De Morgan Algebras - Completeness and Recursion

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An elementary proof is given of a completeness theorem for De Morgan Algebras. The proof involves a construction that associates to a De Morgan algebra B, a new De Morgan algebra $\hat{B}$. The construction of $\hat{B}$ bears a close analogy to the construction of the complex numbers from the real numbers. Similarly, De Morgan algebras may be constructed from Boolean algebras. Relationships with recursion and periodic sequences are discussed.

1. Introduction

A De Morgan algebra is an algebra that satisfies most of the properties of a Boolean algebra except for the law of the excluded middle, expressed as $x + x' = 1$ or $xx' = 0$ in the usual Boolean notation. These algebras have been studied by various authors (see [1],[2],[5]). The purpose of this paper is to give an elementary proof of a completeness theorem for De Morgan algebras, and to indicate some interesting examples of these algebras.

The completeness theorem that we prove may be deduced at once from deeper results about the structure of De Morgan algebras (see Remark 2.10). Nevertheless, I believe that the proof given here is of interest because it is quite elementary, and it is a generalization of a corresponding argument for Boolean algebras. De Morgan algebras grow out of Boolean algebras.

In section 2 we define De Morgan Algebras and give a construction that associates to a De Morgan algebra B a new De Morgan algebra $\hat{B}$. The construction of $\hat{B}$ bears a close analogy to the construction of the complex numbers from the real numbers. With the aid of this construction, the completeness result (Theorem 2.5) is proved. In section 3 we show how $\hat{B}$ is related to recursion in B. Section 4 outlines the construction of an algebra of periodic sequences, and delineates directions for further investigation.

2. The Completeness Theorem

It is possible to choose a very concise set of axioms for the algebras we shall study. I shall give a longer list, and thereby avoid extremely detailed demonstrations.

Definition 2.1. A De Morgan algebra B is a set B together with a unary operation $a \mapsto a'$ (inversion), and two binary operations $a,b \mapsto a+b$, $a,b \mapsto ab$ that satisfy the following axioms:

(i) The binary operations are each commutative and associative.

(ii) $(a')' = a$, $aa = a$, $a+a = a$ for all $a \in B$.

(iii) $(a+b)' = a'b'$, $(ab)' = a'+b'$ for all $a,b \in B$.

(iv) There exist elements $0,1 \in B$ such that $a0 = 0$, $a+0 = a$, $a1 = a$, $a+1 = 1$ for all $a \in B$.

(v) $a(b+c) = (ab)+(ac)$, $a+(bc) = (a+b)(a+c)$ for all $a,b,c \in B$.

A Boolean algebra is a De Morgan algebra that satisfies the axioms above plus

(vi) $a+a' = 1$ and $aa' = 0$ for all $a \in B$.

Definition 2.2. Let B be a De Morgan algebra and let $\hat{B} = B\times B$. Thus $\hat{B} = \{(a,b)|a,b \in B\}$. Define operations in $\hat{B}$ as follows:

$(a,b)(c,d) = (ac,bd)$, $(a,b)+(c,d) = (a+c,b+d)$

$(a,b)' = (b',a')$, $\mathbf{0} = (0,0)$, $\mathbf{1} = (1,1)$, $\mathbf{i} = (1,0)$, $\mathbf{j} = (0,1)$ for $a,b,c,d \in B$.

With these definitions, it is easy to see that B becomes a De Morgan algebra. Note that $\mathbf{i}' = \mathbf{i}$ and $\mathbf{j}' = \mathbf{j}$. Hence, if B is a non-trivial $(0 \neq 1)$ Boolean algebra, then $\hat{B}$ is a De Morgan algebra that is not Boolean.

If $V = \{0,1\}$ is the smallest non-trivial Boolean algebra, then $V = \{\mathbf{0},\mathbf{1},\mathbf{i},\mathbf{j}\}$ is a small De Morgan algebra. Note that $\mathbf{i}' = \mathbf{i}$, $\mathbf{j}' = \mathbf{j}$ and $\mathbf{i}\mathbf{j} = \mathbf{0}$, $\mathbf{i}+\mathbf{j} = \mathbf{1}$.

If, for $a,b,c \in B$, we adopt the convention $a(b,c) = (ab,ac)$, then we may identify $a \in B$ with $(a,a) \in \hat{B}$. In other words, the diagonal map $\Delta: B \mapsto \hat{B}$, defined by $\Delta(a) = (a,a)$, is a De Morgan algebra homomorphism that exhibits B as a subalgebra of $\hat{B}$. Every element of $\hat{B}$ is of the form $a\mathbf{i} + b\mathbf{j}$ for $a,b \in B$.

Definition 2.3. Let S be any set. The free De Morgan algebra on S, denoted $B(S)$, is obtained as follows: The primitive expressions in $B(S)$ are $0,1$ and elements $s \in S$. Let $E(S)$ denote the set of expressions defined by the following rules:

(i) Primitive expressions are in $E(S)$. That is, $0,1 \in E(S)$ and $S \subset E(S)$.

(ii) If $x,y \in E(S)$, then x', $x+y$, and xy belong to $E(S)$.

Let $=$ denote the equivalence relation on $E(S)$ that is generated by the axioms (i) $\to$ (v) of Definition 2.1. Let $B(S) = E(S)/=$. It is easy to verify that $B(S)$ is a De Morgan algebra with operations inherited from the formal operations of rule (ii).

Definition 2.4. A homomorphism $\phi: B \to C$ of De Morgan algebras B and C is a set-mapping such that $\phi(xy) = \phi(x)\phi(y)$, $\phi(x+y) = \phi(x) + \phi(y)$ and $\phi(x') = \phi(x)'$ for all $x,y \in B$.

As usual, a homomorphism from a free algebra $B(S)$, $\phi: B(S) \to C$, is determined uniquely by its values $\phi(s)$ for $s \in S$.

We are now prepared to state the completeness theorem.

Theorem 2.5. Let $\hat{V} = \{\mathbf{0},\mathbf{1},\mathbf{i},\mathbf{j}\}$ be the four element De Morgan algebra described after Definition 2.2. Let $B(S)$ be a free De Morgan algebra on a set S. Then for $\alpha,\beta \in E(S)$, $\alpha = \beta$ if and only if $\phi(\alpha) = \phi(\beta)$ for every homomorphism $\phi: B(S) \to \hat{V}$.

In other words, an equality $\alpha = \beta$ is a consequence of the axioms for a De Morgan algebra if and only if it is true about the model $\hat{V}$.

In order to prove Theorem 2.5 we shall need some preliminary lemmas.

Lemma 2.6. Let $f(x) \in E(S)$ be an expression involving $x \in S$. Then $f(x) = Ax + Bx' + Cxx' + D$ where A,B,C,D are expressions that do not contain the element x.

The proof of this lemma proceeds just as in ordinary Boolean algebra, and will therefore be omitted.

Lemma 2.7. The following equivalence holds in $E(S)$:

$(Ax + Bx' + Cxx')' = A'x + B'x' + xx' + A'B'C'$.

Proof: $(Ax + Bx' + Cxx')'$

$= (Ax)'(Bx')'(Cxx')'$

$= (A' + x')(B' + x'')(C' + (xx')')$

$= (A' + x')(B' + x)(C' + x' + x)$

$= (A'B' + A'x + B'x' + x'x)(C' + x' + x)$

$= A'B'C' + (A'C' + A'B' + A')x$
$\quad + (B'C' + A'B' + B')x'$
$\quad + (A' + B' + C' + 1 + 1)xx'$

$= A'B'C' + A'(C' + B' + 1)x$
$\quad + B'(C' + A' + 1)x' + xx'$

$= A'B'C' + A'x + B'x' + xx'$

This completes the proof of the lemma.

For the next lemma, we shall proceed informally, regarding $f(x)$ as a function of x, and evaluating $f(\mathbf{0})$, $f(\mathbf{1})$, $f(\boldsymbol{i})$, $f(\boldsymbol{j})$. This can all be made more precise in terms of homomorphisms, but only at some loss in clarity.

Lemma 2.8. Let $f(x) = Ax + Bx' + Cxx' + D$ be an element of $E(S)$ as described in Lemma 2.6. Then

$f(\mathbf{0}) = B + D$
$f(\mathbf{1}) = A + D$
$(f(\boldsymbol{i}) + \boldsymbol{i})(f(\boldsymbol{j}) + \boldsymbol{j}) = D$
$f(\boldsymbol{i}) + f(\boldsymbol{j}) = A + B + C + D.$

Here equality means as functions on $\hat{V}$.

Proof: Certainly $f(\mathbf{0}) = B + D$ and $f(\mathbf{1}) = A + D$. Now $f(\boldsymbol{i}) = (A + B + C)\boldsymbol{i} + D$ (since $\boldsymbol{i}' = \boldsymbol{i}$), and $f(\boldsymbol{j}) = (A + B + C)\boldsymbol{j} + D$. Hence $f(\boldsymbol{i}) + \boldsymbol{i} = (A + B + C + 1)\boldsymbol{i} + D = \boldsymbol{i} + D$, and $f(\boldsymbol{j}) + \boldsymbol{j} = \boldsymbol{j} + D$. Therefore

$$
\begin{aligned}
(f(\boldsymbol{i}) + \boldsymbol{i})(f(\boldsymbol{j}) + \boldsymbol{j}) &= (\boldsymbol{i} + D)(\boldsymbol{j} + D) \\
&= \boldsymbol{i}\boldsymbol{j} + \boldsymbol{i}D + \boldsymbol{j}D + D \\
&= 0 + (\boldsymbol{i} + \boldsymbol{j})D + D \\
&= D + D = D.
\end{aligned}
$$

Finally, $f(\boldsymbol{i}) + f(\boldsymbol{j}) = (A + B + C)(\boldsymbol{i} + \boldsymbol{j}) + D = A + B + C + D.$

This completes the proof of the lemma.

Proof of Theorem 2.5. Let $\alpha, \beta \in E(S)$. By Lemma 2.6 we may assume that $\alpha = Ax + Bx' + Cxx' + D$ and that $\beta = \bar{A}x + \bar{B}x' + \bar{C}xx' + \bar{D}$ where A, B, C, D and $\bar{A}, \bar{B}, \bar{C}, \bar{D}$ are expressions that do not contain x. The proof will proceed by induction on the total number N of variables (elements of S) that occur in the two expressions. If $N=0$, then $\alpha = D$ and $\beta = \bar{D}$ where D and $\bar{D}$ are equal to 0 or 1. Since $0 \neq 1$ in a free De Morgan algebra, the theorem is trivial for the case $N=0$. If $N > 0$, then either α or β contains a variable x and we may use the equivalence indicated above.

Thus we may assume by induction (using Lemma 2.8) that the following equivalences hold in $E(S)$:

$$
(*) \quad \begin{cases} B + D = \bar{B} + \bar{D} \\ A + D = \bar{A} + \bar{D} \\ D = \bar{D} \\ A + B + C + D = \bar{A} + \bar{B} + \bar{C} + \bar{D} \end{cases}
$$

We now use (*) to show that $\alpha = \beta$ in $E(S)$.

$\alpha = Ax + Bx' + Cxx' + D$
$\quad = (A'x + B'x' + xx' + A'B'C')' + D,$

by (2.7)

$\quad = (A'x)'(B'x')'(xx')'(A'B'C')' + D$
$\quad = ((A'x)' + D)((B'x')' + D)((xx')' + D)((A'B'C')' + D)$
$\quad = ((A + D) + x')((B + D) + x)((xx')' + D)(A + B + C + D)$

Now make the substitutions indicated by (*), reverse steps, and conclude that $\alpha = \beta$. This completes the induction step and the proof of the theorem.

Note that Theorem 2.5 has the following corollary:

Corollary 2.9. Let $B(S)$ be a free De Morgan algebra. Then $B(S)$ is isomorphic to a sub-algebra of a product of copies of the four element algebra $\hat{V} = \{\mathbf{0}, \mathbf{1}, \boldsymbol{i}, \boldsymbol{j}\}$.

Proof: Let $\Phi = \{\phi : B(S) \to \hat{V} \mid \phi$ is a homomorphism$\}$. Let $\hat{V}_\phi$ be a copy of $\hat{V}$

indexed by an element of Φ, and let $\mathcal{V} = \prod_{\phi \in \Phi} \hat{V}_\phi$. Define $F : B(S) \to \mathcal{V}$ by $F(\alpha) = \prod_{\phi \in \Phi} \phi(\alpha)$ where $\phi : B(S) \to \hat{V}_\phi$. Then by 2.5 $F(\alpha) = F(\beta)$ if and only if $\alpha = \beta$. Hence F injects $B(S)$ as a subalgebra of $\mathcal{V}$.

Remark 2.10. In fact, Corollary 2.9 is true for arbitrary De Morgan algebras. This deeper result may be found in [2] or [5].

3. Recursion and Fixed Points

The $\wedge$ construction leading from a Boolean algebra B to its corresponding De Morgan algebra $\hat{B}$ is closely related to the structure of recursion in the given Boolean algebra.

Let $T : B \to B$ be a mapping of the form $T(x) = ax + bx'$ where $a, b \in B$. T has period two; in fact

$$T^2(x) = T(T(x)) = (a + b)x + abx'$$

$$T^3(x) = ax + bx' = T(x)$$

$$\ldots$$

$$T^{n+2}(x) = T^n(x).$$

Hence there may be no element $X \in B$ such that $T(X) = X$. However, $\hat{B}$ does contain such fixed points. For example, if $T(x) = x'$, then $x = x'$ has no solutions in the Boolean algebra B, but is satisfied by $\dot{\imath}$ and $\dot{\jmath}$ in $\hat{B}$.

Proposition 3.1. Let B be a Boolean algebra, and let $T : B \to B$ be the mapping described above. Let $\mathfrak{T} : \hat{B} \to \hat{B}$ be the corresponding mapping on $\dot{B}$. Then there exist elements X of $\hat{B}$ such that $\mathfrak{T}(X) = X$. In particular, we may take $X = (T(x), T^2(x))$ or $X = (T^2(x), T(x))$ for any $x \in B$.

Proof: The following identities in B are easily verified:

$T(x) = aT(x) + bT^2(x)'$, $T^2(x) = aT^2(x) + bT(x)'$.

Let $X = (T(x), T^2(x))$. Then

$$\begin{aligned}\mathfrak{T}(X) &= aX + bX' \\ &= a(T(x), T^2(x)) + b(T^2(x)', T(x)') \\ &= (aT(x) + bT^2(x)', aT^2(x) + bT(x)') \\ &= (T(x), T^2(x)) \\ &= X .\end{aligned}$$

Thus the algebraic structure of $\hat{B}$ reflects the properties of period two sequences recursively generated from B. The next section describes a more general De Morgan algebra of sequences.

4. Sequence Models

Corresponding to a De Morgan algebra B let $\mathcal{S}(B)$ denote the set of all sequences of elements of B with an assigned even period (possibly of period 0). That is $\mathcal{S}(B)$ consists of sequences $b = \{b_n\}$ such that n ranges over the integers and $b_{n+p} = b_n$ for all n, where $p = p(b)$ is an even non-negative integer associated with the sequence.

$\mathcal{S}(B)$ has the structure of an algebra as follows:

(i) $(ab)_n = (a_n b_n)$, $(a + b)_n = a_n + b_n$ and $p(ab) = p(a + b) = \begin{cases} \mathrm{lcm}(p(a), p(b)) & \text{if } p(a) \neq 0 \text{ and } p(b) \neq 0, \\ 0 & \text{otherwise,} \end{cases}$

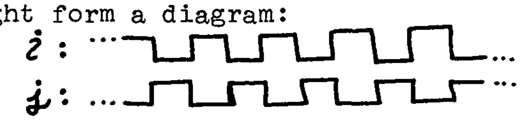
Here the symbol lcm denotes least common multiple.

(ii) $(a')_n = (a_{n-k})'$ where $k = p(a)/2$, $p(a') = p(a)$.

Note that in the sequence algebra, inversion is obtained by ordinary inversion plus a half-period shift. The sub-algebra of period two sequences in $\mathcal{S}(B)$ is isomorphic to B. As it stands, the sequence algebra is not a De Morgan algebra.

Axioms (i), (ii), (iii), and (v) are satisfied but there is no available choice for 0 or 1. If $\mathfrak{s}(B)_p$ denotes the subalgebra of sequences of period p then we may take 0_p and 1_p to be the constant sequences of zeroes and ones respectively, with assigned period p. This gives $\mathfrak{s}(B)_p$ the structure of a De Morgan algebra for each p. We could force $\mathfrak{s}(B)$ into the mold by taking a quotient construction, but I believe it is more interesting to leave it as it stands. Our rules for combining sequences of different periods provide a simple model of interference phenomena that bears investigation on its own grounds.

We may regard $\mathfrak{s}(B)$ as a set of periodic oscillations. In this regard it is interesting to compare our ideas with the suggestions of G. Spencer Brown in his book Laws of Form ([4]). Spencer Brown suggests that an element X satisfying $X' = X$ might be seen as an oscillation (if its 0, then its 1, then its 0, then its 1, ...). Taking this literally, we might form a diagram:



In each case, the spatial sense in which $i' = i$ and $j' = j$ involves ordinary inversion plus a left-right shift. With i and j synchronized (and note that here the synchronization has become the spatial relationship of the two sequences) as above, we have $ij = 0$. This is the motivation for our construction of $\mathfrak{s}(B)$, and also for the construction $B \mapsto \hat{B}$ of section 2.

There are many connections between our discussion and Brown's work. These will be explored in another paper. I would like to remark here that it is striking that once one lifts the law of the excluded middle from Boolean algebra, there is opened up the possibility of infinite models involving simple analogs of waveforms and interference phenomena. This temporal, musical aspect is precisely what is prohibited by the stark all or nothing of two-valued logic. When we drop these restrictions, the result is not fuzziness and ambiguity, but the precise emergence of patterned forms, spatial and temporal.

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