(3) Using (1) and (2), together with the fact that Ψ^* is an algebra homomorphism, the result is established in general by checking it locally on k-forms ω restricted to local coordinate neighborhoods:

$$\omega|_U = \sum a_{i_1 \dots i_k}\, dx_{i_1} \wedge \dots \wedge dx_{i_k}.$$

(Details are left to the reader.) □

Definitions. Let X be a smooth manifold. A smooth differential form ω on X is *closed* if $d\omega = 0$. A form ω is *exact* if it is the differential of another form on X; that is, ω is exact if $\omega = d\tau$ for some smooth form τ. (Note that every exact form is closed, since $d^2 = 0$. The converse question is fundamental to our subject.)

Let $Z^k(X, d)$ denote the vector space of closed k-forms on X. Let $B^k(X, d)$ denote the space of exact k-forms on X. Then $B^k(X, d) \subset Z^k(X, d)$ because $d^2 = 0$. Let $H^k(X, d) = Z^k(X, d)/B^k(X, d)$. $H^k(X, d)$ is called the k-th *De Rham cohomology group* of X. Its dimension, which we shall see is finite for compact X, is called the k-th *Betti number* of X.

Remark. Although these cohomology groups are defined in terms of the manifold structure of X, they are topological invariants; that is, if two manifolds are homeomorphic (by a not necessarily smooth homeomorphism), then they have isomorphic cohomology groups. In fact, these groups can be defined directly using only the topological structure of X.

Example 1. $H^0(X, d) \cong R^1$ if X is connected. For since there are no forms of degree less than 0, $B^0(X, d) = 0$. Thus

$$H^0(X, d) = Z^0(X, d) = [f \in C^\infty(X, R^1);\ df = 0].$$

If U is any coordinate neighborhood of X, with coordinate functions $(x_1, \dots, x_n)$, then $df = 0$ on U means

$$0 = df = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(f)\, dx_i;$$

that is, $(\partial/\partial x_i)(f) = 0$ for all i. But this implies that f is constant on U. Since X is connected, and since f is constant on each coordinate neighborhood in X, then f must be constant on X; that is, $Z^0(X, d) = [\text{constant functions on } X] \cong R^1$.

Example 2. $H^1(S^1, d) \cong R^1$, where S^1 is the circle. For since there are no non-zero k-forms on S^1 for $k > 1$, $Z^1(S^1, d) = C^\infty(S^1, \Lambda^1(S^1))$. Moreover,

$$B^1(S^1, d) = [df;\ f \in C^\infty(S^1, R^1)].$$

Now, if θ denotes the polar coordinate on S^1, then $\partial/\partial\theta$ is a non-zero vector field on S^1 and its dual 1-form $d\theta$ is a non-zero 1-form on S^1. (See Fig. 5.4.) Furthermore, $d\theta$ is not exact (in spite of the notation!)—but, given any 1-form $\omega = g(\theta)\, d\theta$ on S^1, $\omega - (c\, d\theta)$ *is* exact for some $c \in R^1$. Thus

$$Z^1(S^1, d)/B^1(S^1, d) \cong [c\, d\theta;\ c \in R^1] \cong R^1.$$

Exercise: Verify the above facts. Take $c = \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\, d\theta$.

Remarks. Let ψ: $X \to Y$ be smooth. Then

$$\psi^*\colon Z^k(Y, d) \to Z^k(X, d) \quad\text{and}\quad \psi^*\colon B^k(Y, d) \to B^k(X, d).$$

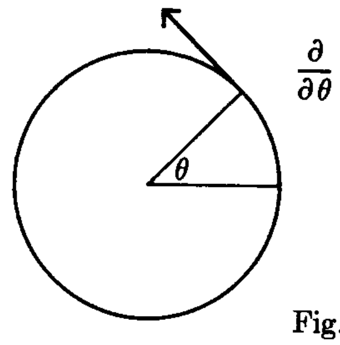

Fig. 5.4

For if ω is a closed k-form on Y, then $d(\psi^*\omega) = \psi^*(d\omega) = \psi^*(0) = 0$. If $\omega = d\tau$ is an exact k-form on Y, then $\psi^*(\omega) = \psi^*(d\tau) = d(\psi^*(\tau))$. Thus ψ^* induces a linear map $\tilde{\psi}$ on cohomology, such that

$$\tilde{\psi}: Z^k(Y, d)/B^k(Y, d) \to Z^k(X, d)/B^k(X, d);$$

that is,

$$\tilde{\psi}: H^k(Y, d) \to H^k(X, d).$$

If $S: W \to X$ and $T: X \to Y$ are smooth, it is easy to check that $(T \circ S)^* = S^* \circ T^*$, and hence $\widetilde{(T \circ S)} = \tilde{S} \circ \tilde{T}$:

$$W \xrightarrow{S} X \xrightarrow{T} Y,$$

$$H^k(W, d) \xleftarrow{\tilde{S}} H^k(X, d) \xleftarrow{\tilde{T}} H^k(Y, d).$$

Thus we have attached to each smooth manifold X new algebraic invariants $H^k(X, d)$ such that given smooth maps between manifolds, there are induced algebraic maps between these algebraic objects. As in the case of the fundamental group, we are thus able to solve certain difficult topological problems by studying their algebraic counterparts.

Now let us show that $H^k(R^n, d) = 0$ for all $k > 0$. Since R^n is diffeomorphic (isomorphic as a smooth manifold) with the unit ball $B_0(1)$ about 0 in R^n, we may as well show that $H^k(B_0(1), d) = 0$ for all $k > 0$. For this we need the following technical lemma.

LEMMA. Let X be a smooth manifold. Then, for each k, consider the maps

$$C^\infty(X, \Lambda^{k-1}(X)) \underset{h_{k-1}}{\overset{d}{\rightleftarrows}} C^\infty(X, \Lambda^k(X)) \underset{h_k}{\overset{d}{\rightleftarrows}} C^\infty(X, \Lambda^{k+1}(X)).$$

Suppose there exist linear maps

$$h_j: C^\infty(X, \Lambda^{j+1}(X)) \to C^\infty(X, \Lambda^j(X)) \qquad (j = k-1 \text{ or } k)$$

such that $h_k \circ d + d \circ h_{k-1}$ is the identity map on $C^\infty(X, \Lambda^k(X))$. Then $H^k(X, d) = 0$; that is, every closed k-form is exact.

Proof. Suppose $\omega \in C^\infty(X, \Lambda^k(X))$ is closed. Then

$$\omega = (h_k \circ d + d \circ h_{k-1})(\omega) = h_k(d\omega) + d(h_{k-1}\omega) = d(h_{k-1}\omega). \qquad \square$$

Remark. If a sequence of such linear maps h_j is defined for all $j \geq 0$, the sequence h_j is called a *homotopy operator.*

THEOREM 3. (Poincaré's lemma) Let $U = B_0(1) \subset R^n$. Then $H^k(U, d) = 0$ for all $k > 0$.

Proof. We shall construct maps h_{k-1}, h_k satisfying the conditions of the lemma. This is done through an integration process. Since these maps are to be linear, it suffices to define h_{k-1} on forms $\omega = g\, dx_{i_1} \wedge \ldots \wedge dx_{i_k}$; similarly for h_k. For such ω, set

$$h_{k-1}(\omega)(x) = \left(\int_0^1 t^{k-1} g(tx)\, dt\right)\mu,$$

where

$$\begin{aligned}\mu &= x_{i_1} dx_{i_2} \wedge \ldots \wedge dx_{i_k} - x_{i_2} dx_{i_1} \wedge dx_{i_3} \wedge \ldots \wedge dx_{i_k} \\ &\quad + \cdots + (-1)^{k-1} x_{i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_{k-1}}.\end{aligned}$$

(Note that $d\mu = k\, dx_{i_1} \wedge \ldots \wedge dx_{i_k}$.)

The map h_k is defined similarly by replacing k everywhere by $k + 1$.

Now, for $\omega = g\, dx_{i_1} \wedge \ldots \wedge dx_{i_k} \in C^\infty(U, \Lambda^k(U))$ and $x \in U$,

$$\begin{aligned}(d \circ h_{k-1})(\omega)(x) &= d\left[\left(\int_0^1 t^{k-1} g(tx)\, dt\right)\mu\right] \\ &= \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(\int_0^1 t^{k-1} g(tx)\, dt\right) dx_j \wedge \mu + \left(\int_0^1 t^{k-1} g(tx)\, dt\right) d\mu \\ &= \sum_{j=1}^{n} \left(\int_0^1 t^{k-1} \frac{\partial}{\partial x_j}(g(tx))\, dt\right) dx_j \wedge \mu + \left(\int_0^1 t^{k-1} g(tx)\, dt\right) d\mu \\ &= \sum_{j=1}^{n} \left(\int_0^1 t^{k} \frac{\partial g}{\partial x_j}(tx)\, dt\right) dx_j \wedge \mu + k\left(\int_0^1 t^{k-1} g(tx)\, dt\right) dx_{i_1} \wedge \ldots \wedge dx_{i_k},\end{aligned}$$

and

$$\begin{aligned}(h_k \circ d)(\omega)(x) &= h_k\left(\sum_{j=1}^{n} \frac{\partial g}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}\right) \\ &= \sum_{j=1}^{n} \left(\int_0^1 t^{k} \frac{\partial g}{\partial x_j}(tx)\, dt\right)[x_j\, dx_{i_1} \wedge \ldots \wedge dx_{i_k} - dx_j \wedge \mu].\end{aligned}$$

Thus,

$$\begin{aligned}(d \circ h_{k-1} + h_k \circ d)(\omega)(x) &= \left[k\left(\int_0^1 t^{k-1} g(tx)\, dt\right) + \sum_{j=1}^{n} \left(\int_0^1 t^{k} \frac{\partial g}{\partial x_j}(tx) x_j\, dt\right)\right] dx_{i_1} \wedge \ldots \wedge dx_{i_k} \\ &= \left\{\int_0^1 \left[k t^{k-1} g(tx) + t^k \frac{d}{dt}(g(tx))\right] dt\right\} dx_{i_1} \wedge \ldots \wedge dx_{i_k} \\ &= \left\{\int_0^1 \frac{d}{dt}[t^k g(tx)]\, dt\right\} dx_{i_1} \wedge \ldots \wedge dx_{i_k} \\ &= t^k g(tx)\Big|_0^1 dx_{i_1} \wedge \ldots \wedge dx_{i_k} \\ &= g(x)\, dx_{i_1} \wedge \ldots \wedge dx_{i_k} \\ &= \omega(x) \qquad (\text{for all } x \in U).\end{aligned}$$

Since $d \circ h_{k-1} + h_k \circ d$ acts as identity on such ω, it acts by linearity as identity on all k-forms. □

Remark 1. The maps h_{k-1} and h_k used in this proof were not just picked out of the air. They were constructed as follows. Given a vector space T and $v \in T$, v defines a map $i(v)$: $\Lambda^k(T^*) \to \Lambda^{k-1}(T^*)$ by

$$[i(v)(\omega)](v_1, \ldots, v_{k-1}) = \omega(v, v_1, \ldots, v_{k-1}).$$

Note that i is a bilinear map $T \otimes \Lambda^k(T^*) \to \Lambda^{k-1}(T^*)$. This map i is called *interior multiplication*. The map h_{k-1} was obtained by applying $i(x)$ to ω and averaging over the line through the origin in the direction x.

Remark 2. Theorem 3 is a special case of a more general result. Let U be a smooth manifold. Suppose there exists a smooth map Ψ: $U \times I_\varepsilon \to U$, where

$$I_\varepsilon = [r \in R^1;\ -\varepsilon < r < 1 + \varepsilon], \text{ such that}$$

$$\Psi(u, 1) = u \qquad (\text{for all } u \in U),$$

$$\Psi(u, 0) = u_0 \qquad (\text{for all } u \in U;\ \text{some } u_0 \in U).$$

Then $H^k(U, d) = 0$ for all $k > 0$.

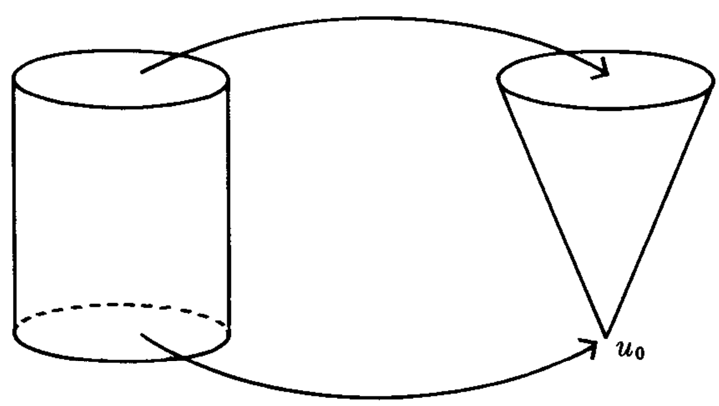

Fig. 5.5

The map Ψ is a *smooth homotopy*. This theorem says that if U is smoothly homotopic to a point, then the cohomology of U is that of a point.

In the case covered by Theorem 3, a smooth homotopy is given by

$$\Psi(x, t) = tx \qquad (t \in I_\varepsilon;\ x \in B_0(1)).$$

Note that the above proof of Poincaré's lemma works equally well for a *star-shaped* region, that is, an open set U such that for some $x_0 \in U$, the line segment joining x_0 to any other point in U lies completely in U.

5.3 MISCELLANEOUS FACTS

THEOREM 1. Let X and Y be smooth manifolds, with X connected, and let ψ: $X \to Y$ be smooth. Assume $d\psi \equiv 0$. Then ψ is a constant map; that is, $\psi(x) = y_0$ for some $y_0 \in Y$ and for all $x \in X$.