Exam 2 - Math 215 - Fall 2009 – With Solutions

Do problems $1, 2, 3, 4$ and choose *one* more problem from problems $5, 6, 7, 8$. Write all your proofs with care, using full sentences and correct reasoning.

1. Prove $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$ for all $n = 1, 2, 3, \cdots$.
Answer. Omitted.

2. Prove that the following two statements are equivalent:

$$(A \Rightarrow C) \wedge (B \Rightarrow C)$$

and

$$(A \vee B) \Rightarrow C.$$

In your proof, do *not* use truth tables. Use the facts that

$$X \Rightarrow Y \ = (\sim X) \vee Y$$

and

$$\sim (X \wedge Y) = \ (\sim X) \vee (\sim Y),$$

and

$$X \vee (Y \wedge Z) = (X \vee Y) \wedge (X \vee Z).$$

and give a completely algebraic proof.
Answer. Omitted.

3. Define the composition of the function $f : X \longrightarrow Y$ and the function $g : Y \longrightarrow Z$ to be the function $g \circ f : X \longrightarrow Z$ with $g \circ f(x) = g(f(x))$ for all $x \in X$. Prove that if f is surjective and g is surjective, then $g \circ f$ is surjective.
Answer. Let $z \in Z$. Then $z = g(y)$ for some $y \in Y$, since g is surjective. And $y = f(x)$ for some $x \in X$ since f is surjective. Therefore $g(f(x)) = g(y) = z$, showing that $g \circ f$ is surjective.

4. Given sets A and B, consider the following two statements about a function $f : A \longrightarrow B$.
(i) $\exists b \in B$ such that $\forall a \in A, f(a) = b$.
(ii) $\forall b \in B, \exists a \in A$ such that $f(a) = b$.

One of these statements is the definition for f to be a surjective mapping from A to B. Which one is it? For the other statement, please explain what it says and give an example of a function from $A = \{1, 2, 3\}$ to $B = \{1, 2\}$ that has this property.

Answer. Statement **(ii)** is the definition of surjective. For an example of **(i)**, let $f(1) = f(2) = f(3) = 1$.

5. **(a)** Let X be any set. Let $P(X)$ denote the set of subsets of X. Let $F : X \longrightarrow P(X)$ be any well-defined mapping from X to its power set $P(X)$. Show that F is not surjective. Hint: Consider the set

$$C = \{x \in X | x \notin F(x)\}.$$

Note that your proof must apply to any set X.

Answer. If $C = F(a)$ for some $a \in X$, then $a \in C$ iff $a \notin F(a)$. But since $C = F(a)$, we conclude that $a \in C$ iff $a \notin C$. This is a contradiction. Therefore C is not equal to $F(a)$ for any a. This shows that F is not surjective.

(b) Make the special assumption that *if X is any set, then it is not the case that X is a member of X*. On the basis of this special assumption prove that there does not exist a set $\mathcal{U}$ such that for all sets $Y, Y \in \mathcal{U}$.

Answer. If the $\mathcal{U}$ was a set, then we would have that $\mathcal{U}$ is a member of itself. Since, by assumption, no set is a member of itself, this is a contradiction. We conclude that $\mathcal{U}$ cannot be a set.

(c) Give an example of infinitely many infinite sets $X_1, X_2, X_3, \cdots$ such that $|X_1| \prec |X_2| \prec |X_3| \prec \cdots$. Here $|A| \prec |B|$ means that the cardinality of A is less than the cardinality of B. Note that each of the sets X_i must have infinite cardinality. Explain how you know that your answer is correct.

Answer. Let N denote the natural numbers. Then

$$|N| \prec |P(N)| \prec |P(P(N))| \prec \cdots.$$

This follows at once from the fact that $|X| \prec |P(X)|$ for any set X, and this is implied by part **(a)** of this problem.

Answers to the remaining questions are omitted.

6. Let C^n_r denote the binomial choice coefficient. Thus C^n_r is equal to the number of r-element subsets of a set with n-elements. This is sometimes phrased as the number of ways to choose r things from n things.

(a) State the binomial theorem for $(x+y)^n$ in terms of the coefficients C_r^n.

(b) Use the identity $(1+x)^{n+m} = (1+x)^n(1+x)^m$ to find and prove a formula for C_r^{n+m} in terms of C_i^n and C_j^m $(0 \leq i \leq n, 0 \leq j \leq m)$. Your formula will be the consequence of applying the binomial theorem to both sides of the above identity.

7. Prove that $(A \wedge B) \vee (A \wedge C) \wedge (B \wedge C) = (A \vee B) \wedge (A \vee C) \wedge (B \vee C)$ in elementary logic. Please make your proof by first translating the statement to be proved into Boolean algebra notation where $X + Y = X \vee Y$ and $XY = X \wedge Y$ and $\bar{X} = \sim X$.

8. Prove that, for a positive integer n, a $2^n \times 2^n$ square grid with any one square removed can be covered using L-shaped non-overlapping tiles. Each tile consists in three adjacent grid squares in an L-shaped pattern.