WINNING WAYS

for your mathematical plays

VOLUME 2: GAMES IN PARTICULAR

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SPROUTS

This game (introduced by M.S. Paterson and J.H. Conway some time ago) has a novel feature which complicates the analysis to such an extent that the normal outcome of the 7-spot game is still unknown. Even the 2-spot game is remarkably complicated.

The move in Sprouts is to join two spots, or a single spot to itself (Fig. 5) by a curve which does not meet any previously drawn curve or spot. But when this curve is drawn, a new spot must be placed upon it. No spot may have more than three parts of curves ending at it.

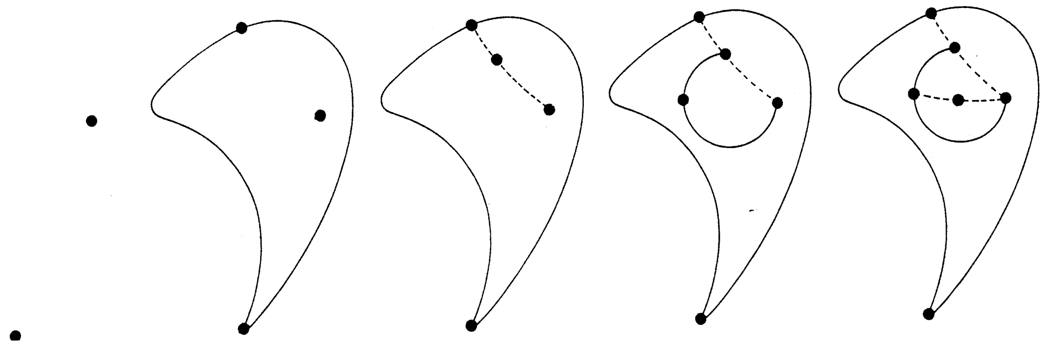


Figure 5. A Short Game of Sprouts.

A typical game is shown in Fig. 5, with the second player's moves drawn as dotted lines. Since the new spots can still be used in later moves, a Sprouts game will last longer than a Cabbages game from the same initial position, and it is perhaps not even obvious that it need ever end. But there is a simple argument which shows that in fact a Sprouts game starting from n spots can last at most $3n-1$ moves. We take the 3-spot game as an example. Each spot has potentially 3 ends of curves available to it, which we shall call its three *lives*, so initially the 3-spot game has 9 lives. But each move takes one life away from the two spots it joins (or two lives away from a spot joined to itself), and adds a new spot which has just one life. Therefore each move reduces the total number of lives by one. Since the very last spot to be created is still alive at the end of the game, the total number of moves is at most $9-1 = 8$. But Fig. 6 shows just how complicated even the 2-spot game really is.

One of the most interesting theorems about Sprouts (due to D. Mollison and J.H. Conway) is the Fundamental Theorem of Zeroth Order Moribundity (FTOZOM). We shall not prove it here, but will at least state it. The FTOZOM asserts that the n-spot Sprouts game must last at least $2n$ moves, and that if it lasts exactly this amount, the final configuration is made up of the insects shown in Fig. 7.

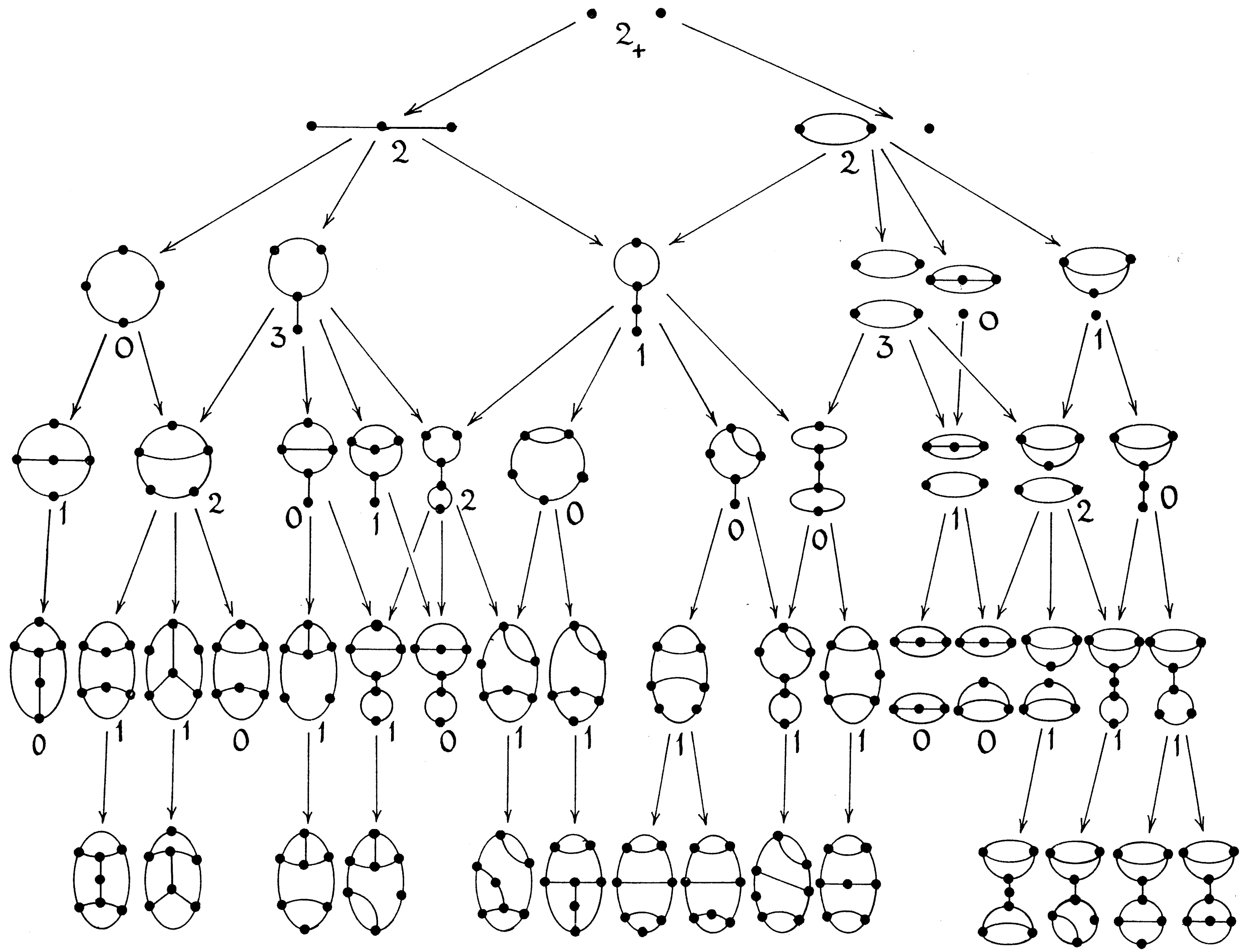


Figure 6. Two-spot Sprouts, with Reduced Forms.

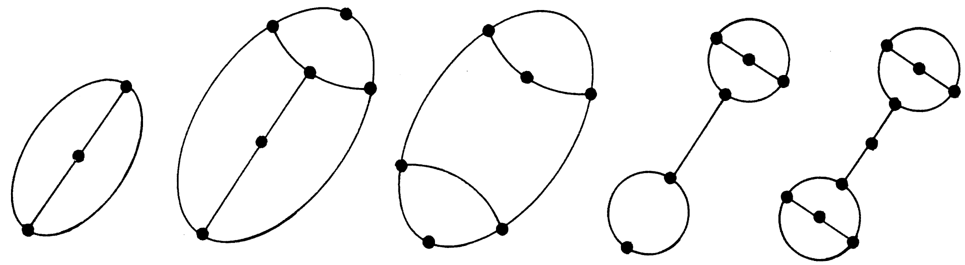


Figure 7. The Five Fundamental Insects.

To be more precise, the final configuration must consist of just one of these insects (which might perhaps be turned inside out in some way) infected by an arbitrarily large number of lice (some of which might infect others). One of the possible configurations is shown as Fig. 8—it consists of an inside-out scorpion inside an inside-out louse, liberally infected with other lice!

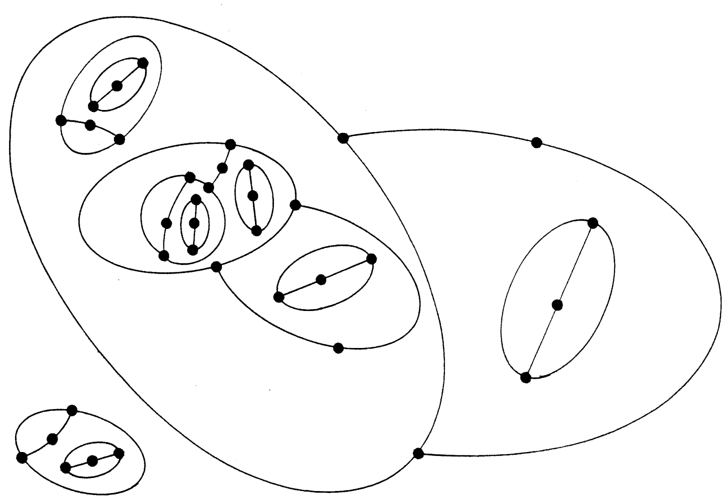


Figure 8. The Lousy End of a Short Sprouts Game.

How should we play if we wish to win a Sprouts game? It is clear that whether the play is normal or misère, the outcome only depends on whether the total number of moves in the game is odd or even, so in some sense winning is controlling the number of moves. Now the 3-spot game necessarily lasts for 6, 7, or 8 moves, and it is very difficult to make it last 8 moves, so that really the fight is between 6 and 7 moves. Apparently the same thing happens in larger games—essentially one player tries to make the game last m moves, while the other tries to drag it out to $m+1$, all other numbers being very unlikely.

To see how to control the number of moves, we examine the situation at the end of the game, which we suppose to have started with n spots and lasted for m moves. The final number of spots is $n+m$, and the total life at the end of the game is $l = 3n-m$, since we started with $3n$ lives, and subtracted one per move. Each of the live spots at the end of the game has two dead spots as its two nearest neighbors, and the remaining dead spots are called *Pharisees*. (The concept of neighbor is quite subtle—in Fig. 9 we show the two different ways in which two dead spots can be neighbors of a live one.)

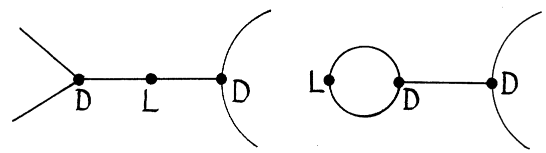


Figure 9. Two Live Spots (L) and their Dead Neighbors (D).

Now no dead spot can be a neighbor of two different live spots, for otherwise we could join these two spots and continue the game. So the number ϕ of Pharisees is given by the equation

$$\phi = (n+m) - (l+2l) = (n+m) - 3(3n-m) = 4m-8n$$

and we have the *Moribundity Equation*:

$$m = 2n + \tfrac{1}{4}\phi.$$

From this equation we can deduce several things:

(i) The number of moves is at least $2n$.
(ii) The number of Pharisees is a multiple of 4.
(iii) If at any time in the game we can ensure that the final position has at least P Pharisees, then the game will last at least $2n+\frac{1}{4}P$ moves.

There is a corresponding result to (iii) in the opposite direction:

(iv) If at any time in the game we can ensure that the final position has at least l live spots, then the game will last at most $3n-l$ moves.

So, according to our previous ideas, one player will try to lengthen the game by producing Pharisees, while his opponent tries to shorten it by producing spots which must remain alive.

There is a useful way to estimate the number of live spots there will be at the end of the game. *If any region defined by curves of the game has a live spot strictly inside, then there will be a live spot inside that region at all later times.* So in Fig. 10 we can regard, if we like, the plane as divided

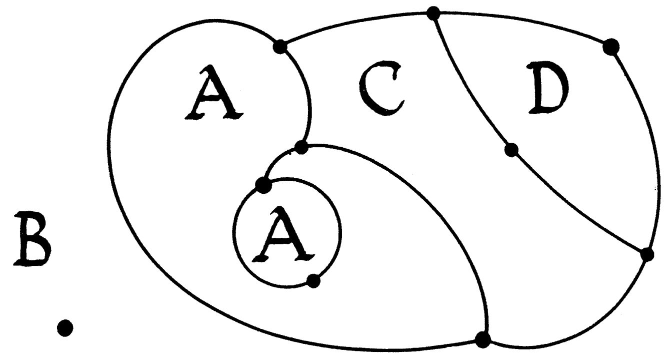


Figure 10. A Sprouts Position with One Pharisee.

into four regions A, B, C, D, and the regions A and B each have live spots strictly inside. Any move made in either of these regions creates a new live spot, and so each of A and B will contain a live spot at the end of the game. We cannot say the same of C and D, whose only live spots lie on their borders, but if we regard C and D together as forming a single region, then this new region has just one spot strictly inside. So we can see that the game will have at least 3 live spots in its final position. It also has presently one Pharisee P, and so (since it developed from an initial position with $n=4$ spots) we can see that it will last *at most* $3n-3 = 9$ moves, and *at least* $2n+\frac{1}{4} = 8\frac{1}{4}$ moves. Since it is difficult to see how the game could last for *exactly* $8\frac{1}{4}$ moves, we conclude that the total length of the game will be 9 moves, however it is played from now on! (Actually, 6 moves have already been made, so just 3 more moves are to follow.) Accordingly, this is either a normal play game about to be won by the first player, or a misère play one being won by the second player.

Using these ideas, it is fairly easy to give analyses of games with small numbers of spots. We give the results we have obtained in Table 10, which shows the number of moves that the winner can arrange for the game to last.

no. of spots:	0	1	2	3	4	5	6
normal play:	0P	2P	4P	7N	9N	11N	14P
misère play:	0N	2N	5P	7P	9P		

Table 10. Outcomes of the Smallest Sprouts Games.

The fact that 6-spot normal Sprouts is a $\mathscr{P}$-position was first proved (to win a bet) by Denis Mollison, whose analysis of the game ran to 47 pages! Using the ideas above, we can shorten this considerably, but we have not yet been able to analyze 5-spot Sprouts with misère play.

BRUSSELS SPROUTS

Here is another game, which should be more interesting than Sprouts. We start with a number of crosses, instead of spots. The move is to continue one arm of a cross by some curve which ends at another arm of the same or a different cross, and then to add a new cross-bar at some point along this curve. A 2-cross game of Brussels Sprouts is shown as Fig. 11. After playing a few games of Brussels Sprouts, the skilful reader will be able to suggest a good starting strategy.

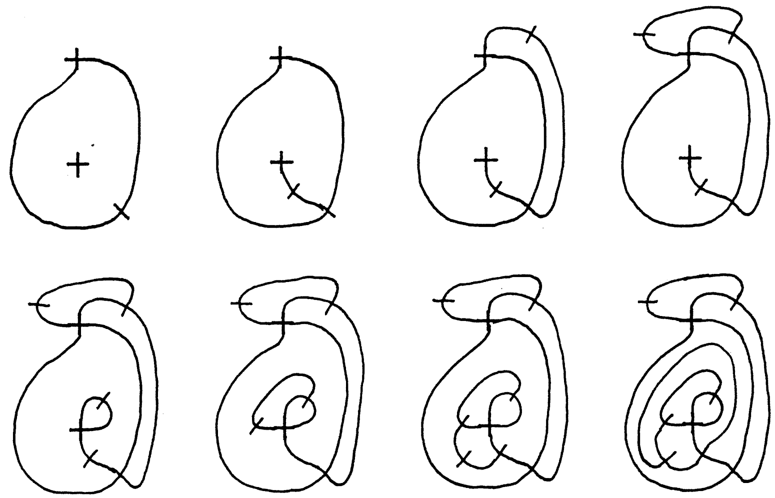
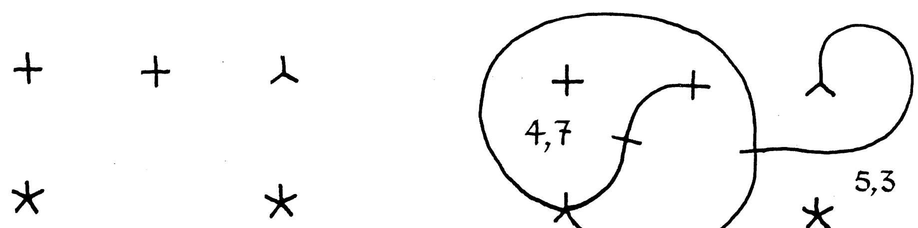


Figure 11. A 2-cross Game of Brussels Sprouts.

STARS-AND-STRIPES



(a) (b)

Figure 12. A Game of Stars-and-Stripes.

Suppose we make addition of the cross-bar optional in Brussels Sprouts. It is natural at the same time to allow "stars" with any number of arms instead of just crosses with exactly 4 arms, and to call the cross-bar a *stripe*. An initial position (5, 5, 4, 4, 3) is shown in Fig. 12(a), along with a position 3 moves later (Fig. 12(b)). In the analysis, the game becomes a disjunctive sum of regions, and we can pretend that each region contains only stars. In general, a connected portion of the picture which has just n arms sticking into the region concerned counts as an n-arm star inside the region. (Even the boundary of the region counts as a star.) In Fig. 12(b) we have therefore labelled each region with numbers showing the sizes of the stars in that region.

The one-star game is isomorphic to the octal game **4·07**, since a move with cross-bar essentially splits an n-arm star into two stars of sizes a and b, with $a+b = n$, $a, b \neq 0$, and the move without crossbar splits it into stars of sizes a and b with $a+b = n-2$. The nim-values for this game (Chapter 4, Tables 7(a), 6(b)) are $0.\dot{0}12\dot{3}$ and the genus appears in Table 11.

n	0	1	2	3	4	5	6	7	8	9	10	11
genus of n	0	0	1	2	3	0	1_a^{431}	2	3_b^{31}	$0_{a_1}^{520}$	1_c^{431}	2_d^{0420}

$a = 2_2320$ $\quad b = a_1a2_220$ $\quad c = b_1ba_3a_12_220$ $\quad d = cb_2ba_2a_1a3_23$

Table 11. The Genus of Stars-and-Stripes Positions.