $A^{-1}$ obtained as follows: Given $A$ $n \times n$ define the $ij$ cofactor $C_{ij}$ by the formula

$$C_{ij} = (-1)^{i+j}\,\mathrm{Det}(A[ij])$$

where $A[ij]$ is the $(n-1) \times (n-1)$ matrix obtained from $A$ by removing row $i$ and column $j$ from $A$.

e.g. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ then $C_{12}$: $\begin{pmatrix} \cancel{1} & \cancel{2} & \cancel{3} \\ 4 & \cancel{5} & 6 \\ 7 & \cancel{8} & 9 \end{pmatrix}$

$$= (-1)^{1+2}\,\mathrm{Det}\begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} = -(36-42) = -(-6) = 6.$$

Let $C = (C_{ij})$. This is the cofactor matrix for $A$.

Then we have the fact that $C^t A = \mathrm{Det}(A)\, I$ where $(C^t)_{ij} = C_{ji}$ is the transpose of $C$ and $I$ is the $n \times n$ identity matrix.

e.g. $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$C=\begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \quad C^T=\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$C^TA=\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$=\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

$$=(ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$C^TA = \text{Det}(A)I$$

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From this it follows that, when $\text{Det}(A) \neq 0$, then

$$\boxed{A^{-1} = \frac{1}{\text{Det}(A)} C^T}.$$

<u>Exercise</u>: Assuming $A=\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$ has $\text{Det}(A) \neq 0$. Find the formula for $A^{-1}$!