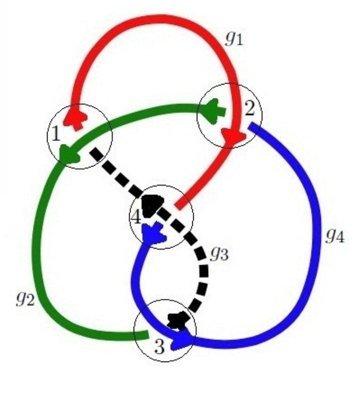
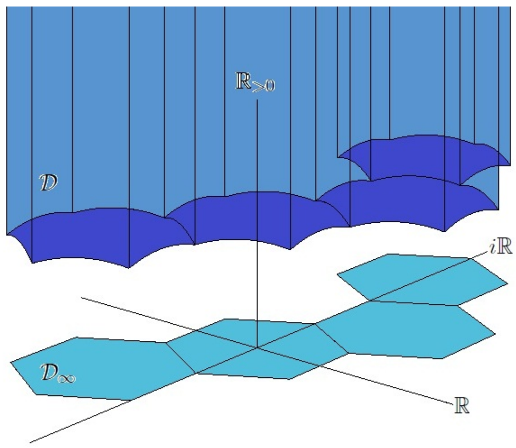
University of Fribourg

Master Thesis

The Discovery of Hyperbolization of Knot Complements





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Abstract

We give a detailed presentation of the first example of hyperbolization of a knot complement, due to Riley [Ri1].

Chapter 1 is dedicated to theoretical background. In particular, we introduce the concepts of knot theory, Kleinian groups, fundamental domains, Poincaré's Polyhedron Theorem, and hyperbolic 3-manifold theory.
In Chapter 2, we give an isomorphism between the fundamental group of the figure-eight complement $\mathbb{R}^3 \setminus 4_1$ and a certain subgroup $\Gamma < PSL(2, \mathbb{C})$. We then make use of Poincaré's Polyhedron Theorem to produce a fundamental domain $\mathcal{D}$ for the action of Γ on the hyperbolic 3-space $\mathcal{U}^3$. As an outcome, we obtain that Γ is discrete and torsion-free, so that the quotient space $\mathcal{D}^* = \mathcal{U}^3/\Gamma$ is a complete oriented hyperbolic 3-manifold of finite volume. We finally use topological arguments to show that $\mathbb{R}^3 \setminus 4_1$ is homeomorphic to $\mathcal{U}^3/\Gamma$.
In Chapter 3, we outline two related results: Mostow-Prasad rigidity, and the Hyperbolization Theorem of Thurston in the case of knots.
Finally, we give as an appendix an unpublished article by Riley, accounting his discovery.

Contents

Introduction

Around 1978, William Thurston held the famous lectures entitled "The Geometry and Topology of 3-Manifolds" at Princeton. The related lecture notes [Th1] are so rich and so deep that it is one of the main references in the study of low-dimensional geometry and topology. Thurston's genius and geometric intuitions, crowned by a Fields Medal in 1982, still feed and stimulate many geometers and topologists worldwide.

Chapter 3 of [Th1] is devoted to geometric structures on manifolds, and begins with a section entitled "A Hyperbolic Structure on the Figure-Eight Knot Complement". There, Thurston states -in one page- that it is possible to "see" the complement of the figure-eight knot "∞" or "4_1" in $\mathbb{R}^3$ as a complete hyperbolic 3-manifold of finite volume constructed by gluing two ideal regular tetrahedra together in a certain way. He also mentions that Robert Riley produced a compatible hyperbolic structure on the figure-eight complement.

In fact, already in 1975, Riley gave in [Ri1] the first proof of this result. Today, we know that most knot and link complements are hyperbolic. This statement was at that moment only conjectured, and is now known as a special case of Thurston's Hyperbolization Theorem.

The aim of this work is to give a detailed presentation of the original proof following Riley [Ri1] and to describe the explicit hyperbolic structure on the figure-eight complement.

Since such a study involves different fields of mathematics, we give in Chapter 1 the necessary theoretical background. In particular, we introduce the concepts of knot theory, Kleinian groups, fundamental domains, Poincaré's Polyhedron Theorem, and hyperbolic 3-manifold theory. This should make this work self-contained and accessible to non-specialists.

In Chapter 2, we make use of these tools to work out Riley's approach. More precisely, we give an isomorphism between the fundamental group of the figure-eight complement $\mathbb{R}^3 \setminus 4_1$ and a certain subgroup $\Gamma < PSL(2, \mathbb{C})$. We then make use of Poincaré's Polyhedron Theorem to produce a fundamental domain $\mathcal{D}$ for the action of Γ on the hyperbolic 3-space $\mathcal{U}^3$. As an outcome, we obtain that Γ is discrete and torsion-free, so that the quotient space $\mathcal{D}^* = \mathcal{U}^3/\Gamma$ is a complete oriented hyperbolic 3-manifold of finite volume. Finally, we use topological arguments based on Waldhausen [Wa] to show that $\mathbb{R}^3 \setminus 4_1$ is homeomorphic to $\mathcal{U}^3/\Gamma$.

In Chapter 3, we give some remarks on two important related results: Mostow-Prasad rigidity, which implies that $\mathbb{R}^3 \setminus 4_1$ is isometric to $\mathcal{U}^3/\Gamma$ and to Thurston's glued tetrahedra, and Thurston's Hyperbolization Theorem applied on knot complements, which gives a criterion to decide whether a knot complement can be hyperbolized or not.

Finally, we give as an appendix an unpublished paper by Riley, in which he describes the context and stages before and after his discovery, and his encounter with Thurston.

Notice that the formal definition of "hyperbolic structure" is the implementation of a Riemannian metric of constant negative sectionnal curvature (which can be normalized to -1). However, we will follow Riley's initial approach, which did not deal with Riemannian geometry.

Good introductions and surveys of hyperbolization of knot complements can be found in [Ad2], [CR], or [Mi2].

Chapter 1

Toolbox

Preliminary remark. The goal of this chapter is to provide all necessary tools to understand how the hyperbolization of a knot complement can be performed. In particular, we have tried to provide a sufficient mathematical background to make the article of Riley [Ri1] accessible.
An experienced reader can easily omit one, several or all sections, and a less experienced (or more interested) reader will find the important definitions and results, and at the begining of each section references to the books and articles which have been used to ellaborate it. Proofs, details and additional material can be found there.

1.1 Fundamental Group

First of all, let us recall some important definitions and facts concerning fundamental groups and topological manifolds. More informations can be found e.g. in [Le].
Given a topological space X, we can consider closed continuous curves in X:

Definition. Let $x_0 \in X$ be a basepoint and $I{:=}[0,1]$ the unit interval of $\mathbb{R}$. A *loop* in X based at x_0 is a continuous map $\alpha : I \to X$ such that $\alpha(0) = \alpha(1) = x_0$.
The product of two loops α_0 and $\alpha_1 : I \to X$ based at x_0 is written $\alpha_0 \circ \alpha_1$, and defined by $(\alpha_0 \circ \alpha_1)(s) := \begin{cases} \alpha_0(2s) & \text{, for } 0 \leq s \leq \frac{1}{2} \\ \alpha_1(2s-1) & \text{, for } \frac{1}{2} \leq s \leq 1 \end{cases}$.
For a loop $\alpha : I \to X$ based at x_0, the loop $\alpha^{-1} : I \to X$ such that $\alpha^{-1}(s) := \alpha(1-s)$, $\forall s \in I$, is based at x_0 and called the inverse of α.

In order to introduce the concept of fundamental group of a manifold, a first notion to mention is the notion of homotopy:

Definition. Let X, Y be topological spaces, and $f, g : X \to Y$ continuous maps.

A *homotopy* from f to g is a continuous map $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$, $\forall x \in X$.
If there exists a homotopy from f to g, then f and g are said to be *homotopic*, written $f \simeq g$.

Proposition 1.1. *Loop homotopy is an equivalence relation.*

So, given a topological space X and a basepoint $x_0 \in X$, we can sort the loops based at x_0 in homotopy classes:

Definition. The *fundamental group* of X based at x_0 is defined to be the set of all homotopy classes of loops based at x_0, and denoted $\pi_1(X, x_0)$.

Proposition 1.2. $\pi_1(X, x_0)$*, together with the product of classes defined by* $[\alpha_0] * [\alpha_1] := [\alpha_0 \circ \alpha_1]$*, is a group, with neutral element* $c_{x_0} := [c_{x_0}]$*, where* $c_{x_0} : I \to X$ *such that* $c_{x_0}(s) := x_0 \; \forall s \in I$.
The inverse of $[\alpha]$ *is* $[\alpha]^{-1} := [\alpha^{-1}]$.

In spite of its quite simple definition, the fundamental group is a powerful tool to study topological and geometric properties of manifolds.
As an example and a preparation for the following, here is a result that shows how the fundamental group can be used to work with topological spaces (and especially with manifolds):

Definition. Two topological spaces X, Y are *homotopic equivalent* $:\Leftrightarrow$ there exists continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq Id_Y$ and $g \circ f \simeq Id_X$.

Proposition 1.3. *If* X, Y *are homotopic equivalent spaces, and if* $f : X \to Y$ *is continuous, then* $\pi_1(X, x_0) \cong \pi_1(Y, f(x_0)) \; \forall \; x_0 \in X$.

Remark. It is important to notice that the converse doesn't hold: an isomorphism between fundamental groups does not imply a homotopy equivalence between the space. The importance of this fact will be clarified in section 2.4.

The objects we will mainly study are topological spaces that "locally look like" a certain "standard space", e.g. $\mathbb{R}^3$ or $\mathbb{H}^3$. We now introduce the (more accessible) notion of topological n-manifold, and will deal with hyperbolic 3-manifolds in Section 1.7.

Definition. A (topological) *n-manifold* is a second-countable, Hausdorff topological space M such that each point of M has a neighborhood homeomorphic to $\mathbb{R}^n$.

As the manifolds we will work with are all path connected, let us finally recall the

Proposition 1.4. *If* X *is path connected, then* $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ *for all* $x_0, x_1 \in X$*, and so we simply write* $\pi_1(X)$.

1.2 Group presentation

Topological groups (in particular, certain finitely presented groups) will be an important part of the following discussions. For example, we have already encountered the notion of fundamental group, and the next section will contain an important theorem about the fundamental group of a knot, producing a presentation of this group.
Let us recall what a group presentation is, beginning from a very abstract and general point:

Definition. Let $S := \{a, b, c, ...\}$ be a set of elements, and $\overline{S} := \{\overline{a}, \overline{b}, \overline{c}, ...\}$, the set of all *barred* symbols of S.
A *word* in S is a finite string of elements from $S \cup \overline{S}$.

Definition. Let $W(S)$ be the set of all words in S.
$\emptyset$ denotes the *empty word* in $W(S)$.
The *concatenation (composition)* of two words $w_1, w_2 \in W(S)$ is the word $w_1w_2 \in W(S)$.
For $w \in W(S)$ the word $\overline{w}$ is the word w written backwards, with all bars and unbars exchanged.

Let $R \subseteq W(S)$. We define the following relation on $W(S)$:
For $w, w' \in W(S)$, $w \sim w'$:$\Leftrightarrow$ there exists a finite sequence of words $w =: w_0, w_1, ..., w_{n-1}, w_n := w'$ such that each pair w_i, w_{i+1} is related by one of the following operations (stated for elements $v, v' \in W(S)$)

- Cancellation: $va\overline{a}v' \leftrightarrow v\overline{a}av' \leftrightarrow vv'$, $\forall a \in W(S)$
- Relation: $vrv' \leftrightarrow vv'$ $\forall r \in R$

This relation is an equivalence relation on $W(S)$.

Definition. The set of all equivalence classes is written $\Pi := W(S)/\sim$. Elements of Π are written $[w]$.

Proposition 1.5. Π*, together with the multiplication* $[w] * [w'] := [ww']$ *is a group, with neutral element* $1 := [\emptyset]$ *and inverse* $[w]^{-1} := [\overline{w}]$.

Remark. $\overline{w} \in W(S)$ is often written w^{-1}. Furthermore, one usually write w instead of $[w]$ for elements of Π when the context is clear enough.

Definition. A *presentation* for Π is given by $\langle S : R \rangle$. S is called the set of the *generators* of Π, and R the set of *relators* of Π.

Examples.
1. C_n, the cyclic group of order n, has the presentation $C_n = \langle a \mid a^n \rangle$.
2. $D_n = \left\langle r, s \mid r^n, s^2, (rs)^n \right\rangle$ is a presentation of the dihedral group of order $2n$.

Remark. A group may have many (equivalent) presentations.

1.3 Knots, Knot Complements

Now that we have some minimal background of topology and group presentation, we can start with knots.
The concept of knot is quite intuitive, but recent theory, related e.g. to quantum physics and low-dimensional topology. We will see that the investigations on knot invariants can provide information about objects that seem to have no relation at first sight, e.g. hyperbolic manifolds.
The following section will give a short introduction to knot theory, following the books of Adams [Ad1], Lickorish [Li] and Prasolov & Sossinsky [PS], and the lecture notes of Roberts [Ro]. The main goal will be to state a theorem about the Wirtinger presentation of a knot group.
Let us begin with basic definitions.

Definition. A *knot* K is a subset of $\mathbb{R}^3$ or $\mathbb{S}^3$ homeomorphic to the circle $\mathbb{S}^1$.

Remark. Thinking of knots in $\mathbb{R}^3$ or $\mathbb{S}^3 \cong \mathbb{R}^3 \cup \{\infty\}$ doesn't make any fundamental difference in knot theory, because knots and sequences of deformations of knots may always be assumed not to hit ∞.
From now on, knots will be considered as lying in $\mathbb{R}^3$, only for "imagination comfort".

Definition. An *oriented knot* is a knot with an orientation specified.

In a similar way that any closed knotted elastic can always be twisted and stretched without changing its "structure", knots are topological objects that are invariant under such operations. This motivates the following definition.

Definition. Let K be a knot and $\triangle$ a planar triangle in $\mathbb{R}^3$ that intersects K in exactly one edge of $\triangle$. A Δ-*move* is the operation of replacing K by a new knot K' consisting in K with the intersecting edge of $\triangle$ removed, and the two other edges of $\triangle$ added.

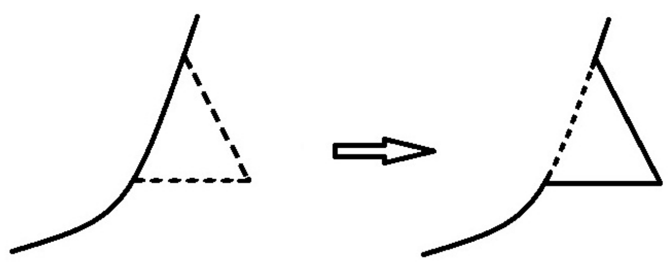

Figure 1.1: A Δ-move

Definition. Two knots K and $\tilde{K}$ are said to be *isotopic* if there exists a finite sequence of knots $K =: K_0, ..., K_n := \tilde{K}$ such that each pair K_i, K_{i+1} is related by a Δ-move.

Proposition 1.6. *Isotopy is an equivalence relation.*

Remarks. 1. Topologically, two knots K and $\tilde{K}$ are isotopic if and only if there exists an orientation preserving piecewise linear homeomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that $f(K) = \tilde{K}$.

2. From now on, "a knot K" will refer to the isotopy class represented by K, which will be written K instead of $[K]$.

In order to visualise knots, a natural idea is to produce a projection of the knot onto a drawing plane. This idea works, and in fact we will see that if the knot is "well placed" (or if we choose a convenient projection plane), one can have a one-to-one correspondance between a knot and a knot diagram.

Definition. Let $p : \mathbb{R}^3 \to \mathbb{R}^2$ denote the standard vertical projection. K is *in general position* if the preimage of each point of $p(K)$ consists of either one or two points of K, in the latter case neither of the two points being a vertex of K.

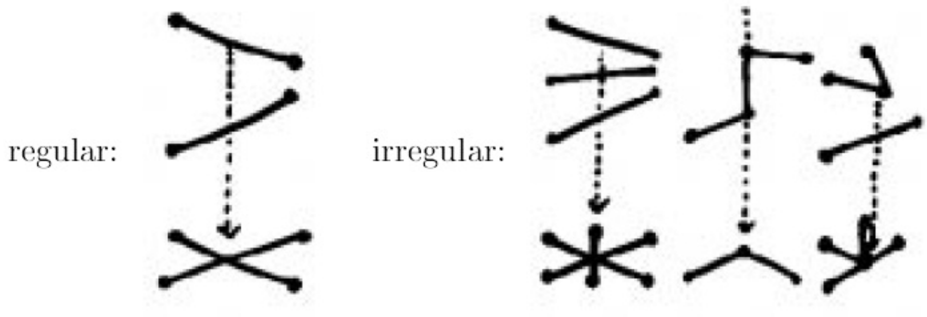

Figure 1.2: Regular and irregular projections (Picture: [Ro], p.9)

Definition. Let K be in general position. A *(knot) diagram* $D(K)$ of K is the projection $p(K)$ together with the "over-under" information at each crossing.
The projection of an oriented knot gives an *oriented diagram.*
An *arc* of $D(K)$ is any finite segment between two consecutive under-passes of $D(K)$.

The following proposition motivates the usual identification between a knot and one of his diagrams and is not hard to prove (see e.g. [Ro], p.10).

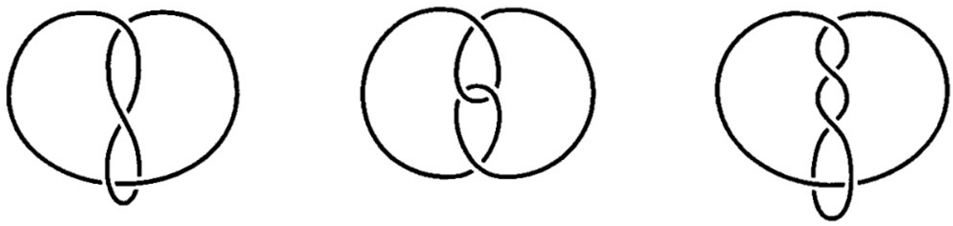

Figure 1.3: Knots diagrams (all of the same knot (!))

Proposition 1.7. *1. Any knot has a diagram.*

2. From any diagram one can reconstruct the knot up to isotopy.

Now that we have sorted the knots in isotopy classes and provided a way to represent them in diagrams, the next "natural" problem is to find a way to combine knots with each other. This can be done quite easily.

Definition. A knot K is said to be the *unknot* if K bounds an embedded piecewise linear disc in $\mathbb{R}^3$.

Definition. The *sum* $K_1 + K_2$ of two oriented knots K_1 and K_2 is defined as follows:
Consider knots diagrams $D(K_1)$ and $D(K_2)$, and regard them as being in distincts copies of $\mathbb{R}^2$. Remove from each $\mathbb{R}^2$ a small disc that meets the given knot in an arc and no crossing, and then identify together the resulting boundary circles, and their intersections with the knots, so that all orientations match up. The unknot is the neutral element of the knot addition.

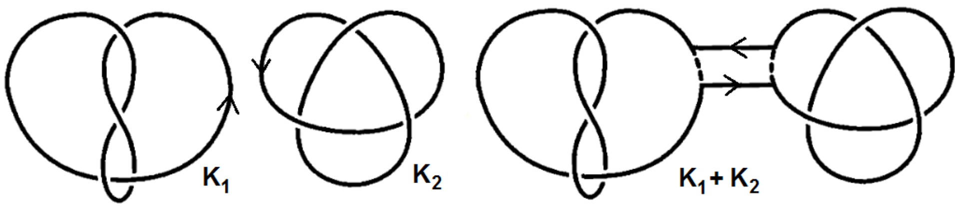

Figure 1.4: The sum $K_1 + K_2$

As in the case of $\mathbb{Z}$, the operation "+" defined above leads to the concept of prime knots.

Definition. A knot K is a *prime knot* if is not the unknot, and

$$K = K_1 + K_2 \Rightarrow K_1 \text{ or } K_2 \text{ is the unknot.}$$

We now come to one of the main objects of the present work, the knot complement and its fundamental group.

Definition. A *knot invariant* is an assignment to each knot of some algebraic or topological object (e.g. number, polynomial, group, etc.) that depends only on the isotopy class of the knot.

Definition. Let $K \subset \mathbb{R}^3$ be a knot. The *complement* of K is defined to be $\mathbb{R}^3 \setminus K$.

Theorem 1.1. $\mathbb{R}^3 \setminus K$ *is a (non-compact) path-connected 3-manifold.*

Definition. Let K be a knot. The *group of* K, written $\pi_1(K)$, is the fundamental group $\pi_1(\mathbb{R}^3 \setminus K)$.

Remarks.

1. Considering K as lying in $\mathbb{S}^3$ and considering the complement $\mathbb{S}^3 \setminus T_K$, where T_K is a tubular non-self-intersecting neighborhood of K (homeomorphic to a torus), gives a compact version of the knot complement: in this case $\mathbb{S}^3 \setminus T_K$ is a compact path-connected 3-manifold with boundary a torus.
2. $\pi_1(\mathbb{S}^3 \setminus T_K)$ and $\pi_1(\mathbb{R}^3 \setminus K)$ are isomorphic, so the choice of the compact or non-compact case is not very important in our context.

Theorem 1.2.

1. *The knot determines the complement as follows: if K_1 and K_2 are isotopic, then $\pi_1(K_1) \cong \pi_1(K_2)$.*
2. *Knots are determined by their complement as follows: if $\pi_1(K_1) \cong \pi_1(K_2)$, then K_1 and K_2 are isotopic.*
3. *The knot group is an invariant of knots.*
4. *The group determines the knot.*

The two theorems stated above seem elementary, but certain assertions, in particular the third and fourth of the latter theorem, are quite difficult to prove (for references see e.g. [Ro], p.62). They have many interesting consequences. For example we can now try to find connections between knots and manifolds. The natural hope is to get information about manifolds directly by looking at knots, and vice-versa. A famous conjecture, the so-called Volume Conjecture of Kashaev, states that the volumes of certain manifolds can be obtained as values related to a given invariant on the suitable knot (see e.g. [Sch] for more details).
We now come to the main goal of this section.

Definition. Let K be a knot and $D(K)$ an oriented diagram for K.
To each crossing, assign a sign $+1$ or -1, according to the following figure:

If the crossing has sign $+1$, then it is called a positive crossing, and a negative crossing otherwise.

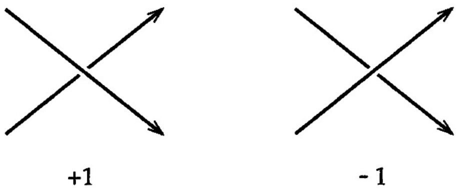

Figure 1.5: Positive and negative crossings

Theorem 1.3 (Wirtinger Presentation)**.** *Let K be a knot with corresponding oriented diagram $D(K)$ with n crossings.*
Let $g_1, ..., g_n$ be the arcs of $D(K)$ and denote $S := \{g_1, ..., g_n\}$.
At each signed crossing, one has three incident arcs, labelled as follows:

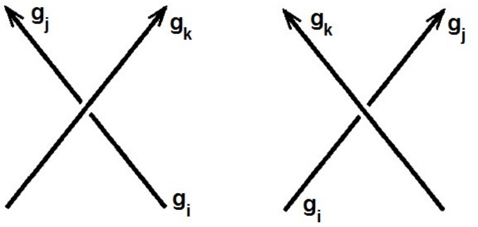

Figure 1.6: Labelled incident arcs at positive and negative crossings

To each positive crossing, associate the relation $g_k^{-1} g_i g_k = g_j$, and to each negative crossing, the relation $g_k g_i g_k^{-1} = g_j$. Denote by R the set of all n relations obtained in this way.
Then the fundamental group $\pi_1(K)$ has presentation $\langle S : R \rangle$.
This presentation is called the Wirtinger Presentation of $\pi_1(K)$.

So, given a knot K, we have a very elementary way to obtain a presentation of $\pi_1(K)$.

In short, we now begin to see that there are connections between knots and manifolds, and that groups play an important role in both settings.

1.4 Kleinian Groups of Hyperbolic Isometries

We now study another tool and investigate isometry groups, more precisely direct isometries of the hyperbolic space $\mathbb{H}^3$. References for this sections are the books of Ford [Fo], Katok [Kt], Matsuzaki & Taniguchi [MaTa], and

Ratcliffe [Ra], and the articles of Riley [Ri1] and [Ri3].
First of all, we introduce the

Definition. Let $\mathcal{H}^3 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\}$ the upper half-space, equipped with the metric $ds^2_{\mathcal{H}} = \frac{dx_1^2 + dx_2^2 + dx_3^2}{x_3^2}$.
Then, $(\mathcal{H}^3, ds^2_{\mathcal{H}})$ is called POINCARE*'s upper half-space model of the hyperbolic space* $\mathbb{H}^3$.
For the metric $d_{\mathcal{H}}(x, y)$, $x, y \in \mathcal{H}^3$, one deduces $cosh(d_{\mathcal{H}}(x, y)) = 1 + \frac{|x-y|^2}{2x_3y_3}$.
The geodesics of $\mathcal{H}^3$ are the vertical lines of $\mathcal{H}^3$ and the half-circles centered in points of $\mathbb{R}^2 \times \{0\}$ and orthogonal to $\mathbb{R}^2$.

Remark. In fact, the hyperbolic space $\mathbb{H}^3$, as defined as the simply connected complete Riemannian manifold of constant sectional curvature -1, admits other models. All are equivalent, but each one has its own advantages. For example, the half-space model is a conformal model, and the boundary $\partial\mathcal{U}^3$ can be visualized as $\mathbb{R}^2 \cup \{\infty\}$.
In the following, our prefered model for the hyperbolic space $\mathbb{H}^3$ will be the "complex" version $\mathcal{U}^3 \subset \mathbb{C} \times \mathbb{R}_{>0}$ of $\mathcal{H}^3$.
We will see that the isometries of $\mathcal{U}^3$ can be expressed in a very simple, tractable way.

1.4.1 Hyperbolic Isometries and Möbius Transformations

We first give some definitions and notations.

For each point $(x_1, x_2, x_3) \in \mathcal{H}^3$, identify the couple $(x_1, x_2) \in \mathbb{R}^2$ with the point $z := x_1 + ix_2 \in \mathbb{C}$, so that $\mathcal{H}^3 \cong \{(z, t) \in \mathbb{C} \times \mathbb{R} \mid t > 0\} = \mathbb{C} \times \mathbb{R}_{>0} =: \mathcal{U}^3$.
From now on, we will use $\mathcal{U}^3$ as model for the hyperbolic 3-space $\mathbb{H}^3$.

Remark. The reason of introducing $\mathcal{U}^3$ instead of $\mathcal{H}^3$ is that we will consider transformations that act on the complex plane, in order to make the discussion more comfortable.
The metric $ds^2_{\mathcal{H}}$ on $\mathcal{H}^3$ is easily identified with the metric $ds^2_{\mathcal{U}} = \frac{|dz|^2 + dt^2}{t^2}$ on $\mathcal{U}^3$.

Definition. Let $t \in \mathbb{R}_+$, and set $\mathbb{C}_t := \mathbb{C} \times \{t\} \subset \mathbb{C} \times \mathbb{R}$, such that $\mathbb{C}_0 \cup \{\infty\}$ is the boundary of $\mathcal{U}^3$, and $P^1(\mathbb{C}) \cong \mathbb{C}_0 \cup \{\infty\}$ the Riemannian sphere.
Often we do not distinguish between $\mathbb{C}_0$ and $\mathbb{C}$. Let $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ and identify $P^1(\mathbb{C})$ with $\widehat{\mathbb{C}}$.

Definition. Let $\widehat{\mathbb{R}^3} := \mathbb{R}^3 \cup \{\infty\}$ and $\mathcal{M}ob(\widehat{\mathbb{R}^3})$ denote the group of all orientation-preserving homeomorphisms of $\widehat{\mathbb{R}^3}$. An element of $\mathcal{M}ob(\widehat{\mathbb{R}^3})$ is called a *Möbius transformation* of $\widehat{\mathbb{R}^3}$.
For a subset $E \subset \widehat{\mathbb{R}^3}$, we define $\mathcal{M}ob(E) := \{T \in \mathcal{M}ob(\widehat{\mathbb{R}^3}) \mid T(E) = E\}$, the group of all Möbius transformations of $\widehat{\mathbb{R}^3}$ which preserve E.

Definition. Let $Isom^+(\mathcal{U}^3)$ be the group of all orientation-preserving isometries of the hyperbolic 3-space $\mathcal{U}^3$.

The following proposition is not difficult to prove (see e.g. [MaTa] p. 19).

Proposition 1.8. *$Isom^+(\mathcal{U}^3) \cong \mathcal{M}ob(\mathcal{U}^3)$.*

Now set $\mathcal{M}ob := \{T : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}} \mid T(z) = \frac{az+b}{cz+d},\ a,b,c,d \in \mathbb{C},\ ad-bc=1\}$, the set of all linear fractional transformations on $\widehat{\mathbb{C}}$.
It is not difficult to prove that $\mathcal{M}ob$ is a group, which can be identified with a very particular matrix group as follows.

Proposition 1.9. *$\mathcal{M}ob \cong PSL(2,\mathbb{C}) := SL(2,\mathbb{C})/\{\pm I_2\}$, I_2 the identity matrix of $Mat(2\times 2,\mathbb{C})$.*

Remark. In the sequel, we do not distinguish between a class $[T] \in PSL(2,\mathbb{C})$ and its representative $T \in SL(2,\mathbb{C})\ mod \pm I_2$.
The action of $\mathcal{M}ob \cong PSL(2,\mathbb{C})$ on the boundary $\widehat{\mathbb{C}}$ of $\mathcal{U}^3$ is the following:

Definition. For a $z \in \widehat{\mathbb{C}}$ and an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{C})$, let $T(z) := \frac{az+b}{cz+d}$ with the following special cases.

- for $c = 0 : T(\infty) = \infty$
- for $c \neq 0 : T\left(-\frac{d}{c}\right) = \infty$, and $T(\infty) = \frac{c}{a}$.

We state and prove a theorem that links up hyperbolic isometries and fractional linear transformations, and makes the notation $\mathcal{M}ob$ clear:

Theorem 1.4. *$Isom^+(\mathcal{U}^3) \cong \mathcal{M}ob$.*

Proof. $\boxed{\text{"}\supseteq\text{"}}$ One can see (see e.g. [Fo]) that any element A of $\mathcal{M}ob$ can be expressed as a finite composition of (an even number of) reflections of $\widehat{\mathbb{C}}$ with respect to circles or lines $K_j \subset \widehat{\mathbb{C}}$.
For each K_j, one can find a unique sphere or plane $\widetilde{K_j}$ in $\mathbb{R}^3 \cong \widehat{\mathbb{C}_0}$ which is perpendicular to $\mathbb{C} \cong \mathbb{R}^2$ along K_j. One can show that the reflection of $\mathbb{R}^3$ with respect to $\widetilde{K_j}$ is conjugate to the fundamental inversion $z \mapsto \frac{1}{z}$ by an element in $\mathcal{M}ob(\widehat{\mathbb{R}^3})$.
Hence, the composition $\widetilde{A}$ of all these reflections of $\widehat{\mathbb{R}}^3$ with respect to the spheres or planes $\widetilde{K_j}$ is an element of $\mathcal{M}ob(\widehat{\mathbb{R}}^3)$.
Moreover, $\widetilde{A} \in \mathcal{M}ob(\mathcal{U}^3)$, because a reflection with respect to any K_j preserves $\mathcal{U}^3$. So $\widetilde{A}_{|\widehat{\mathbb{C}}} = A$, which shows that one can embed $\mathcal{M}ob$ into $\mathcal{M}ob(\mathcal{U}^3)$, and $\mathcal{M}ob \subset Isom^+(\mathcal{U}^3)$.
$\boxed{\text{"}\subseteq\text{"}}$ We now consider an element $T \in Isom^+(\mathcal{U}^3) \cong \mathcal{M}ob(\mathcal{U}^3)$.
Let $p := T(\infty) \in \widehat{\mathbb{C}}$, and A be the Möbius transformation $A(z) := \frac{1}{z-p}$,

corresponding to the matrix $\begin{pmatrix} 0 & 1 \\ 1 & -p \end{pmatrix}$. Furthermore, let $\widetilde{A}$ be the element of $\mathcal{M}ob(\mathcal{U}^3)$ which corresponds to A under the embedding described above. Then, $\widetilde{A} \circ T \in \mathcal{M}ob(\mathcal{U}^3)$ fixes ∞, and is therefore a similarity of $\mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3$. Hence, $(\widetilde{A} \circ T)_{|\widehat{\mathbb{C}}} = A \circ T_{|\widehat{\mathbb{C}}} \in \mathcal{M}ob$, and so is $T_{|\widehat{\mathbb{C}}}$ an element of $\mathcal{M}ob$, which proves that any element of $Isom^+(\mathcal{U}^3)$ can be seen as an element of $\mathcal{M}ob$. ■

Definition. The extension of an element $A \in \mathcal{M}ob$ acting on $\widehat{\mathbb{C}}$ to an element $\widetilde{A} \in \mathcal{M}ob(\mathcal{U}^3)$ described in the proof of the Theorem above is called the *Poincaré extension* of A.

Summarising, we have several equivalent ways to think of orientation-preserving hyperbolic isometries: either as homeomorphisms that preserve the hyperbolic metric of $\mathcal{U}^3$, or as fractional linear transformations over $\hat{C}$, or as particular $2x2$ matrices. In the following, we will use one or the other point of view, depending on the context. Furthermore, the following proposition provides a classification of Möbius transformations.

Proposition 1.10. *Let $z \in \widehat{\mathbb{C}}$. Any element $\gamma \neq id$ of $\mathcal{M}ob$ can be transformed by conjugation into either*

1. *$\gamma_1 \in \mathcal{M}ob$ given by $\gamma_1(z) := z + 1$ with matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, or*
2. *$\gamma_2 \in \mathcal{M}ob$ given by $\gamma_2(z) := \lambda z$, for a $\lambda \in \mathbb{C} \setminus \{0, 1\}$, with matrix $\begin{pmatrix} \lambda^{\frac{1}{2}} & 0 \\ 0 & \lambda^{-\frac{1}{2}} \end{pmatrix}$.*

Having this characterization, we can now classify Möbius transformations.

Definition. Let $\gamma \in \mathcal{M}ob$, $\gamma \neq id$. Then,

- γ is called *parabolic* $:\Longleftrightarrow$ γ is conjugate to γ_1;
- γ is called *elliptic* $:\Longleftrightarrow$ γ is conjugate to γ_2 with $|\lambda| = 1$;
- γ is called *loxodromic* $:\Longleftrightarrow$ γ is conjugate to γ_2 with $|\lambda| \neq 1$.

In fact, the type of a Möbius transformation can be read off from the trace of its matrix representative.

Proposition 1.11. *Let $A_\gamma \in PSL(2, \mathbb{C})$ corresponding to $\gamma \in \mathcal{M}ob$. Then,*

- *$tr^2(A_\gamma) = 4 \Longleftrightarrow \gamma$ is parabolic ;*
- *$0 \leq tr^2(A_\gamma) < 4 \Longleftrightarrow \gamma$ is elliptic ;*

- $tr^2(A_\gamma) \in \mathbb{C} \setminus [0,4] \iff \gamma$ *is loxodromic.*

A_γ *is also called parabolic, elliptic, or loxodromic.*

In particular, parabolic transformations will play an important role in our further discussion.

1.4.2 Kleinian Groups

Remember that the hyperbolic space $\mathbb{H}^3$ is a topological space. We will consider only groups of isometries that act on $\mathbb{H}^3$ in a certain discretized nice way.

Definition. A group G of homeomorphisms acts on a topological space X *properly discontinuously* if for each compact subset $K \subset X$, there exists only finitely many elements $g \in G$ which satisfy the condition $g(K) \cap K \neq \emptyset$.

Definition. A subgroup Γ of $Isom^+(\mathbb{H}^3)$ is called *Kleinian* $:\iff \Gamma$ acts properly discontinuously on $\mathbb{H}^3$.
Under the identification of $Isom^+(\mathbb{H}^3)$ with $\mathcal{M}ob(\mathcal{U}^3)$, $\mathcal{M}ob$, or $PSL(2,\mathbb{C})$, the corresponding (sub)groups are also called Kleinian groups.

From now on, Γ will be the prefered notation for a Kleinian group.
In the sequel, we will be able to characterize every Möbius transformation by looking at its fixed points. We first give following

Definition. Let $S_\infty := \partial\mathbb{H}^3$ denote the sphere at infinity of the hyperbolic space $\mathbb{H}^3$, i.e. the set of the points whose distance to the origin is infinite. Especially, when $\mathbb{H}^3$ is realized by $(\mathcal{U}^3, d_\mathcal{U})$, $S_\infty = \widehat{\mathbb{C}}$.

Now we can state the following

Proposition 1.12. *Let* $\gamma \in Isom^+(\mathbb{H}^3)$, $\gamma \neq id$.

$$\text{Then } \gamma \text{ is } \left\{ \begin{array}{c} \textit{parabolic} \\ \textit{loxodromic} \\ \textit{elliptic} \end{array} \right\} \iff \gamma \text{ has } \left\{ \begin{array}{c} \textit{a unique} \\ \textit{two distinct} \\ \textit{infinitely many} \end{array} \right\} \textit{fixed point(s)}$$

in $\mathbb{H}^3 \cup S_\infty$.
Moreover, if γ *has any fixed point in* $\mathbb{H}^3$, *then* γ *is elliptic.*

Finally, we give following proposition, which will be useful in section 1.7 (more details can be found in [MaTa], chapter 1.2).

Proposition 1.13. *An element* γ *of a Kleinian group has finite order* $\iff$ γ *has a fixed point in* $\mathbb{H}^3$.

1.4.3 Geometric Action of Möbius Transformations on $\mathcal{U}^3$

Our next aim is to understand how a Möbius transformation acts on $\mathcal{U}^3$. We look more precisely at the action of a Möbius transformation $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{C})$. From previous discussions, we know that T provides a hyperbolic isometry on $\mathcal{U}^3$. We study now the effect of T on euclidean lengths.
We distinguish the two cases $T(\infty) \neq \infty$ and $T(\infty) = \infty$, which is equivalent to distinguish the two cases $c \neq 0$ and $c = 0$.

First case: $c \neq 0$

We first look at the action of T on $\partial\mathcal{U}^3 \setminus \{\infty\} = \mathbb{C}_0 \cong \mathbb{C}$.
Let $z \in \mathbb{C}$. The function $T(z) = \frac{az+b}{cz+d}$ has the derivative $T'(z) = \frac{1}{(cz+d)^2}$, which means that euclidean lengths are multiplied by $|T'(z)| = |cz+d|^{-2}$. In particular, if $|cz+d| = 1$, T acts as a euclidean isometry. This consideration leads to the

Proposition 1.14. *The set of all points $z \in \partial\mathcal{U}^3 \setminus \{\infty\}$ such that $|T(z)| = |z|$ is equal to the set $\{z \in \mathbb{C} |\ |cz+d| = 1\}$.*

Since $c \neq 0$, $\{z \in \mathbb{C} \mid |cz+d| = 1\} = \left\{z \in \mathbb{C} \mid |z + \frac{d}{c}| = |\frac{1}{c}|\right\}$, and is a euclidean circle in $\mathbb{C}$ with center $-\frac{d}{c}$ and radius $\frac{1}{|c|}$.
This result motivates the following

Definition. The *isometric circle* of the transformation $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{C})$ is the set $I_0(T) := \{z \in \mathbb{C} \mid |T(z)| = |z|\}$.

Remark. Since $T^{-1}(z) = \frac{-dz+b}{cz-a}$, the isometric circle of T^{-1} is $I_0(T^{-1}) = \{z \in \mathbb{C} \mid |cz-a| = 1\}$, with center $\frac{a}{c}$ and radius $\frac{1}{|c|}$.

Proposition 1.15. *There exists a euclidean line $R_0(T) \subset \mathbb{C}$ which is the euclidean bisector of $I_0(T)$, i.e. $R_0(T)$ contains the point $-\frac{d}{c}$ and separates $I_0(T)$ into two half-circles with the same euclidean area, such that the action of T on $\mathbb{C}$ is the product of the following three operations*

1. *Euclidean reflection of $\mathbb{C}$ in $R_0(T)$,*
2. *Euclidean inversion of $\widehat{\mathbb{C}}$ in $I_0(T)$, and*
3. *Euclidean translation of $\mathbb{C}$ carrying $I_0(T)$ on $I_0(T^{-1})$,*

in the given order.

There is a natural extension of these considerations to $\mathcal{U}^3$.

Definition. The *isometric sphere* of T is the hyperbolic plane $I(T) \subset \mathcal{U}^3$ whose euclidean boundary is the isometric circle $I_0(T)$. Similarly, the hyperbolic plane $R(T)$ is defined to be the plane in $\mathcal{U}^3$ having $R_0(T)$ as euclidean boundary.

Proposition 1.16. *T acts on $\mathcal{U}^3$ by reflection in $R(T)$, then by inversion in $I(T)$, and finally by translation carrying $I(T)$ on $I(T^{-1})$.*

Thus, T is entirely determined by $I(T)$, $I(T^{-1})$ and $R(T)$.

Proposition 1.17. *Let $Jac(T)$ denote the euclidean Jacobian of T.*
Then $Jac(T)$ is either < 1, $= 1$ or > 1 in a point $p \in \mathcal{U}^3$, according to p being outside $I(T)$, on $I(T)$, or inside $I(T)$ with respect to $\mathbb{C}_0 \cong \mathbb{C}$.
In other words, T acts as euclidean expansion in the interior of $I(T)$, as euclidean isometry carrying $I(T)$ on $I(T^{-1})$, and as euclidean contraction outside $I(T)$.

Second case: $c = 0$

Here, $d = a^{-1}$, and so $T = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$.
From the previous case, one deduces that T has no isometric sphere and that the Jacobian $Jac(T)$ is constant on $\mathcal{U}^3$. We suppose without loss of generality that $Jac(T) = 1$. Then we obtain the following

Proposition 1.18. *T is either*

- *parabolic of the form $T = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, and corresponds to the euclidean translation $z \mapsto z + b$, or*
- *elliptic of the form $T = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$, $a \neq \pm 1$, and corresponds to a euclidean rotation of $\mathcal{U}^3$ about the axis whose euclidean endpoint is the finite fixed point of T in $\mathbb{C}$, followed by a translation.*

Since the characteristic property of transformations T with $c = 0$ is that T fixes ∞, we now define

Definition. Let $PSL(2,\mathbb{C})_\infty = \{T \in PSL(2,\mathbb{C}) \mid T(\infty) = \infty\}$ be the set of Möbius transformations fixing ∞.
For a subgroup $G < PSL(2,\mathbb{C})$, one defines $G_\infty := G \cap PSL(2,\mathbb{C})_\infty$.

Proposition 1.19. *If Γ is a Kleinian group, then Γ_∞ acts properly discontinuously on $\widehat{\mathbb{C}}$.*

In summary, we have seen that a Möbius transformation T not fixing ∞ acts on $\mathcal{U}^3$ as a product of a reflection, an inversion and a translation. In this case, we have identified 3 geometric objects characterizing the transformation T, the two isometric spheres $I(T)$ and $I(T^{-1})$ and the hyperplane $R(T)$. Furthermore, we have seen that a Möbius transformation T fixing ∞ acts on each *horosphere* $\mathbb{C}_t, t > 0$, in $\mathcal{U}^3$ as combination of a translation with as a euclidean rotation.

1.5 Discrete Groups and Fundamental Domains

We will now refine our knowledge of Möbius transformations and examine particular geometric objects that can been attached to Kleinian groups.
This section deals with a further topological characterization of Kleinian groups, and the concept of fundamental domain and fundamental polyhedron. In a first time, we will present definitions and basic facts, and in a second time we will state a theorem of Poincaré about a way to produce such a polyhedron.
References for the first part can be found in the books of Ford [Fo], Kapovich [Ka], and Matsuzaki & Taniguchi [MaTa]. The second part is based on the article of Maskit [Ma1] which was used by Riley, and of the more recent books of Maskit [Ma2], and Ratcliffe [Ra].

1.5.1 Discrete Groups, Ford Domains

Let G be a topological group, i.e. a topological space and a group such that for all $g, h \in G$ the operations $(g, h) \mapsto gh$ and $g \mapsto g^{-1}$ are continuous.
We begin with a topological definition.

Definition. A subgroup $H < G$ is called *discrete subgroup* $:\Longleftrightarrow \; \forall\, h \in H,\ \{h\}$ is open in H.

The following theorem gives us another way to think of Kleinian groups.

Theorem 1.5. *A Kleinian group is a discrete subgroup of $PSL(2, \mathbb{C})$, and vice versa.*

Now, to each discrete subgroup Γ of $Isom^+(\mathbb{H}^3)$ one can assign a geometric-combinatorial object.

Definition. Let $G < Isom^+(\mathbb{H}^3)$. A set $F \subset \mathbb{H}^3$ is a *fundamental domain* for $G :\Longleftrightarrow$

1. $F \subset \mathbb{H}^3$ is a domain (i.e. an open and connected subset),
2. $\forall\, g_1,\ g_2 \in G,\ g_1 \neq g_2 :\ g_1F \cap g_2F = \emptyset$, and
3. $\bigcup_{g \in G} \overline{gF} = \mathbb{H}^3$.

If F is a polyhedron, then F is called *fundamental polyhedron.*

A fundamental domain F can be seen as a pattern in $\mathbb{H}^3$ that will be copied, modified, and moved into $\mathbb{H}^3$ under the action of G, and finally -by passing to its closure- cover the entire space $\mathbb{H}^3$, without having any intersection with another "transformed copy" of itself. In fact, if F satisfies certain matching properties, the orbit space $\mathbb{H}^3/G$ can be identified with the image of F by a gluing of its faces, and be seen as a smoothy hyperbolic manifold. That motivates the study of fundamental domains.
The existence of a fundamental domain for a group acting on $\mathbb{H}^3$ is not assured. But if the group is discrete, then one has following result.

Proposition 1.20. *Every discrete subgroup $\Gamma < Isom^+(\mathbb{H}^3)$ admits a fundamental domain.*

There are several ways to construct particular fundamental domains in such situations. We will be concerned with so-called Ford domains.

Definition. Let Γ be a discrete subgroup of $PSL(2, \mathbb{C})$. Let $D_\infty \subset \mathbb{C}_0$ be a fundamental domain for Γ_∞ whose boundary is either empty or a finite union of euclidean polygons.
The *Ford domain* D_Γ corresponding to D_∞ is the intersection of the portion of $\mathcal{U}^3$ outside all isometric spheres of $\Gamma \setminus \Gamma_\infty$ with the euclidean cylinder $\big\{(z,t) \in \mathcal{U}^3 |\ z \in D_\infty\big\}$.

In other words, D_Γ is the portion of $\mathcal{U}^3$ outside all isometric spheres of Γ which projects orthogonally onto D_∞ in $\mathbb{C}_0$.
Before giving further properties of a Ford domain, we give another way to see it, mentioned in [Bo] as a definition, but given here as a proposition. First, we have to give following

Definition. The point $(z_1, t_1) \in \mathcal{U}^3$ is said to *lie above* the point $(z_2, t_2) \in \mathcal{U}^3 :\Longleftrightarrow\ t_1 \geq t_2$.

With this definition, we can state the following result.

Proposition 1.21. *A Ford domain D_Γ of a Kleinian group $\Gamma < PSL(2, \mathbb{C})$ can be identified as the set $\{p \in \mathcal{U}^3 |\ p$ lies above $\gamma(p),\ \forall\, \gamma \in \Gamma\}$.*

Now we give some important properties of the Ford domain D_Γ. For a formal definition of a polyhedron, we refer to the next section.

Theorem 1.6. *Let Γ a Kleinian group and D_Γ the Ford domain of Γ. Then:*

1. *D_Γ is a hyperbolic polyhedron.*
2. *For the vertical projection $\mathcal{U}^3 \longrightarrow \mathbb{C}$, every face of D_Γ projects to a euclidean polygon in $\mathbb{C}_0$.*
3. *D_Γ is invariant under Γ_∞.*
4. *The boundary of $\overline{D_\Gamma}$ is the union of polygons with finitely many edges.*

1.5.2 Poincaré's Polyhedron Theorem

There is a very nice result due to Poincaré about sufficiently nice polyhedra $D \subset \mathbb{H}^3$, providing a group Γ that operates freely discontinuously by isometries on $\mathbb{H}^3$ with fundamental polyhedron D. We will also get from this construction a complete presentation of Γ.
However, D needs to satisfy several non-trivial conditions we are going to enumerate now.
Let us first recall the notion of a polyhedron.

Definition. A *polyhedron* $D \subset \mathbb{H}^3$ is an open connected subset of $\mathbb{H}^3$ as follows:

1. $\partial D = \bigcup_{i \in I} S_i$, I a finite set, and for all $i \in I$, S_i is a subset of a hyperbolic plane H_i and the closure of a polygon in H_i, called *side* of D.

2. The sides of the polygons S_i are called *edges* of D, denoted e_j ; the endpoints of the e_j are called the *vertices* of D.

3. For each edge e_j, there are only 2 sides S_i and S_k such that $S_i \cap S_k = e_j$.
   Any two sides are either disjoint, intersect in a common edge, or intersect in a common vertex.
   An edge is either a subset of a side, meets the side in a common vertex, or is disjoint from a side.
   Two edges are either disjoint or meet in one common vertex.

4. For all $x \in \partial D$, and all $\delta > 0$ sufficiently small, the ball of radius δ centered in x has a connected intersection with D (i.e. the set $B_\delta(x) \cap D$ is connected).

We will now describe three conditions that will constitute the hypothesis of the Theorem of Poincaré.

Definition. An *identification on a polyhedron* D is defined as follows. To each side S of D, one assigns another side S' of D and an isometry $T(S, S')$, under the following conditions:
(I1) $T(S, S') : \mathbb{H}^3 \to \mathbb{H}^3$, $S \mapsto S'$, sometimes indicated $(S)' := S'$, is an isometry,
(I2) $(S')' = S$, and $T(S', S) = (T(S, S'))^{-1}$,
(I3) $S = S' \Longrightarrow T(S, S)_{|S} = id$,
(I4) For all sides S of D, there exists a neighborhood $V_S \subset \mathbb{H}^3$ of S such that for $T := T(S, S')$, $T(V_S \cap D) \cap D = \emptyset$.

Consider the group $G :=< \{T(S, S') \mid S \text{ side of D}\} >$ generated by the isometries $T(S, S')$. Each isometry $T(S, S')$ is called a *generators* of G.
It follows by **(I3)** that $T(S, S)$ has order 2, i.e. $T(S, S)$ is a reflection with respect to the side S. The corresponding relations $(T(S, S))^2 = 1$ are called *reflection relations.*

Definition. Let D be a polyhedron with identification.
We identify the points of $\overline{D}$ modulo the action of G. In other words, if $x \in D$, then x is equivalent only to itself, and if $x \in \partial D$, then x is equivalent to all other points $x' \in \partial D$ such that there exists an element $T := T(S, S')$ of G with $T(x) = x'$. This relation is clearly an equivalence relation.
One can then form the *identified polyhedron* $D^* := \overline{D}/G$, with the usual quotient topology.

Let $p : \overline{D} \longrightarrow D^*$ the canonical projection of $\overline{D}$ onto $\overline{D}/G$. Since our aim is obtain a geometric object, it is natural to ask D^* to be a metric space. In other words, the action of G on D has to satisfy a certain properness condition, defined as follows.

Definition. A polyhedron D with identification is said to satisfy the *properness condition* $:\Longleftrightarrow$
(P) $\forall\, x \in D^*$, $p^{-1}(x)$ is a finite set (and then D^* is a metric space).

Considering the identification above, we can produce a list of edges which are identified in a successive order, the pairs of sides which are glued, and the isometries which are involved in the process.
Let D be a complete polyhedron with identification. Consider some edge e_1 of D, S_1 one of the two sides of D containing e_1.
Then one can produce the side S_1' corresponding to S_1 under the identification, and the corresponding generator $T(S_1, S_1') =: T_1$.
Set $e_2 := T_1(e_1)$. One can then obtain S_2, the unique other side of D containing e_2, the corresponding side S_2' and the corresponding isometry $T(S_2, S_2') =: T_2$.
Repetition of the process gives a sequence $\{e_i\}$ of edges of D, a sequence $\{(S_i, S_i')\}$ of pairs of sides, and a sequence $\{T_i\}$ of generators.
Notice that the 3 sequences need not to have the same period.

Definition. The *period of the identification* is the least common multiple m of the periods m_e, m_S and m_T of the sequences described above.
The set $\mathcal{E} := \{e_1, ..., e_m\}$ is called an *edge cycle.*

Remark. The edges in a cycle need not be pairwise distinct. In fact, there are repetitions if at least two out of the set of generators $\{T_1, ..., T_m\}$ are reflections (and in this case each edge shows up twice in the cycle).

Repeating the procedure for all edges of $\mathcal{D}$, we can produce j edge cycles $\mathcal{E}^j = \{e_1^j, ..., e_{m_j}^j\}$ of identified edges, each one related to a sequence of pairs of sides and a sequence of isometries.
In the following discussion, we sometimes omit the index j in the notations, and just consider any edge cycle $\mathcal{E}$ and the corresponding sequences $\{(S_i, S_i')\}$ of pairs of sides, and $\{T_i\}$ of generators.
Let us now consider the product of all isometries in a period.

Definition. The isometry $C := T_m \circ ... \circ T_1$, which satisfies $C(e_i) = e_i$ ($i = 1, ..., m$), is called *cycle transformation* at the edge cycle $\mathcal{E} = \{e_1, ..., e_m\}$.

The next condition we want the polyhedron D to satisfy is related with the total angle made by glued sides in a cycle of fixed orientation.

Let $\alpha(e_i)$ be the interior angle formed by the two sides meeting at e_i in D.

Definition. A polyhedron D with identification is said to *satisfy the cycle conditions* $:\Longleftrightarrow$
(CC1) For all edge cycles $\mathcal{E}^j = \{e_1^j, ..., e_{m_j}^j\}$, there exists an integer $\nu_j \in \mathbb{N}_+$ such that $\sum_{i=1}^{m_j} \alpha(e_i^j) = \frac{2\pi}{\nu_j}$.
(CC2) For each edge cycle $\mathcal{E}$, the cycle transformation C at $\mathcal{E}$ preserves the natural orientation of $D \subset \mathbb{H}^3$.
(CC3) If the edge e_i has no finite endpoint, then the cycle transformation C at $\mathcal{E}$ is the identity on e_i.

As a consequence of this condition, we have the following result.

Proposition 1.22. *The cycle conditions **(CC1)**, **(CC2)**, and **(CC3)** imply that each C is orientation preserving, and that $C^\nu = 1$.*

Definition. For each cycle, the relation $C^\nu = 1$ is called *cycle relation*.

The last condition that has to be verified by the polyhedron D to serve as model for a complete manifold is obviously a condition of completeness. This condition is technical, but helps to avoid "pathological" situations.

Definition. A polyhedron D with identification is called *complete* $:\Longleftrightarrow$
(CP) D^* is complete.

The case of a finite-sided polyhedron of finite volume with some *ideal vertices* (i.e. vertices on the sphere at infinity) is of special interest. There, the condition **(CP)** can be hard to check directly. However, there are several ways to check it indirectly (see e.g. [Ma2] p. 79, [Vi] p. 164). Here we follow Ratcliffe ([Ra], p. 440).
Notice that the finite-sided polyhedron D might be decomposed as disjoint union of simpler polyhedra.

Definition. Denote $P_x \subset D$ the polyhedron in D which contains the ideal vertex x.
The *link* of the ideal vertex x is the set $L(x) := P_x \cap \mathbb{C}_t(x)$, where $\mathbb{C}_{t_x}$ denotes the horosphere $\mathbb{C}_t$ with $t > 0$ sufficiently large such that $\mathbb{C}_t$ intersects no other polyhedron of D than P_x.
It is clear that $L(x)$ is a compact euclidean polygon in $\mathbb{C}_t(x)$.

The side-pairing on D induces a pairing of its ideal vertices, and consequently an equivalence relation over them. Let us call the equivalence classes of this relation *cusp points* of D^*. For each cusp point $[x]$, one can form the set $\{L(y) \mid y \in [x]\}$ of euclidean polygons related to $[x]$.
For an element T of G, we consider elements $y, y' \in [x]$ such that $T(y') = y$. Then, for sides $S \ni y$ and $S' \ni y'$ of D, the intersections $\mathbb{C}_{t_y} \cap S$ and $C_{t_{y'}} \cap S'$ are sides of $L(y)$, resp. $L(y')$.
Thus the restriction of T on $\mathbb{C}_{t_{y'}}$ is an isometry sending $\mathbb{C}_{t_{y'}}$ to $T(\mathbb{C}_{t_{y'}})$ and preserving the euclidean metric on the horospheres. Up to a change of scale, T sends the side $\mathbb{C}_{t_{y'}} \cap S'$ of $L(y')$ to the side $\mathbb{C}_{t_y} \cap S$ of $L(y)$. It is clear that the set τ of all such (possibly rescaled) transformations T induces a side-pairing of the polygons of $\{L(y) \mid y \in [x]\}$.

Definition. Let $[x]$ be a cusp point of D^*. We define $L[x]$ as the space obtained by identifying the edges of the polygons of $\{L(y) \mid y \in [x]\}$ under the action of τ.

At this point, we are able to state the following propositions, which will allow us to prove completeness indirectly.

Proposition 1.23. *Let x be an ideal vertex of D.*
Then, $L[x]$ is complete if and only if the set $\{L(y) \mid y \in [x]\}$ can be chosen such that the action of G restricts to a side-pairing on $\{L(y) \mid y \in [x]\}$.

Notice that the "right-left" direction is obivous from previous discussion.

Proposition 1.24. *The metric space D^* is complete if and only if $L[x]$ is complete for each cusp point $[x]$ of D^*.*

Remark. One can see that in the oriented case, $L[x]$ is a torus for all cusp points $[x]$.

We now consider polyhedra verifying all conditions mentionned above.

Definition. A complete polyhedron with identification satisfying the properness condition **(P)** and the cycle conditions **(CC1)**, **(CC2)**, and **(CC3)** is called *Poincaré polyhedron.*

We finally come to the main theorem this section.

Theorem 1.7 (The Poincaré Polyhedron Theorem)**.** *Let D be a Poincaré polyhedron, and let $\Gamma = \Gamma_D$ be the group generated by the identifications T of its sides.*
Then,

1. *Γ acts properly discontinuously on $\mathbb{H}^3$,*
2. *D is a fundamental polyhedron for Γ, and*

3. *the cycle relations together with the reflection relations form a complete set of relations for* Γ.

In summary, we have a tool that will allow us to find a Kleinian group directly from an identification constructed from a certain polyhedron.

An application of the present material is the following. If we find an identified polyhedron $D^* = \pi(\overline{D})$ such that D is a Poincaré polyhedron, then we have an explicit Kleinian group Γ_D, given by a geometric presentation, and acting on $\mathbb{H}^3$ with fundamental polyhedron D, such that $M = \mathbb{H}^3/\Gamma_D$ is a hyperbolic manifold.
This connection will be explained in the next section.

Remark. There are several versions of Poincaré's Polyhedron Theorem, depending on the ambiant space and the configuration we are looking for. The difficulties encountered in the hyperbolic case with ideal vertices don't appear in the euclidean or spherical cases.
Epstein and Petronio have given in [EP] a (consequent) survey of the theorem, compatible with an algorithmic approach.

1.6 Hyperbolic 3-Manifolds

It is now time to give a formal definition of oriented hyperbolic 3-manifolds, and to explain why Kleinian groups are so important in their study. References for this section are the books of Benedetti & Petronio [BP], Matsuzaki & Taniguchi [MaTa] and Ratcliffe [Ra].

Definition. A connected Hausdorff space M is called an oriented *hyperbolic* 3-*manifold* if there exists a family (or *atlas*) $\{(U_j, \varphi_j)\}_{j\in J}$, J an index set for M, such that:

1. each U_j is an open subset of M, and $\{U_j\}_{j\in J}$ is a covering of M,
2. each φ_j is a homeomorphism of U_j onto $\mathbb{H}^3$, and
3. each non-empty intersection $U_k \cap U_j$ is connected, and $\varphi_k \circ \varphi_j^{-1}$: $\varphi_j(U_k \cap U_j) \longrightarrow \varphi_k(U_k \cap U_j)$ is an orientation-preserving diffeomorphism which preserves the hyperbolic metric.

In other words, we ask a hyperbolic manifold to "locally look like" the hyperbolic space $\mathbb{H}^3$.

Proposition 1.25. *An orientation-preserving isometric homeomorphism from a domain* D *of* $\mathbb{H}^3$ *into* $\mathbb{H}^3$ *is necessarily the restriction of an element of* $Isom^+(\mathbb{H}^3)$ *to* D.

As a consequence, the chart changes in the definition of a hyperbolic manifolds turn out to be nothing else than the restriction of Möbius transformations on M.
Now we come to the relations with Kleinian groups.

Definition. Let X be a locally compact Hausdorff space, and G a group of homeomorphisms of X.
G operates *freely* on $X :\Longleftrightarrow (\forall x \in X, g \in G : g(x) = x \Rightarrow g = id)$.

Remark. Observe that $\mathbb{H}^3$ is a locally compact Hausdorff space.

In order to give nice properties of groups acting freely and properly discontinuously on geometric spaces, we need the following definition.

Definition. A group G is said to be *torsion-free* $:\Longleftrightarrow$ there are no elements of finite order in $G \setminus \{e\}$.

Proposition 1.26. *Let $X \in \{\mathbb{E}^3, \mathbb{H}^3, \mathbb{S}^3\}$, and let G be a group of orientation-preserving isometries of X.*
Then the following conditions are equivalent:

1. *G operates freely and properly discontinuously on X.*
2. *G is a torsion-free discrete subgroup of $Isom^+(X)$.*

The importance of Kleinian groups becomes now clearer. But we have even more, as the following theorem states.

Theorem 1.8. *Let X be either $\mathbb{E}^3$ or $\mathbb{H}^3$.*
Then, M is a complete connected euclidean or hyperbolic oriented 3-manifold $\Longleftrightarrow$ the fundamental group $\pi_1(M)$ is a discrete torsion-free subgroup of $Isom^+(X)$, and $M \cong X/\pi_1(M)$.

Let us now concentrate on $X = \mathbb{H}^3$ with a fixed orientation.

Theorem 1.9. *For any complete oriented hyperbolic 3-manifold M, there exists a torsion-free Kleinian group Γ such that $M = \mathbb{H}^3/\Gamma$. The group Γ is unique up to conjugation by elements of $Isom^+(\mathbb{H}^3)$.*
Conversely, for any torsion-free Kleinian group Γ, $M = \mathbb{H}^3/\Gamma$ is a complete oriented hyperbolic 3-manifold.

Let us use the notations of section 1.5.2. The theorem we have just stated is very helpful in our context: the completeness condition will be clearly verified if we apply the theorem of Poincaré to a complete polyhedron D leading to $D^* \cong \mathbb{H}^3/\Gamma$ for a certain Kleinian group Γ. We have even more. Because of uniqueness up to conjugation, Γ can in fact be identified with the fundamental group $\pi_1(D^*)$.

The strategy we will follow to realize an explicit hyperbolic structure on a knot complement is now clear: given a knot K, we compute the fundamental group $\pi_1(\mathbb{R}^3 \setminus K)$ of the complement, and represent $\pi_1(\mathbb{R}^3 \setminus K)$ in $PSL(2,\mathbb{C})$. If this group $\Gamma < PSL(2,\mathbb{C})$ is discrete and torsion-free, then we produce a hyperbolic 3-manifold $\mathbb{H}^3/\Gamma$ with fundamental group $\pi_1(\mathbb{H}^3/\Gamma) \cong \pi_1(\mathbb{R}^3 \setminus K)$. For this, we make use of Poincaré's Polyhedron Theorem by considering a polyhedron D adapted to $\pi_1(K)$.

1.7 Waldhausen's Theorem

We have seen in section 1.1 that an isomorphism of fundamental groups of manifolds does not imply the existence of a homeomorphism between the manifolds. At the very end of the proof, we will need some topological arguments related to a result of Waldhausen giving sufficient conditions for this statement to hold. References for this section are [Ha], [Ja] (especially Chapters I, III, VII), [Mn], [Th1] (especially section 4.10 p. 71), and [Wa].

We begin with some definitions. In the sequel M will denote an orientable, connected, compact 3-manifold (non necessarily hyperbolic).

Definition. A *surface* is a connected 2-manifold. In the sequel, a *surface F in the manifold M* is supposed to be compact and properly embedded, i.e. $F \cap \partial M = \partial F$.
A *surface in ∂M* is a surface in ∂M.
A *system of surfaces* in M or ∂M consists of finitely many, mutually disjoint surfaces in M or in ∂M.

Definition. A system $\mathcal{F}$ of surfaces in M or ∂M is *compressible* (in M) if $\mathcal{F}$ satisfies at least one of the following conditions.

- There exists a non-contractible simple closed curve γ in $\mathring{F}$, $F \in \mathcal{F}$, and a disc $D \subset M$ such that $\mathring{D} \subset \mathring{M}$ and $D \cap \mathcal{F} = \partial D = \gamma$.
- There exists a ball $B \subset M$ such that $B \cap \mathcal{F} = \partial B$.

If F is not compressible (in M), then F is called *incompressible* (in M)

Definition. The manifold M is called *irreducible* if every 2-sphere in M is compressible.
The manifold M is called *boundary-irreducible* if ∂M is incompressible.

Definition. Let M be irreducible and not a ball. If M contains an incompressible surface, then M is called *sufficiently large.*

Remark. It is hard to find explicit examples of irreducible 3-manifolds which are not sufficiently large. Observe that if M is a compact oriented 3-manifold which is not a ball and which has non-trivial boundary, then M is sufficiently large (CF. [Th1], p. 71-72)

A further condition deals with fundamental groups. We introduce the following definition.

Definition. Let M and N be 3-manifolds, and $\psi : \pi_1(N) \longrightarrow \pi_1(M)$ a homomorphism.
ψ is said to *respect the peripheral structure* if for each boundary surface F of N, there exists a boundary surface G of M such that $\psi(i_*^N(\pi_1(F)))$ is contained in some subgroup $A < \pi_1(M)$, with A conjugate to $i_*^M(\pi_1(G))$ in $\pi_1(M)$. Here, $i_*^N : \pi_1(F) \to \pi_1(N)$ and $i_*^M : \pi_1(G) \to \pi_1(M)$ are the inclusions.

In the case of knots k, the situation is simplified by the fact that the only boundary surface of $\mathbb{S}^3 \setminus T_k$ is a knotted torus.

We now come to the result of Waldhausen. For a proof, see [Wa], p.80.

Theorem 1.10. *(Waldhausen) Suppose M and N are oriented 3-manifolds which are irreducible and boundary-irreducible such that M is sufficiently large.*
If there exists an isomorphism $\psi : \pi_1(N) \longrightarrow \pi_1(M)$ which respects the peripheral structure, then there exists a homeomorphism $f : N \longrightarrow M$ which induces ψ.

Remark. A compact 3-manifold M which is irreducible, boundary-irreducible and sufficiently large is called a *Haken manifold*.

The following result (which admits several formulations) is related to our context.

Theorem 1.11. *(Dehn's Lemma) Let M be a 3-manifold with boundary and let γ be a closed curve in ∂M.*
If there exists an immersed disc D in M such that $\partial D = \gamma$, then there exists an embedded disc $D' \subset M$ with the same boundary $\partial D' = \gamma$.

Finally, we add two statements which will be useful later. For the formulation of the first one, we need additional definitions.

Definition. Let X be a (path-connected) topological space, and $i \in \mathbb{N}^*$.
The *$i-th$ homotopy group* of X, denoted $\pi_i(X)$, is the set of homotopy equivalence classes of continuous maps $S^i \to X$.

Remark. This general definition is compatible with the definition of $\pi_1(X)$ given in section 1.1.

Definition. Let X be a (path-connected) topological space. If for $i > 1$ one has $\pi_i(X) = 0$, then X is called *aspherical*.

The following proposition can be found in [Ha] (p.342).

Proposition 1.27. *Let X a topological space and $\widetilde{X}$ the universal cover of X.*
Then, $\pi_i(\widetilde{X}) \cong \pi_i(X)$ for all $i > 1$.

Aspherical manifolds have the following nice property (CF. [Lu]).

Proposition 1.28. *Let M be a closed oriented 3-manifold.*
Then, M is aspherical if and only if M is irreducible and $\pi_1(M)$ is torsion free.

These definitions and results are complicated, but as we will see in Section 2.4, in our concrete situation there will be some simplifications.

Chapter 2

The Hyperbolization of the Figure-Eight Complement

We now have enough material to deal with the main subject of this work. As said in the Introduction, Riley's work [Ri1] is the first example of the computation of an explicit hyperbolic structure on a knot complement. In this chapter, we will give a detailed explanation of the process in [Ri1], using only material that has been developed in Chapter 1, except for the very end of the proof. There, we only give the outlines of the topological argumentation of Riley, based on the work of Waldhausen [Wa] and Armstrong [Ar].

2.1 The Figure-Eight Knot Group

The particular knot complement studied by Riley is the complement of the so-called "figure-eight" knot, defined as follows.

Definition. Consider the knot depicts in following figure.

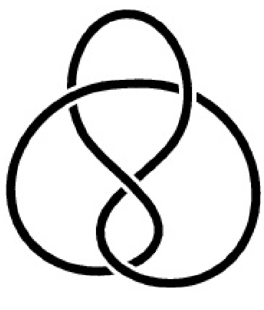

Figure 2.1: The Figure-Eight

This knot is called "*figure-eight*", sometimes denoted by 4_1.
In the following, we will denote it simply by K.

Remark. The watchful reader has probably noticed that the Figure 1.1 in Section 1.3 depicts the figure-eight.

Using Wirtinger's Theorem 1.4, we now compute the fundamental group $\pi_1(K)$.
We first orient K and label the arcs and the crossings of K as follows.

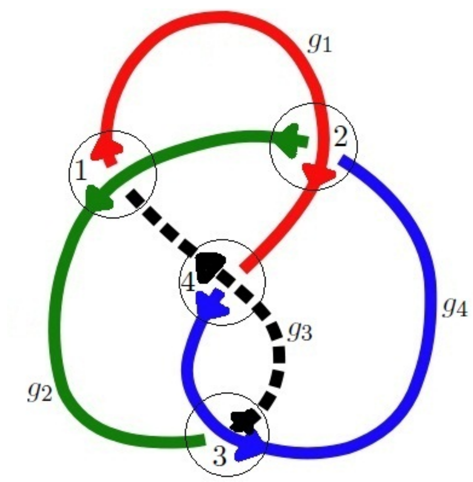

Figure 2.2: Labelled arcs and crossings of K

The relation r_1 at crossing 1 is $g_2 g_3 g_2^{-1} = g_1$ and the relation r_2 at crossing 2 is $g_1 g_4 g_1^{-1} = g_2$ since both crossings are negative crossings.
The relation r_3 at crossing 3 is $g_4^{-1} g_2 g_4 = g_3$ and the relation r_4 at crossing 4 is $g_3^{-1} g_1 g_3 = g_4$ since both crossings are positive crossings.
Then, Theorem 1.4 implies that

$$\pi_1(K) =< g_1, g_2, g_3, g_4 \mid r_1, r_2, r_3, r_4 > .$$

We set $g_1 =: a$, $g_2 =: b$, $g_3 =: c$, $g_4 =: d$ and rewrite the relations r_i, $i = 1, ..., 4$, in order to make the reading easier. Then,

$$\pi_1(K) = \left\langle a, b, c, d \;\middle|\; \begin{array}{ll} bcb^{-1} = a & d^{-1}bd = c \\ ada^{-1} = b & c^{-1}ac = d \end{array} \right\rangle .$$

We now reduce the redundant generators and relations, in order to have a reduced presentation.
The relation r_1 is equivalent to the relation $\widetilde{r_1}:\ c = b^{-1}ab$, and from relation r_2 one gets the relation $\widetilde{r_2} : d = a^{-1}ba$.

We can use $\widetilde{r_1}$ and $\widetilde{r_2}$ in relations r_3 and r_4 to get $\widetilde{r_3}:\ (a^{-1}ba)^{-1}b(a^{-1}ba) = b^{-1}ab$ and $\widetilde{r_4}:\ (b^{-1}ab)^{-1}a(b^{-1}ab) = a^{-1}ba$.
Furthermore, $\widetilde{r_3}$ reduces to

$$a^{-1}b^{-1}aba^{-1}ba = b^{-1}\underline{a}b,$$

and $\widetilde{r_4}$ to

$$b^{-1}a^{-1}bab^{-1}\underline{a}b = a^{-1}ba.$$

Isolating the underlined a in $\widetilde{r_3}$ and $\widetilde{r_4}$, we get $r'_3:\ a = ba^{-1}b^{-1}aba^{-1}bab^{-1}$ and $r'_4:\ a = ba^{-1}b^{-1}aba^{-1}bab^{-1}$, and observe that r'_3 is the same relation as r'_4.

Thus, eliminating the redundant generators c and d and the redundant relations, we get that a and b suffice to generate $\pi_1(K)$, and that the single relation is

$$r'_3:\ a = b\underline{a^{-1}}b^{-1}aba^{-1}bab^{-1}.$$

We now make the following change, which will be helpful in the sequel. In the relation r'_3, isolate the underlined a^{-1}. Then r'_3 becomes

$$b^{-1}aba^{-1}b^{-1}ab^{-1}a^{-1}b = a^{-1},$$

which is equivalent to

$$a = b^{-1}aba^{-1}bab^{-1}a^{-1}b.$$

Finally, we have the following presentation for $\pi_1(K)$:

$$\pi_1(K) =< a, b \mid a = \underbrace{(b^{-1}aba^{-1})}_{=:w}\, b\, \underbrace{(ab^{-1}a^{-1}b)}_{=w^{-1}} > \qquad (*)$$

2.2 The Polyhedron $\mathcal{D}$

As we have said at the end of Chapter 1, the goal now is to find a hyperbolic manifold whose fundamental group is isomorphic to $\pi_1(K)$. The main tool that will be used is Poincaré's Theorem 1.7.
First of all, we are going to construct a polyhedron $\mathcal{D}$ which will be our candidate to apply Poincaré's Theorem.

The only piece of information we have is that the fundamental group $\Gamma_{\mathcal{D}}$ we are supposed to get from Poincaré's Theorem has to be isomorphic to $\pi_1(K)$.
Thus, we are going to produce a representation θ of $\pi_1(K)$ in $PSL(2,\mathbb{C})$, and construct the Ford domain associated to the image $\theta(\pi_1(K))$, which will be our polyhedron $\mathcal{D}$. Then, we will show that the image $\theta(\pi_1(K))$ is isomorphic to $\pi_1(K)$.

2.2.1 Representation of $\pi_1(K)$ in $PSL(2,\mathbb{C})$

Let $\omega := -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. It is not hard to check that ω is a primitive cube root of the unity, i.e. ω is such that $|\omega| \overset{(o1)}{=} 1$, $\omega^3 \overset{(o2)}{=} 1$ and $\omega^2 + \omega + 1 \overset{(o3)}{=} 0$.

We define the homomorphism $\theta:\ \pi_1(K) \longrightarrow PSL(2,\mathbb{C})$ as follows: for the generators a and b of $\pi_1(K)$, we define

$$\theta(a) := A = \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix}, \text{ and } \theta(b) := B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and extend θ to $\pi_1(K)$ in the natural way. Notice that both A and B are parabolic (CF. Proposition 1.11). Put

$$\theta(w) =: W = B^{-1}ABA^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix},$$

according to $(*)$. A short computation using property $(o1)$ shows that

$$W = \begin{pmatrix} 0 & \omega \\ -\omega^2 & 1-\omega \end{pmatrix} \text{ and that } W^{-1} = \begin{pmatrix} 1-\omega & -\omega \\ \omega^2 & 0 \end{pmatrix}.$$

Another computation using property $(o2)$ shows that $WBW^{-1} = A$.

Thus, $\theta(wbw^{-1}) = \theta(a)$, from which we deduce that

$$\theta:\ \pi_1(K) \to < A, B \mid WBW^{-1} = A >$$

is a surjective homomorphism (the injectivity will only be proved at the end of chapter 2).

Notation. Let $\mathcal{K} :=< A, B \mid WBW^{-1} = A > < PSL(2,\mathbb{C})$.

2.2.2 Related Geometry

The transformation B corresponds to the translation $z \mapsto z+1$, which yields a hyperbolic isometry of $\mathcal{U}^3$ fixing ∞, and has therefore no isometric circle.

Furthermore, we know from section 1.4.3 that $X = A$, A^{-1}, W, and W^{-1} have isometric circles $I_0(X)$ in $\mathbb{C}$.
By the properties $(o1)$, $(o2)$ and $(o3)$, we compute their centers and radii in $\mathbb{C}$ as follows.

- $I_0(A)$ has center $c_A := \frac{1}{\omega} = \omega^2 = -1-\omega = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$, and radius $\frac{1}{|-\omega|} = 1$.
- $I_0(A^{-1})$ has center $c_{A^{-1}} := -\frac{1}{\omega} = 1+\omega = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, and radius $\frac{1}{|\omega|} = 1$.

- $I_0(W)$ has center $c_W := \frac{\omega-1}{-\omega} = -\frac{1}{\omega} + \frac{1}{\omega^2} = -\omega^2 + \omega = 1 + 2\omega = \sqrt{3}i$, and radius $\frac{1}{|-\omega^2|} = \frac{1}{|-\frac{1}{\omega}|} = |\omega| = 1$.

- $I_0(W^{-1})$ has center $c_{W^{-1}} := 0$, and radius $\frac{1}{|\omega^2|} = 1$.

Hence, the isometric spheres $I(A)$, $I(A^{-1})$, $I(W)$ and $I(W^{-1})$ are the half-spheres of respective centers $(c_A, 0)$, $(c_{A^{-1}}, 0)$, $(c_W, 0)$ and $(c_{W^{-1}}, 0)$, and have all radius 1 in $\mathbb{C}_0 \subset \mathcal{U}^3$.

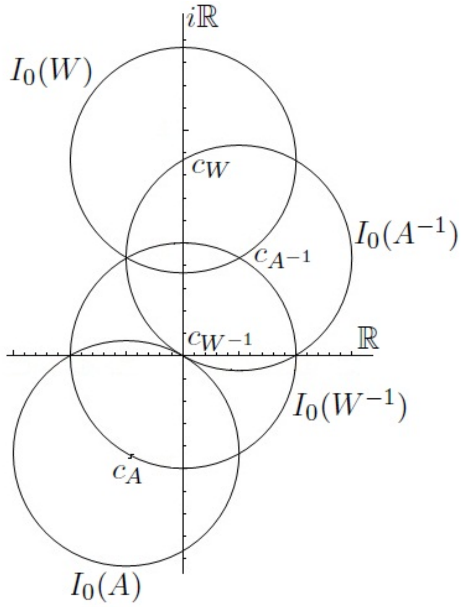

Figure 2.3: The isometric spheres of A, A^{-1}, W and W^{-1}

In order to describe the isometric spheres of all elements of $\mathcal{K}$, we first introduce the following notation and state the following proposition.

Notation. Let $a, b, x, y \in \mathbb{Z}[\omega]$. We write $a \equiv b \mod (x, y)$ if there exists $m, n \in \mathbb{Z}$ such that $a = b + mx + ny$.

Proposition 2.1. *Let $\alpha \in \mathbb{Z}[\omega]$.*
Then, α is congruent to either c_A, $c_{A^{-1}}$, c_W or $c_{W^{-1}}$ mod $(1, 2 + 4\omega)$.

Proof. Let $\alpha_n = a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + ... + a_n\omega^n \in \mathbb{Z}[\omega] \subset \mathbb{C}$, with $a_i \in \mathbb{Z}$, $i \in \mathbb{N}$, and $n \in \mathbb{N}$ fixed. We suppose without loss of generality that $\frac{n}{3} \in \mathbb{N}$.
Then property $(o2)$ implies that $\alpha_n = (a_0 + a_3 + ... + a_{n-3} + a_n) + (a_1 + a_4 + ... + a_{n-2})\,\omega + (a_2 + a_5 + ... + a_{n-1})\,\omega^2$.
From property $(o3)$ one deduces $\omega^2 = -1 - \omega$, and setting $a^{(0)} := a_0 + ... + a_n$,

$a^{(1)} := a_2 + ... + a_{n-2}$, and $a^{(2)} := a_1 + ... + a_{n-1}$, one gets

$$\alpha_n = (a^{(0)} - a^{(1)}) + (a^{(2)} - a^{(1)})\,\omega.$$

Since $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, one can write $\alpha_n = Re(\alpha_n) + i\,Im(\alpha_n)$ as follows.

$$\alpha_n = a^{(0)} - a^{(1)} - \frac{1}{2}(a^{(2)} - a^{(1)}) + i\,\frac{\sqrt{3}}{2}\,(a^{(2)} - a^{(1)}),$$

with

$$Re(\alpha_n) = a^{(0)} - a^{(1)} - \frac{1}{2}(a^{(2)} - a^{(1)}), \text{ and } Im(\alpha_n) = \frac{\sqrt{3}}{2}\,(a^{(2)} - a^{(1)}).$$

We now distinguish 2 cases, depending on $Re(\alpha_n)$ being an integer or not.

- First case: $Re(\alpha_n) \in \mathbb{Z}$: Then $-\frac{1}{2}(a^{(1)} + a^{(2)}) \in \mathbb{Z}$, and obviously $a^{(1)} + a^{(2)} \equiv 0 \pmod 2$.
  Furthermore, since $a^{(1)} \in \mathbb{Z}$, $Im(\alpha_n) = \sqrt{3}\,(\frac{1}{2}a^{(1)} - \frac{1}{2}a^{(2)}) = \sqrt{3}\,(a^{(1)} - \frac{1}{2}(a^{(1)} + a^{(2)}))$ implies $Im(\alpha_n) \in \mathbb{Z}\sqrt{3}$.
  As a consequence, we deduce that either $Im(\alpha_n) \equiv 0 \pmod{2\sqrt{3}}$, or $Im(\alpha_n) \equiv \sqrt{3} \pmod{2\sqrt{3}}$, depending on $\frac{1}{2}a^{(1)} - \frac{1}{2}a^{(2)}$ being even or not.
  Thus, if $Re(\alpha_n) \in \mathbb{Z}$, then α_n is congruent to either c_W or $c_{W^{-1}}$ mod $(1, 2 + 4\omega)$.

- Second case: $Re(\alpha_n) \notin \mathbb{Z}$: Then $Re(\alpha_n) + \frac{1}{2} \in \mathbb{Z}$, and $Re(\alpha_n) \equiv \frac{1}{2} \pmod 1$ (which is equivalent to $Re(\alpha_n) \equiv -\frac{1}{2} \pmod 1$).
  From $Re(\alpha_n) + \frac{1}{2} \in \mathbb{Z}$ one deduces that $a^{(1)} + a^{(2)} - 1$ is even. Using this fact, we write $\frac{1}{2}a^{(1)} - \frac{1}{2}a^{(2)} = \frac{1}{2}(a^{(1)} + a^{(2)} - 1) - a^{(2)} + \frac{1}{2} =: \lambda + \frac{1}{2}$, for $\lambda := \frac{1}{2}(a^{(1)} + a^{(2)} - 1) - a^{(2)} \in \mathbb{Z}$.
  With this notation, $-Im(\alpha) = \sqrt{3}\,(\frac{1}{2}a^{(1)} - \frac{1}{2}a^{(2)})$ becomes $\sqrt{3}\,\lambda + \frac{\sqrt{3}}{2}$, $\lambda \in \mathbb{Z}$.
  We now distinguish the cases λ even and λ odd:
  - if λ is even, then $\sqrt{3}\,\lambda \equiv 0 \pmod{2\sqrt{3}} \Leftrightarrow \sqrt{3}\,\lambda + \frac{\sqrt{3}}{2} \equiv \frac{\sqrt{3}}{2} \pmod{2\sqrt{3}}$.
  - if λ is odd, then $\sqrt{3}\,\lambda \equiv -\sqrt{3} \pmod{2\sqrt{3}} \Leftrightarrow \sqrt{3}\,\lambda + \frac{\sqrt{3}}{2} \equiv -\frac{\sqrt{3}}{2} \pmod{2\sqrt{3}}$.

  Thus, if $Re(\alpha_n) \notin \mathbb{Z}$, then α_n is congruent to either c_A or $c_{A^{-1}}$ mod $(1, 2 + 4\omega)$.

In summary, we have seen that $\alpha_n = a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + ... + a_n\omega^n$, with $a_i \in \mathbb{Z}$, $i \in \mathbb{N}$ is congruent to either c_A, $c_{A^{-1}}$, c_W or $c_{W^{-1}}$ mod $(1, 2 + 4\omega)$

for all $n \in \mathbb{N}$ fixed, which achieves the proof. ■

As a consequence, each $\alpha \in \mathbb{Z}[\omega]$ is the center of an isometric sphere of an element of $\mathcal{K}$ with radius 1.

Definition. For $t \in \mathbb{C}$ fixed, we define the transformation $[B(t)] \in PSL(2, \mathbb{C})$ by its representative $B(t) := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. $B(t)$ corresponds to the translation $z \mapsto z + t$ in $\mathbb{C}$.

In particular, $B(2+4\omega)$ depicts the translation $z \mapsto z + 2\sqrt{3}\,i$ in the perpendicular direction of the translation induced by $B(1) = B$.
Using these notations, we can state following proposition which is in fact a corollary of the proof of Proposition 2.1.

Proposition 2.2. *Let* $\mathcal{L} := \{$*half-spheres* $S_1(\alpha)$ *in* $\mathcal{U}^3$*, with center* $\alpha \in \mathbb{Z}[\omega]$ *and radius* $1\}$.
Then, $\mathcal{L}$ *is a lattice of half-spheres in* $\mathcal{U}^3$ *which is stable under the action of the* 2*-generator group* $< B,\, B(2+4\omega) >$.

We now look at the geometric situation of a half-sphere of $\mathcal{L}$.

Proposition 2.3. *Each half-sphere of* $\mathcal{L}$ *meets 6 other half-spheres of* $\mathcal{L}$ *along the edges of a regular hyperbolic hexagon, with intersection angle* $\frac{2\pi}{3}$.

Proof. Let $\alpha \in \mathbb{Z}[\omega]$ be the center of a half-sphere $S_1(\alpha) \in \mathcal{L}$.
Then $\alpha+1$, $\alpha-1$, $\alpha+\omega$, $\alpha-\omega$, $\alpha+\omega^2 = \alpha-\omega-1$ and $\alpha-\omega^2 = \alpha+\omega+1$ are on the circle $\mathcal{C}_1(\alpha) \subset \mathbb{C}$ of radius 1 centered at α.
Because of the definition of ω and Proposition 2.1, there is no other element of $\mathbb{Z}[\omega]$ on $\mathcal{C}_1(\alpha)$, and no element of $\mathbb{Z}[\omega] \setminus \{\alpha\}$ inside the disc of $\mathbb{C}$ bounded by $\mathcal{C}_1(\alpha)$.
Thus, each half-sphere of $\mathcal{L}$ meets exactly 6 other half-spheres of $\mathcal{L}$.

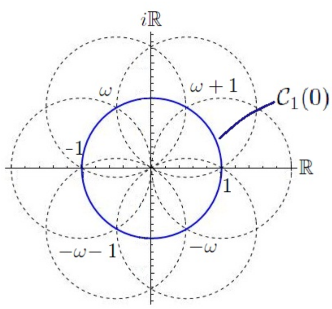

Figure 2.4: Intersecting half-spheres of $\mathcal{L}$

For the following, we suppose without loss of generality that $\alpha = 0$.
We first observe that the centers 1, -1, ω, $\omega+1$, $-\omega$ and $-\omega-1$ of the half-spheres intersecting the half-sphere $S_1(0)$ centered at 0 and with radius 1 are the vertices of a regular euclidean hexagon in $\mathbb{C}$.
Each intersecting half-sphere can be obtained as the rotation of the half-sphere $S_1(1)$ of radius 1 centered in 1 of an angle $k \cdot \frac{\pi}{3}$, for $k = 1, ..., 5$, and because all spheres have the same radius as $S_1(0)$ and $S_1(1)$, their intersection with $S_1(0)$ is the image of the intersection $S_1(0) \cap S_1(1)$ under a rotation of the same angle, as indicated above.
Since the intersection $S_1(0) \cap S_1(1)$ consists of a vertical half-circle centered at $\frac{1}{2}$ and with radius $\frac{\sqrt{3}}{2}$ (short computation), one deduces that the boundary of the region $S_1(0) \setminus (S_1(1) \cup S_1(\omega+1) \cup S_1(\omega) \cup S_1(-1) \cup S_1(-\omega-1) \cup S_1(-\omega))$ is a union of circle arcs projecting onto a regular euclidean hexagon in $\mathbb{C}$.
We remind that geodesics of $\mathcal{U}^3$ are vertical lines and vertical half-circles centered in $\mathbb{C}$ in order to deduce that the intersection $S_1(0) \cap S_1(1) \cap S_1(\omega) \cap S_1(\omega+1) \cap S_1(-1) \cap S_1(-\omega-1) \cap S_1(-\omega)$ is a regular hyperbolic hexagon.

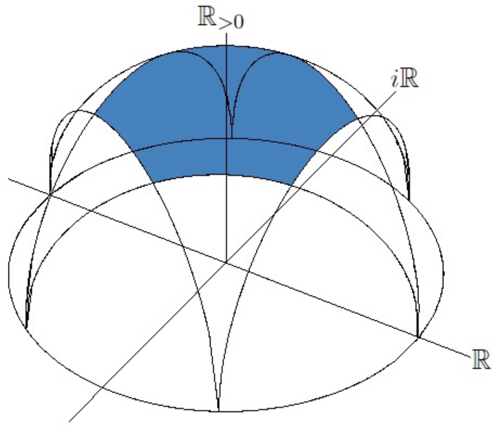

Figure 2.5: The hyperbolic hexagon on $S_1(0)$

Recall that the upper half-space model $(\mathcal{U}^3, d_{\mathcal{U}})$ is conformal. In particular, one gets that the interior angles of the hyperbolic hexagon are all $\frac{2\pi}{3}$.
Now, we compute the intersection angles between intersecting half-spheres. This time, we only look at the intersection $S_1(0) \cap \{(z,t) \in \mathcal{U}^3 \,|\, Re(z) = \frac{1}{2}\}$, and compute the half angle.
For the point $q := (\frac{1}{2}, 0, \frac{\sqrt{3}}{2}) \in S_1(0) \cap \{z \in \mathbb{C} \,|\, Re(z) = \frac{1}{2}\}$, we have the following situation by considering $\mathcal{U}^3$ as $\mathbb{R}^2 \times \mathbb{R}_{>0}$ with its euclidean structure: the unitary vector $v_S = (-\frac{\sqrt{3}}{2}, 0, \frac{1}{2})$ is tangent to the half-sphere in q, and the unitary vector $v_P := (0,0,1)$ is a unitary vertical vector in $\mathbb{R}^2 \times \mathbb{R}_{>0}$.
Then, the intersection angle between $S_1(0)$ and $S_1(1)$ is twice the angle φ

between the vectors v_S and v_P (remember that the intersection $S_1(0) \cap S_1(1)$ is a half-circle in the vertical plane $\{(z,t) \in \mathcal{U}^3 \mid Re(z) = \frac{1}{2}\}$).

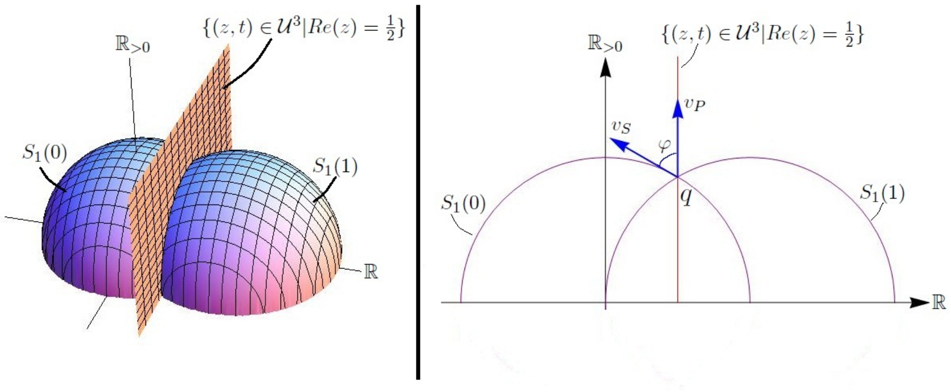

Figure 2.6: The angle φ

We know that $\varphi \in [0, \pi]$ is given by the formula $\varphi = arccos\left(\frac{<v_S, v_P>}{||v_S|| \, ||v_P||}\right) = arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$, with respect to the standard scalar product.
Thus, the intersection angles between intersecting half-spheres are all equal to $\frac{2\pi}{3}$, and we have proved the last part of the assertion. ■

2.2.3 Definition of $\mathcal{D}_\infty$

We construct now a polyhedron $\mathcal{D}$ which will be the candidate for a fundamental domain of $\mathcal{K} < PSL(2, \mathbb{C})$ to apply Poincaré's Polyhedron Theorem 1.7.
Observe that $\mathcal{K}$ contains the matrix B fixing ∞. Therefore, it is natural to study first a horospherical neighbourhood of ∞, and to begin by defining a certain euclidean polygon $\mathcal{D}_\infty$.

Definition. Let $\mathcal{D}_\infty \subset \mathbb{C}$ be the interior of the closure of the union of the euclidean projections on $\mathbb{C}$ of the regular hexagons on the isometric spheres $I(A)$, $I(A^{-1})$, $I(W)$ and $I(W^{-1})$ described in Proposition 2.3. (CF. Figure 2.7, left)

Proposition 2.4. *The region $\mathcal{D}_\infty$ is a fundamental polygon for the group $< B,\, B(2+4\omega) > < Isom^+(\mathbb{C})$.*

<u>*Proof.*</u> First of all, $< B,\, B(2+4\omega) >$ is a subgroup of $Isom^+(\mathbb{C})$, since it is generated by two translations of $\mathbb{C}$ (CF. Section 1.5.1).
We now check the three points of the definition of a fundamental domain.

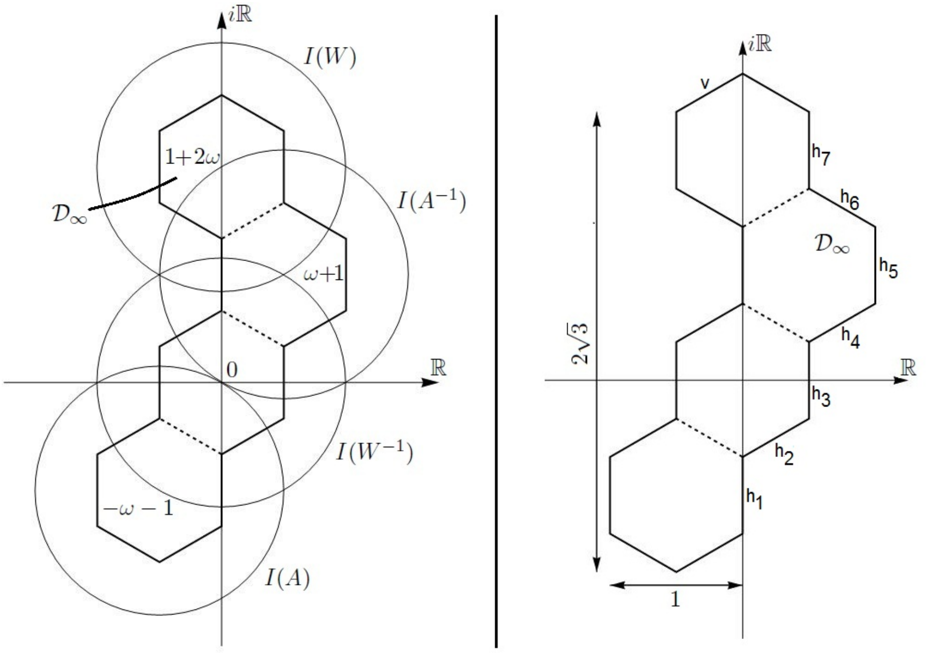

Figure 2.7: The polygon $\mathcal{D}_\infty$ and distances on $\mathcal{D}_\infty$

1. Obviously $\mathcal{D}_\infty$ is an open polygon and a domain in $\mathbb{C}$.

2. We notice that $\mathcal{D}_\infty$ can be constructed from the regular euclidean hexagon centered in 0 with inradius $\frac{1}{2}$ and 2 vertices on $i\mathbb{R}$. A short computation shows that the radius of the circumscribed circle to this hexagon is $\frac{\sqrt{3}}{3}$.
Hence, the configuration is such that $B\mathcal{D}_\infty \cap \mathcal{D}_\infty = \emptyset$ (because B is the horizontal translation of length 1), and $B(2+4\omega)\mathcal{D}_\infty \cap \mathcal{D}_\infty = \emptyset$(because $B(2+4\omega)$ is the vertical translation of length $2\sqrt{3}$; CF. Figure 2.7, right).
Thus, for all distinct $\widetilde{B}_1,\ \widetilde{B}_2 \in < B,\ B(2+4\omega) >$, the intersection $\widetilde{B}_1\mathcal{D}_\infty \cap \widetilde{B}_2\mathcal{D}_\infty$ is empty, because B and $B(2+4\omega)$ generate $< B,\ B(2+4\omega) >$.

3. From point 2. above and the configuration, we deduce that $\overline{B\mathcal{D}_\infty} \cap \overline{\mathcal{D}_\infty}$ consists in the consecutive edges $h_1, ..., h_7$, and that $\overline{B(2+4\omega)\mathcal{D}_\infty} \cap \overline{\mathcal{D}_\infty}$ consists in the single edge v of Figure 2.7.
Here again, a careful look at the picture and the fact that B and $B(2+4\omega)$ are translations of respective length 1 and $2\sqrt{3}$ in orthogonal directions and generate $< B,\ B(2+4\omega) >$ allow us to conclude that $\bigcup_{\widetilde{B} \in < B,\, B(2+4\omega) >} \overline{\widetilde{B}\mathcal{D}_\infty} = \mathbb{C}$.

Hence, $\mathcal{D}_\infty$ is a fundamental domain for the group $< B,\, B(2+4\omega) >$ acting on $\mathbb{C}$. ■

We come finally to the definition of $\mathcal{D}$.

Definition. Let $\mathcal{D}$ be the set of the points of $\mathcal{U}^3$ lying above all half-spheres of $\mathcal{L}$ and whose orthogonal projection on $\mathbb{C}$ is $\mathcal{D}_\infty$.

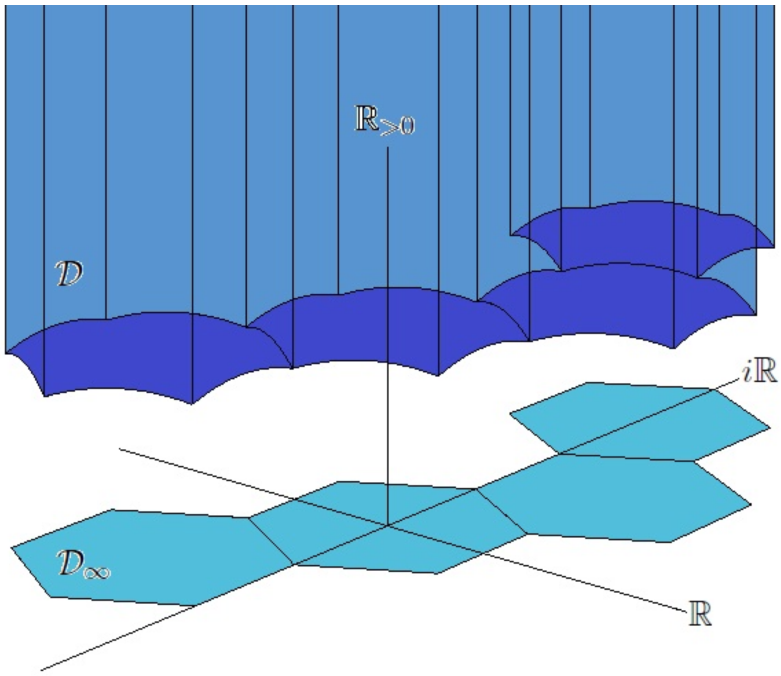

Figure 2.8: The polyhedron $\mathcal{D}$ and its projection $\mathcal{D}_\infty$

From our construction, the necessary conditions 2. and 3. of Theorem 1.7 are satisfied. It remains to prove that $\mathcal{D}$ is compatible with the conditions 1. and 4. of the same theorem. In other words, we have to show that $\mathcal{D}$ is a hyperbolic polyhedron which is bounded by polygons having finitely many edges.

It is clear from the construction that $\mathcal{D}$ is an open connected subset of $\mathcal{U}^3$. We check that $\mathcal{D}$ matches the definition of section 1.5.2.

- We notice that $\mathcal{D}$ has 22 sides. 18 sides of $\mathcal{D}$ are triangles with one ideal vertex (∞), which project onto the sides of $\mathcal{D}_\infty$. The 4 remaining sides are the hexagons projecting onto the interior of $\mathcal{D}_\infty$.
  Each side is a subset of either a vertical half-plane or a half-sphere centered in $\mathbb{C}$, and are all hyperbolic planes. Furthermore, the edges

of $\mathcal{D}$ are either vertical half-lines or arcs of half-circles centered in $\mathbb{C}$ and orthogonal to $\mathbb{C}$, and therefore geodesics of $\mathcal{U}^3$. Hence, each side of $\mathcal{D}$ is a (geodesic) hyperbolic polygon.

- It is clear by construction that each edge of $\mathcal{D}$ is the intersection of exactly two sides of $\mathcal{D}$, the vertical edges being the intersection of two triangular sides, and the "circle" edges the intersection of either a triangular and a hexagonal side, or two hexagonal sides.
  Furthermore, the relative position of two sides, a side and an edge, or two edges, as described in point 3. of the definition of a polygon, are correctly realized in $\mathcal{D}$.

- Finally, it is clear that the intersection of $\mathcal{D}$ with any open ball $B_\delta(x)$ of radius $\delta > 0$ and centered in $x \in \partial\mathcal{D}$ is a connected subset of $\mathcal{D}$.

Thus, $\mathcal{D}$ is a hyperbolic polyhedron bounded by polygons having finitely many edges, and is therefore admissible in our context. Furthermore, $\mathcal{D}$ is non-compact, but of finite volume (short computation). Indeed, $\mathcal{D}$ is the hyperbolic convex hull of finitely many points in $\overline{\mathcal{U}^3}$.

2.3 Application of Poincaré's Theorem

We now come to the central point of the proof, where we make use of Poincaré's Theorem for $\mathcal{D}$.
We are going to define an identification on $\mathcal{D}$ using elements of $\mathcal{K}$. That is why we first have to look more in details at the geometric effect of A, A^{-1}, W and W^{-1} on $\mathcal{U}^3$. The effect of $B(t)$, $t \in \mathbb{C}$, is well-know, as it is a translation.

2.3.1 Geometric Action of A, A^{-1}, W and W^{-1} on $\mathcal{U}^3$

In the following, we will think without distinction at an element in $PSL(2, \mathbb{C})$ as a matrix representative (modulo $\pm I_2$) or as an isometry of $\mathcal{U}^3$. Furthermore, we won't distinguish them from their Poincaré extensions, in order to make the discussion more easily accessible.

Action of A and A^{-1}

We remember that $A = \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix}$ has the isometric sphere $I(A) \subset \mathcal{U}^3$ centered in $c_A = -1 - \omega$ and with radius 1.
According to Proposition 1.16, A can be seen as the composition

$$A = \tau_A \circ \iota_A \circ \rho_A$$

where ρ_A is the euclidean reflection in $R(A)$, ι_A is the euclidean inversion in $I(A)$, and τ_A is the translation carrying $I(A)$ on $I(A^{-1})$.

Let us now discuss these three transformations.

- From the previous section, we now that $I(A)$ has center $c_A = -1-\omega$ and that $I(A^{-1})$ has center $c_{A^{-1}} = 1+\omega$. Thus, the translation τ_A is given by $\tau_A(z) = z+2+2\omega$, $\forall z \in \mathcal{U}^3$.
- ι_A is the euclidean inversion in $I(A)$. In particular, ι_A fixes all points of $I(A)$ (this fact is the only property of ι_A we will need, so we don't go any further into the analysis of ι_A).
- To characterize ρ_A, we have to find $R(A)$. First, notice that $A(0) = 0$. This is leads to $(\iota_A \circ \rho_A)(0) = \tau_A^{-1}(0) = -2-2$, i.e. $\rho_A(0) = \iota_A^{-1}(-2-2\omega)$. Since $-2-2\omega = 2\omega^2$ and since $|\omega^2| = 1$, we deduce that $2\omega^2$ is on the circle of radius 1 centered in ω^2, i.e. $2\omega^2 \in I(A)$. Since ι_A fixes the points of $I(A)$ pointwise, one concludes that $\rho_A(0) = -2-2\omega$. Thus, $R(A)$ is the vertical half-plane in $\mathcal{U}^3$ bounded by the line $R_0(A) \subset \mathbb{C}$ through $-1-\omega$ bissecting the segment between 0 and $-2-2\omega$, and ρ_A is the reflection in $R(A)$ (CF. Figure 2.9).

We furthermore recall that $A^{-1} = \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix}$ has isometric sphere $I(A^{-1}) \subset \mathcal{U}^3$ with center $c_{A^{-1}} = 1+\omega$ and radius 1.
As for A, one gets that we have the decomposition $A^{-1} = \tau_{A^{-1}} \circ \iota_{A^{-1}} \circ \rho_{A^{-1}}$, where $\tau_{A^{-1}}$ is the translation $z \mapsto z-2-2\omega$, $\iota_{A^{-1}}$ is the inversion in $I(A^{-1})$, and $\rho_{A^{-1}}$ is the reflection in the plane $R(A^{-1}) = \tau_A(R(A))$.

Action of W and W^{-1}

We proceed in a similar way as above, first recalling that $W = \begin{pmatrix} 0 & \omega \\ -\omega^2 & 1-\omega \end{pmatrix}$ with isometric sphere $I(W)$ centered in $c_W = 1+2\omega$ and with radius 1.

As above, we have the decomposition $W = \tau_W \circ \iota_W \circ \rho_W$, given by the following transformations:

- Since $c_W = 1+2\omega$ and $c_{W^{-1}} = 0$, we deduce that $\tau_W(z) = z-1-2\omega$, $\forall z \in \mathcal{U}^3$.
- ι_W is the inversion at $I(W)$. In particular, the points of $I(W)$ are fixed pointwise by ι_W.
- We notice that $W(\omega) = -1$, and consider $(\iota_W \circ \rho_W)(\omega) = \tau_W^{-1}(\omega) = 2\omega \iff \rho_A(\omega) = \iota_A^{-1}(2\omega)$. Since $2\omega = 1+\omega$ is in $I(W)$, $\iota_A^{-1}(2\omega) = 2\omega$, and we obtain $\rho_W(\omega) = 2\omega$.
  As a consequence, $R(W)$ is the hyperbolic half-plane in $\mathcal{U}^3$ bounded by the line $R_0(W) \subset \mathbb{C}$ through $1+2\omega$ and bissecting the segment between 2ω and $1+\omega$, and ρ_W is the reflexion in $R(W)$ (CF. Figure 2.9).

Finally, remember that $W^{-1} = \begin{pmatrix} 1-\omega & -\omega \\ \omega^2 & 0 \end{pmatrix}$, with isometric sphere $I(W^{-1})$ centered in $c_{W^{-1}} = 0$ and radius 1.
As for W, we have the decomposition $W^{-1} = \tau_{W^{-1}} \circ \iota_{W^{-1}} \circ \rho_{W^{-1}}$, where $\tau_{W^{-1}}$ is the translation $z \mapsto z + 1 + 2\omega$, $\iota_{W^{-1}}$ is the inversion in $I(W^{-1})$, and $\rho_{W^{-1}}$ is the reflection in the plane $R(W^{-1}) = \tau_W(R(W))$.

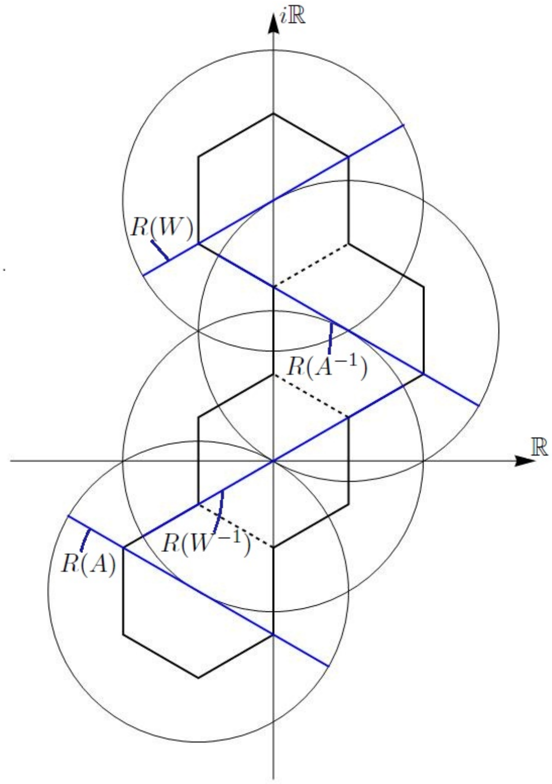

Figure 2.9: The reflection planes

2.3.2 Definition of an Identification on $\mathcal{D}$

Now we have all necessary tools and information to define an identification on the polyhedron $\mathcal{D}$, using only the generators A and B of $\mathcal{K} \subset PSL(2, \mathbb{C})$, and the elements W, $B(2 + 4\omega)$ and $B(3 + 4\omega)$ of $PSL(2, \mathbb{C})$.
Let us label the sides and edges of $\mathcal{D}$ as follows (the edges of the form e_i^j for $j = 1, ..., 8$, are the vertical edges, and the remaining edges are the "hexagonal" edges). The edge labelling will be made clear in the next paragraphs.

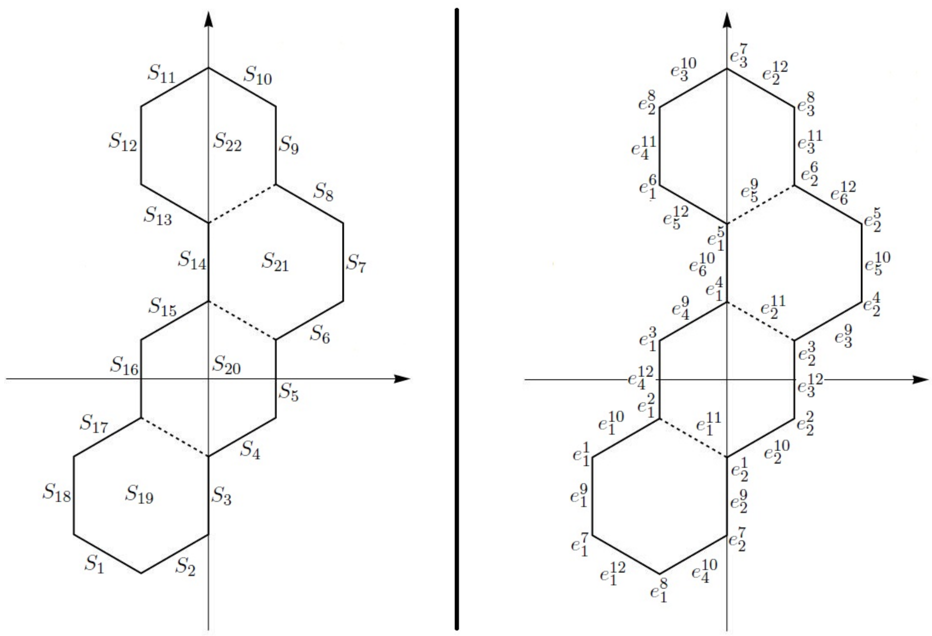

Figure 2.10: Labelled sides and edges

We identify the following pairs of sides:

- Using $B(3+4\omega)^{\pm 1}$: $(S_1\,;S_{10})$.
- Using $B(2+4\omega)^{\pm 1}$: $(S_2\,;S_{11})$.
- Using $B^{\pm 1}$: $(S_3\,;S_{18})$, $(S_4\,;S_{17})$, $(S_5\,;S_{16})$, $(S_6\,;S_{15})$, $(S_7\,;S_{14})$, $(S_8\,;S_{13})$, and $(S_9\,;S_{12})$.
- Using $A^{\pm 1}$: $(S_{19}\,;S_{21})$
- Using $W^{\pm 1}$: $(S_{20}\,;S_{22})$

Looking at the successive "moves" of the edges induced by the sides identification, one can deduce that the edges are identified within the 12 following classes, using the fact that any edge is contained in two different sides (CF. Figure 2.10, right):

- $e_1^j \xrightarrow{B} e_2^j \xrightarrow{B^{-1}} e_1^j$, for $j = 1, ..., 6$ (vertical edges).
- $e_1^7 \xrightarrow{B} e_2^7 \xrightarrow{B(2+4\omega)} e_3^7 \xrightarrow{B(3+4\omega)^{-1}} e_1^7$ (vertical edges).
- $e_1^8 \xrightarrow{B(2+4\omega)} e_2^8 \xrightarrow{B} e_3^8 \xrightarrow{B(3+4\omega)^{-1}} e_1^8$ (vertical edges).

- $e_1^9 \xrightarrow{B} e_2^9 \xrightarrow{A} e_3^9 \xrightarrow{B^{-1}} e_4^9 \xrightarrow{W^{-1}} e_5^9 \xrightarrow{A^{-1}} e_1^9$ ("hexagonal edges").
- $e_1^{10} \xrightarrow{B} e_2^{10} \xrightarrow{W^{-1}} e_3^{10} \xrightarrow{B(2+4\omega)^{-1}} e_4^{10} \xrightarrow{A} e_5^{10} \xrightarrow{B^{-1}} e_6^{10} \xrightarrow{A^{-1}} e_1^{10}$ ("hexagonal edges").
- $e_1^{11} \xrightarrow{A} e_2^{11} \xrightarrow{W^{-1}} e_3^{11} \xrightarrow{B^{-1}} e_4^{11} \xrightarrow{W} e_1^{11}$ ("hexagonal edges").
- $e_1^{12} \xrightarrow{B(3+4\omega)} e_2^{12} \xrightarrow{W} e_3^{12} \xrightarrow{B^{-1}} e_4^{12} \xrightarrow{W^{-1}} e_5^{12} \xrightarrow{B} e_6^{12} \xrightarrow{A^{-1}} e_1^{12}$ ("hexagonal edges").

As an example, we give the detailed procedure for the set $\{e_i^9\}$, $i = 1, ..., 5$.
We begin with e_1^9. The identification $S_{18} \xrightarrow{B} S_3$ sends e_1^9 on e_2^9.
The identification $S_{19} \xrightarrow{A} S_{21}$ identifies e_2^9 with e_3^9 (remember the decomposition of A as composition of a reflection, an inversion, and a translation and notice that all "hexagonal edges" are on the isometric spheres).
Furthermore, e_3^4 is sent on e_4^9 by the identification $S_6 \xrightarrow{B^{-1}} S_{15}$, and the identification $S_{20} \xrightarrow{W^{-1}} S_{22}$ sends e_4^9 on e_5^9.
Finally, e_5^9 is identified with e_1^9 by the side identification $S_{21} \xrightarrow{A^{-1}} S_{19}$.

The procedure described above is unique (up to the choice of the starting edge in a class) because any edge is contained in exactly two different sides.

2.3.3 The Poincaré's Theorem $\mathcal{D}$

We now show that $\mathcal{D}$ is a Poincaré polyhedron. In other words, we check the identification conditions **(I1)**, **(I2)**, **(I3)** and **(I4)**, the completeness conditions **(CP1)** and **(CP2)**, and the cycle conditions **(CC1)**, **(CC2)** and **(CC3)** of section 1.5.2.

The identification conditions

Since the identification defined above is explicitly given, the verification of these conditions is easy.

- Ad **(I1)** : Since the transformations A, B, W, $B(2+4\omega)$ and $B(3+4\omega)$ and their inverses are elements of $PSL(2, \mathbb{C})$, they are isometries of $\mathcal{U}^3$ (CF. section 1.4). Since no other transformation is used to define the identification, the condition is satisfied.
- Ad **(I2)** : From the construction described in the previous section, it is clear that each side is assigned to a unique other side, and that $T(S', S) = (T(S, S'))^{-1}$ for each isometry $T(S, S')$ sending S on S'.

- Ad **(I3)** : Since there is no side S such that $S = S'$, the condition is trivially true.

- Ad **(I4)** : The situation induced by the identification is such that each side is sent to a "non-adjacent" side which is at a euclidean distance at least 1. Therefore, for all side S of $\mathcal{D}$, one can find a neighbourhood $V_S \subset \mathcal{U}^3$ of S such that for $T = T(S, S')$, the intersection $T(V_S \cap \mathcal{D}) \cap \mathcal{D}$ is empty.

As a consequence, we have now that $\mathcal{D}$ is a polyhedron with identification. Furthermore, the group induced by the identification is generated by A, B, W, $B(2+4\omega)$ and $B(3+4\omega)$, with no reflections (empty condition **(I3)**).

The properness condition

Since $\mathcal{D}$ has finitely many vertices, edges and sides, and refering to the explicit side-pairing described above, it is obvious that all for all $x \in \mathcal{D}^*$, the set $p^{-1}(x)$ is finite.

Therefore, the properness condition (P) is satisfied.

The cycle conditions

We refer to the identification described in section 2.3.2. As we have seen, the side pairing induces 12 edge sequences of different lengths.
Let us give the explicit corresponding edge cycles $\mathcal{E}^j$ and generator sequences $\mathcal{T}^j$, $j = 1, ..., 12$. We don't give the related sequences of pairs of sides, because we don't need them in the discussion. They are easy to find out, however.

- $\mathcal{E}^j = \{e_1^j, e_2^j\}$, $j = 1, ..., 6$. The related generators sequences are all the same: $\mathcal{T}^j = \{B, B^{-1}\}$, $j = 1, ..., 6$.

- $\mathcal{E}^7 = \{e_1^7, e_2^7, e_3^7\}$, with $\mathcal{T}^7 = \{B, B(2+4\omega), B(3+4\omega)^{-1}\}$.

- $\mathcal{E}^8 = \{e_1^8, e_2^8, e_3^8\}$, with $\mathcal{T}^8 = \{B(2+4\omega), B, B(3+4\omega)^{-1}\}$.

- $\mathcal{E}^9 = \{e_1^9, e_2^9, e_3^9, e_4^9, e_5^9\}$, $\mathcal{T}^9 = \{B, A, B^{-1}, W^{-1}, A^{-1}\}$.

- $\mathcal{E}^{10} = \{e_1^{10}, e_2^{10}, e_3^{10}, e_4^{10}, e_5^{10}, e_6^{10}\}$, with $\mathcal{T}^{10} = \{B, W^{-1}, B(2+4\omega)^{-1}, A, B^{-1}, A^{-1}\}$.

- $\mathcal{E}^{11} = \{e_1^{11}, e_2^{11}, e_3^{11}, e_4^{11}\}$, with $\mathcal{T}^{11} = \{A, W^{-1}, B^{-1}, W\}$.

- $\mathcal{E}^{12} = \{e_1^{12}, e_2^{12}, e_3^{12}, e_4^{12}, e_5^{12}, e_6^{12}\}$, with $\mathcal{T}^{12} = \{B(3+4\omega), W, B^{-1}, W^{-1}, B, A^{-1}\}$.

We directly deduce for the corresponding periods m_j, $j = 1, ..., 12$:

$$m_1 = ... = m_6 = 2 \; ; \; m_7 = m_8 = 3 \; ; \; m_9 = 5 \; ; \; m_{10} = m_{12} = 6 \; ; \; m_{11} = 4.$$

Furthermore, the sequences above induce following cycle transformations.

- $C_1 = ... = C_6 = B^{-1}B$,
- $C_7 = B(3 + 4\omega)^{-1}B(2 + 4\omega)B$,
- $C_8 = B(3 + 4\omega)^{-1}BB(2 + 4\omega)$,
- $C_9 = A^{-1}W^{-1}B^{-1}AB$,
- $C_{10} = A^{-1}B^{-1}AB(2 + 4\omega)^{-1}W^{-1}B$,
- $C_{11} = WB^{-1}W^{-1}A$,
- $C_{12} = A^{-1}BW^{-1}B^{-1}WB(3 + 4\omega)$.

We now check the cycle conditions:

- Ad **(CC1)** : We refer to the proof of Proposition 2.3 in section 2.2.2. We deduce that the interior angle between two vertical sides is either $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$, depending on the situation. Furthermore, we have seen that the interior angle between a vertical side and a "spherical" side (one of the sides S_{19}, S_{20}, S_{21} or S_{22}) is $\frac{\pi}{3}$ and that the interior angle between two "spherical" sides is $\frac{2\pi}{3}$.
  Thus, we have
  - For $j = 1, ...6$, $m_j = 2$: $\alpha(e_1^j) + \alpha(e_2^j) = \frac{2\pi}{3} + \frac{4\pi}{3} = \frac{2\pi}{\nu_j}$, with $\nu_j = 1$.
  - For $j = 7, 8$, $m_j = 3$: $\alpha(e_1^j) + \alpha(e_2^j) + \alpha(e_3^j) = \frac{2\pi}{3} + \frac{2\pi}{3} + \frac{2\pi}{3} = \frac{2\pi}{\nu_j}$, with $\nu_j = 1$.
  - For $j = 9$, $m_9 = 5$: $\sum_{i=1}^{5} \alpha(e_i^9) = 4 \cdot \frac{\pi}{3} + \frac{2\pi}{3} = \frac{2\pi}{\nu_9}$, with $\nu_9 = 1$.
  - For $j = 10$, $m_{10} = 6$: $\sum_{i=1}^{6} \alpha(e_i^{10}) = 6 \cdot \frac{\pi}{3} = \frac{2\pi}{\nu_{10}}$, with $\nu_{10} = 1$.
  - For $j = 11$, $m_{11} = 4$: $\sum_{i=1}^{4} \alpha(e_i^{11}) = 2 \cdot \frac{2\pi}{3} + 2 \cdot \frac{\pi}{3} = \frac{2\pi}{\nu_{11}}$, with $\nu_{11} = 1$.
  - For $j = 12$, $m_{12} = 6$: $\sum_{i=1}^{6} \alpha(e_i^{12}) = 6 \cdot \frac{\pi}{3} = \frac{2\pi}{\nu_{12}}$, with $\nu_{12} = 1$.

  Thus, the condition is satisfied, and we deduce that $\nu_j = 1$ for $j = 1, ..., 12$.

- Ad **(CC2)** : All cycle transformations C_j, $j = 1, ..., 12$ are orientation preserving, because they are the composition of the orientation preserving transformations $A^{\pm 1}$, $B^{\pm 1}$, $W^{\pm 1}$, $B(2+4\omega)^{\pm 1}$ and $B(3+4\omega)^{\pm 1}$ (A and W are orientation preserving because they both have a matrix representative with positive determinant).

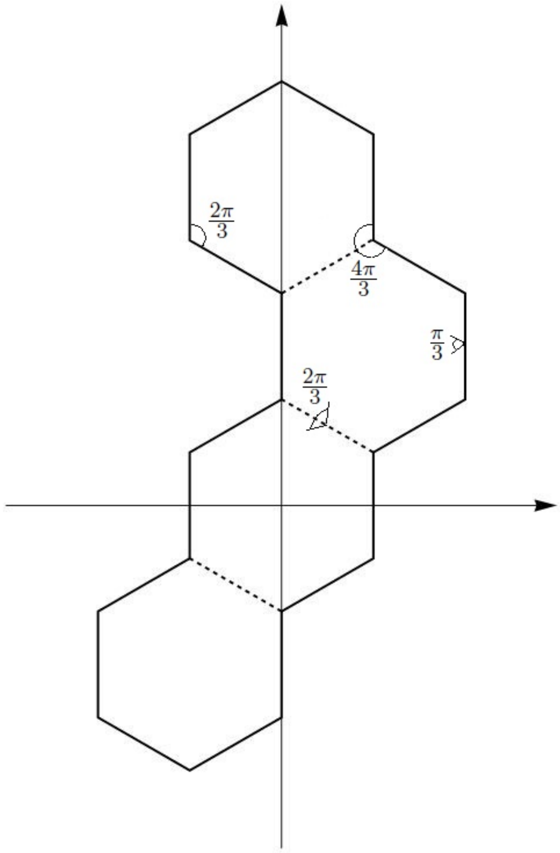

Figure 2.11: Interior angles in $\mathcal{D}$

- Ad **(CC3)** : In our situation, all edges have at least one finite endpoint. Thus, the condition is trivially true.

The completeness condition

$\mathcal{D}$ is a finite-sided polyhedron constructed from 4 pyramids with hexagonal base and ideal apex. Thus, we apply the related propositions stated in section 1.5.2.
The intersection of $\mathcal{D}$ with any horosphere $\mathbb{C}_t$ for $t > 1$ is the union of 4 copies of the same euclidean regular hexagon. Furthermore, the side-pairing on $\mathcal{D}$ is such that one has only one cusp point, formed by the 4 ideal apices of the pyramids.
The restriction of the action of the generators on the hexagons provides obviously a side-pairing of the hexagons sides (we don't even need to rescale). Therefore, the situation here is so nice that we can apply the Propositions 1.23 and 1.24 directly, and deduce that the metric space $\mathcal{D}^*$ is complete, i.e.

the completeness condition **(CP)** is satisfied.

As an outcome, we have that $\mathcal{D}$ is a Poincaré polyhedron.
Furthermore, since $\nu_j = 1$ for all $j = 1, ..., 12$, the cycle relations induced by the cycle transformations listed above are $C_j = I$, for $j = 1, ..., 12$ (for I the identity matrix of $PSL(2, \mathbb{C})$).

Poincaré's Theorem on $\mathcal{D}$

Now that we have seen that $\mathcal{D}$ is a Poincaré polyhedron, we can apply Poincaré's Theorem 1.18 on $\mathcal{D}$.

First, we define the group $\Gamma_{\mathcal{D}} < PSL(2, \mathbb{C})$ as follows.

$$\Gamma_{\mathcal{D}} := \left\langle \begin{array}{c} A, B, W \\ B(2+4\omega), B(3+4\omega) \end{array} \;\middle|\; \begin{array}{c} B^{-1}B \overset{(1)}{=} I \\ B(3+4\omega)^{-1}B(2+4\omega)B \overset{(2)}{=} I \\ B(3+4\omega)^{-1}BB(2+4\omega) \overset{(3)}{=} I \\ A^{-1}W^{-1}B^{-1}AB \overset{(4)}{=} I \\ A^{-1}B^{-1}AB(2+4\omega)^{-1}W^{-1}B \overset{(5)}{=} I \\ WB^{-1}W^{-1}A \overset{(6)}{=} I \\ A^{-1}BW^{-1}B^{-1}WB(3+4\omega) \overset{(7)}{=} I \end{array} \right\rangle .$$

Before stating the conclusions of Poincaré's Theorem, we give a reduced presentation of $\Gamma_{\mathcal{D}}$.
Let us rewrite the relations $(2) - (7)$ as follows:

- $(2) \Leftrightarrow B(3+4\omega) \overset{(2')}{=} B(2+4\omega)B$,
- $(3) \Leftrightarrow B(3+4\omega) \overset{(3')}{=} BB(2+4\omega)$,
- $(4) \Leftrightarrow W \overset{(4')}{=} B^{-1}ABA^{-1}$,
- $(5) \Leftrightarrow B(2+4\omega)^{-1} = A^{-1}BAB^{-1}W \Leftrightarrow B(2+4\omega) \overset{(5')}{=} W^{-1}BA^{-1}B^{-1}A$,
- $(6) \Leftrightarrow A \overset{(6')}{=} WBW^{-1}$,
- $(7) \Leftrightarrow B(3+4\omega) \overset{(7')}{=} W^{-1}BWB^{-1}A$.

Now we eliminate the redundant relations and generators.

- The relation (1) holds in all groups and is therefore trivial.
- Since

$$\begin{aligned} W &= A^{-1} \underbrace{A}_{use\ (6')} W = A^{-1}WBW^{-1}W \\ &= A^{-1} \underbrace{W}_{use\ (4')} B = A^{-1}B^{-1}ABA^{-1}B, \end{aligned}$$

one gets

$$WB^{-1}AB^{-1} = A^{-1}B^{-1}A.$$

Multiplying each side to the left by $W^{-1}B$ and to the right by B, one gets

$$W^{-1}BWB^{-1}AB^{-1}B = W^{-1}BA^{-1}B^{-1}AB.$$

The left term reduces in $W^{-1}BWB^{-1}A$. Using relation (5′) on the right term, one gets $B(2+4\omega)B$, which is $B(3+4\omega)$ according to relation (2′).
Finally, this leads to the relation $W^{-1}BWB^{-1}A = B(3+4\omega)$, which is relation (7′). Therefore, the relation (7′) is redundant.

- The definition of $B(t)$ implies that the relations (2′) and (3′) already holds in $PSL(2,\mathbb{C})$, and are therefore trivial. This implies that the generator $B(3+4\omega)$ is redundant.
- A short computation shows that the relation (5′) is redundant, and therefore that the generator $B(2+4\omega)$ is redundant.
- The relation (4′) can be integrated into relation (6′) to get a relation (6″) involving only A and B.

In summary, Γ has only 2 generators and the relation (6″), and we get following presentation:

$$\Gamma_{\mathcal{D}} = \left\langle A, B \ \mid \ A = \underbrace{B^{-1}ABA^{-1}}_{=W} B \underbrace{AB^{-1}A^{-1}B}_{=W^{-1}} \right\rangle. \qquad (**)$$

Then, the consequences of Poincaré's Polyhedron Theorem are the following:

1. $\Gamma_{\mathcal{D}}$ acts on $\mathcal{U}^3$ properly discontinuously (and is therefore a Kleinian group). By Theorem 1.6, $\Gamma_{\mathcal{D}}$ is discrete.
2. $\mathcal{D}$ is a fundamental polyhedron for $\Gamma_{\mathcal{D}}$.

A further consequence is that $(\Gamma_{\mathcal{D}})_\infty = \langle B, B(2+4\omega)\rangle$ with fundamental polygon $\mathcal{D}_\infty$, as mentioned in section 2.2.3.

2.4 $\mathbb{R}^3 \setminus K$ is a hyperbolic 3-manifold

We now come to the central result of this work. We will use the consequences of Poincaré's Polyhedron Theorem 1.7 stated in the previous section, Waldhausen Theorem 1.10 and other results mentioned in the first chapter of this work.

Theorem. *$\mathbb{R}^3 \setminus K$ is a complete oriented hyperbolic 3-manifold, which is non-compact (but of finite volume, CF. Section 3.1).*

Proof. The proof consists in an algebraic-geometric part, followed by a topological part.

- At the end of section 2.3, we saw that $\Gamma_{\mathcal{D}}$ is a discrete group.
- From Proposition 1.11, one deduces that the generators of $\Gamma_{\mathcal{D}}$ are parabolic, since $tr(A) = tr(B) = 2$. Since A and B are triangular matrices, the transformations A^n and B^n have the same trace as A and B, for all $n \in \mathbb{Z}$.
  Furthermore, a short computation shows that the elements AB, $A^{-1}B$, AB^{-1} and $A^{-1}B^{-1}$ of $\Gamma_{\mathcal{D}}$ have all non-real trace and are loxodromic by Proposition 1.11.
  Since every element of $\Gamma_{\mathcal{D}}$ can be expressed as product of A and B, we deduce that the elements of $\Gamma_{\mathcal{D}}$ are either parabolic or loxodromic. In particular, $\Gamma_{\mathcal{D}}$ doesn't contain any elliptic element.
  Then, Proposition 1.12 shows that $\Gamma_{\mathcal{D}}$ operates freely on $\mathcal{U}^3$, and Proposition 1.26 shows that $\Gamma_{\mathcal{D}}$ is torsion-free.
- Theorem 1.9 hence implies that $\mathcal{D}^* = \mathcal{U}^3/\Gamma_{\mathcal{D}}$ is a complete oriented hyperbolic 3-manifold. Furthermore, it is clear from construction that $\mathcal{D}^*$ is path-connected, and therefore connected.
- The Theorems 1.8 and 1.9 imply that $\Gamma_{\mathcal{D}} \cong \pi_1(\mathcal{D}^*)$.
- On the other side, it is clear from the presentation $(*)$ of $\pi_1(K)$, represented in $PSL(2,\mathbb{C})$ by $\mathcal{K}$, and the presentation $(**)$ of $\Gamma_{\mathcal{D}}$ in previous section that all groups are isomorphic.
- Finally $\Gamma_{\mathcal{D}} \cong \pi_1(K)$, and Theorem 1.9 implies that $\mathcal{U}^3/\mathcal{K}$ and $\mathcal{U}^3/\Gamma_{\mathcal{D}}$ are homeomorphic.

We now come to the topological part of the proof.

- For $t > 0$, let $\mathcal{D}(t)$ be the portion of $\mathcal{D}$ which doesn't lie above the horosphere $\mathbb{C}_t$, and $\mathcal{D}^*(t)$ be the portion of $\mathcal{D}$ whose pre-image in $\mathcal{D}$ lies in $\mathcal{D}(t)$. From section 1.5, we know that for $t > 1$, the boundary $\partial\mathcal{D}^*(t)$ of $\mathcal{D}^*(t)$ is a torus.
- It is clear that for all $t > 1$, $\mathcal{D}^*(t)$ is homeomorphic to $\mathcal{D}^*$. Since $\mathcal{D}^*(2)$ is a compact set, one has to find a homeomorphism between $\mathcal{D}^*(2)$ and $\mathbb{S}^3 \setminus T_K$.
- From the first part of the proof, one has $\pi_1(\mathcal{D}^*(2)) \cong \pi_1(\mathcal{D}^*) \cong \Gamma_{\mathcal{D}} \cong \pi_1(K)$.

We apply Waldhausen Theorem 1.10 for $M = \mathbb{S}^3 \setminus T_K$ and $N = \mathcal{D}^*(2)$.

- Observe that T_K is a knotted torus. $\mathbb{S}^3$ is known to be irreducible. Then Dehn's Lemma 1.11 implies that $\mathbb{S}^3 \setminus T_K$ is irreducible, boundary irreducible (because its boundary is a torus), and sufficiently large (CF. first remark in Section 1.7).

- Furthermore, $\mathcal{D}^*(2)$ has $\mathcal{U}^3$ as universal cover. Since obviously $\pi_i(\mathcal{U}^3) = 0$ for all $i > 1$, we deduce from Proposition 1.27 that $\pi_i(\mathcal{D}^*(2)) = 0$ for all $i > 1$, i.e. $\mathcal{D}^*(2)$ is aspherical. Proposition 1.28 implies then that $\mathcal{D}^*(2)$ is irreducible.

- Since the boundary of $\mathcal{D}^*(2)$ is a (knotted) torus, $\mathcal{D}^*(2)$ is boundary irreducible.

- From section 2.3, we deduce that $\theta^{-1} : \pi_1(\mathcal{D}^*(2)) \longrightarrow \pi_1(\mathbb{S}^3 \setminus T_K)$ is an isomorphism. Furthermore, $\pi_1(\mathcal{D}^*(2)) =< A, B \,|\, WBW^{-1} = A >$ and $\pi_1(\partial\mathcal{D}^*(2)) =< B, B(2+4\omega) >$. A short computation shows that $B(2+4\omega) = W^{-1}BA^{-1}B^{-1}A$.
Set $\widetilde{b} := \theta^{-1}(B(2+4\omega)) = w^{-1}ba^{-1}b^{-1}a \in \pi_1(\mathbb{S}^3 \setminus T_K)$. The group $\theta^{-1}(< B, B(2+4\omega) >) =< b, \widetilde{b} >$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ and therefore conjugate to $\pi_1(\partial(\mathbb{S}^3 \setminus T_K))$ in $\pi_1(\mathbb{S}^3 \setminus T_K)$, since $\partial(\mathbb{S}^3 \setminus T_K)$ is a torus. Hence, the isomorphism θ^{-1} respects the peripheral structure.

- Then, we apply Waldhausen's Theorem 1.11 and deduce that $\mathcal{D}^*(2)$ and $\mathbb{S}^3 \setminus T_K$ are homeomorphic.

Thus, one concludes that $\mathbb{S}^3 \setminus K \approx \mathcal{D}^*(t)$ for all $t > 1$.
Switching to the non-compact case, one deduces that $\mathbb{R}^3 \setminus T_K$ is homeomorphic to $\mathcal{D}^*$, and is therefore a complete non-compact oriented hyperbolic 3-manifold as required. ■

Remark. As a consequence of the Theorem, we deduce by Mostow-Prasad rigidity that $\mathbb{R}^3 \setminus K$ is even isometric to $\mathcal{U}^3/\Gamma_{\mathcal{D}}$ (CF. section 3.1).

Chapter 3

Some Remarks on Important Related Results

3.1 Mostow-Prasad Rigidity

In this section, we give a short introduction to the computation of the volume of $\mathcal{D}^*$ and give an illustration of Mostow-Prasad rigidity applied on $\mathcal{D}^*$, in relation with Thurston's approach of $\mathbb{R}^3 \setminus K$. References for this section are [Mi1], [Th1] and [Vi].

3.1.1 Volume of Hyperbolic Orthoschemes and Ideal Tetrahedra

In order to compute the volume of $\mathcal{D}^*$, we are going to split $\mathcal{D}$ into polyhedra -called orthoschemes- whose volume can be easily computed. For such polyhedra (and other), the usual volume formula $vol(T) = \int_T dvol^3_{\mathbb{H}}$ can be simplified, using the so-called Lobatchevsky function defined as follows.

Definition. The *Lobatchevsky function* is the function Л: $\mathbb{R} \longrightarrow \mathbb{R}$ given by

$$Л(x) := -\int_0^x log|2sin(t)|dt.$$

From the definition and small computations, one can deduce that Л satisfies the following properties.

Properties. *1. Л is continuous.*

2. Л is an odd function, i.e. $Л(-x) = -Л(x)$ *for all* $x \in \mathbb{R}$.

3. Л is periodic of period π, *i.e.* $Л(x+\pi) = Л(x)$, *for all* $x \in \mathbb{R}$.

4. $Л(x) = 0 \iff x = k\,\frac{\pi}{2},\ k \in \mathbb{Z}$.

5. $Л(nx) = n\sum_{j=0}^{n-1} Л\big(x + \frac{k\,\pi}{n}\big)$ *for all* $n \in \mathbb{N}^*$.

In particular, for $n = 2$, Property 5 becomes $Л(2x) = 2\,Л(x) + 2\,Л\left(x + \frac{\pi}{2}\right)$.

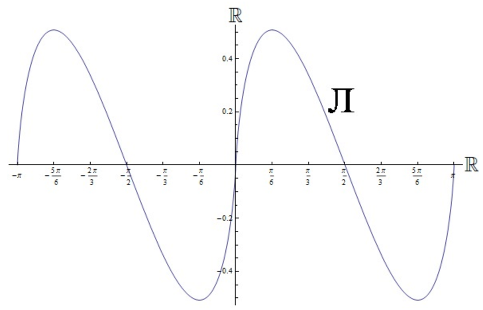

Figure 3.1: The Lobatchevsky function

We now give the definition of an orthoscheme in $\mathbb{H}^3$.

Definition. Let P_1, P_2, P_3, $P_4 \subset \mathbb{H}^3$ be four hyperbolic half-spaces bounded by hyperplanes H_1, H_2, H_3, H_4.
If $H_i \perp H_k$ for $|i - k| > 1$, $i, k = 1, ..., 4$, we call the tetrahedron $O = \bigcap_{i=1}^{4} P_i \subset \mathbb{H}^3$ a (3-)*orthoscheme*.

Denote α_{ij} the dihedral angle of O formed by the hyperplanes H_i and H_j.
A consequence of the definition is that one has $\alpha_{13} = \alpha_{14} = \alpha_{24} = \frac{\pi}{2}$.
In the sequel, we use the following notation for the other dihedral angles. Let $\alpha_{12} =: \alpha$, $\alpha_{23} =: \beta$, and $\alpha_{34} =: \gamma$.

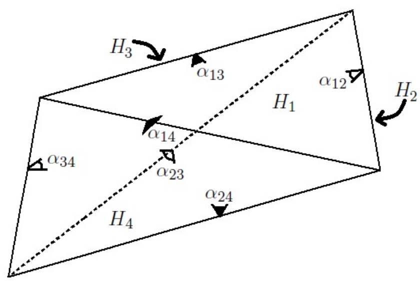

Figure 3.2: An orthoscheme

Proposition 3.1. $\alpha, \beta, \gamma < \frac{\pi}{2}$.

Definition. Let O be an orthoscheme with non-right dihedral angles α, β and γ.

The *principal parameter* of O is the (unique) angle $\delta \in [0, \frac{\pi}{2}]$ such that

$$tan^2\delta = \frac{cos^2\beta - sin^2\alpha \, sin^2\gamma}{cos^2\alpha \, cos^2\gamma}.$$

We furthermore introduce the notion of ideal polyhedron.

Definition. A polyhedron $P \subset \mathbb{H}^3$ is called *n-ideal* if n of its vertices are in $\partial\mathbb{H}^3$. If all vertices of P are in $\partial\mathbb{H}^3$, we call P an ideal polyhedron.

One can see that in an ideal tetrahedron, the dihedral angles split in three pairwise equal angles sitting at opposite edges.

The following theorems give useful formulae for the volume of orthoschemes and ideal tetrahedra.

Theorem 3.1. *Let $O \subset \mathbb{H}^3$ be an orthoscheme with dihedral angles α, β and γ and principal parameter δ.*
Then, the volume of O is given by

$$vol(O) = \frac{1}{4}[Л(\alpha+\delta) - Л(\alpha-\delta) - Л\left(\frac{\pi}{2} - \beta + \delta\right) + Л\left(\frac{\pi}{2} - \beta - \delta\right) + Л(\gamma+\delta) - Л(\gamma-\delta) + 2\,Л\left(\frac{\pi}{2} - \delta\right)].$$

Theorem 3.2. *Let T be an ideal tetrahedron with dihedral angles $\alpha^{(1)}$, $\alpha^{(2)}$ and $\alpha^{(3)}$.*
Then, the volume of T is given by

$$vol(T) = Л(\alpha^{(1)}) + Л(\alpha^{(2)}) + Л(\alpha^{(3)}).$$

3.1.2 Mostow-Prasad Rigidity on $\mathbb{R}^3 \setminus K$

The so-called Rigidity Theorem is a result due to George Mostow (compact case), and extended by Gopal Prasad (non-compact case). There are several formulations of the theorem. We give here a geometric version adapted to our situation (CF. [Th1], Chapter 5.7).

Theorem. *(Mostow-Prasad Rigidity) Let M and N be complete oriented hyperbolic 3-manifolds of finite volume.*
If $\pi_1(M) \cong \pi_1(N)$, then M and N are isometric.

It is a very deep and important result, showing the hard link between the topology and the geometry of 3-manifolds. There are several proofs of the theorem. [Th1] for example gives two different proofs.

We have mentioned in the Introduction that one can also see $\mathbb{R}^3 \setminus K$, for $K = 4_1$ the figure-eight knot, as the manifold τ_2 obtained by gluing two regular ideal hyperbolic tetrahedra. The following picture gives a glimpse of the construction.

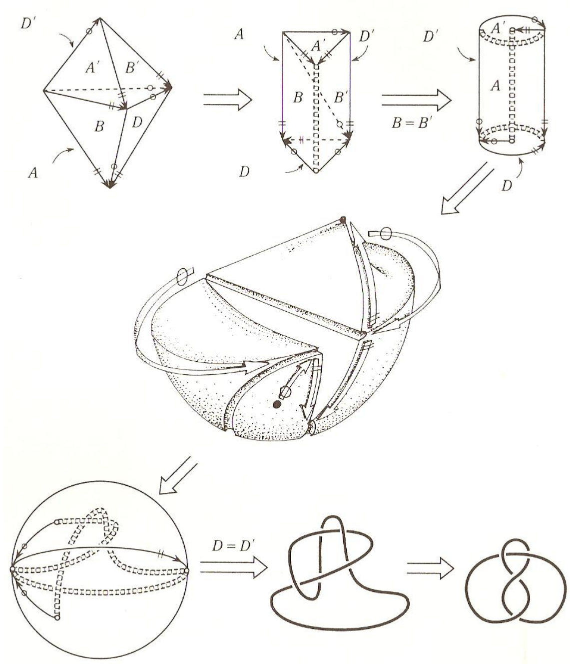

Figure 3.3: The construction of τ_2 (Picture: [MaTa], p.34)

Then, since $\tau_2 \approx \mathbb{S}^3 \setminus T_K$ and $\mathcal{U}^3/\Gamma_{\mathcal{D}} \approx \mathbb{R}^3 \setminus K$, Mostow-Prasad rigidity implies that the volume of τ_2 must be the same as $\mathcal{U}^3/\Gamma_{\mathcal{D}}$.
As an illustration, we give the explicit computation of the volume of τ_2 and $\mathcal{U}^3/\Gamma_{\mathcal{D}}$, and check the equality.

We first define the volume of a quotient manifold.

Definition. Let $\Gamma < Isom^+(\mathbb{H}^3)$ a torsion-free discrete group which acts properly discontinuously on $\mathbb{H}^3$ with fundamental domain D_Γ. Then, the *volume of the manifold* $\mathbb{H}^3/\Gamma$ is defined by $vol(\mathbb{H}^3/\Gamma) := vol(D_\Gamma)$.

Thus, in our situation we have to compute the volume of $\mathcal{D}$.

- We call $\mathcal{H}$ the pyramid with hexagonal base on $S_1(0)$ and ideal apex. By symmetry, one has that $vol(\mathcal{D}) = 4\, vol(\mathcal{H})$.
- $\mathcal{H}$ can be decomposed into 6 isometric 1-ideal tetrahedra, all having the point $(0, 1) \in \mathcal{U}^3$ as vertex. Let $\mathcal{T}$ be one of these tetrahedra. Then $vol(\mathcal{D}) = 24\, vol(\mathcal{T})$.

- Finally, taking a suitable vertical hyperplane bissecting $S_1(0)$, one can see that a tetrahedron splits into two isometric orthoschemes.
  Calling $\mathcal{O}$ one of theses two orthoschemes, we deduce that $vol(\mathcal{D}) = 48\, vol(\mathcal{O})$.

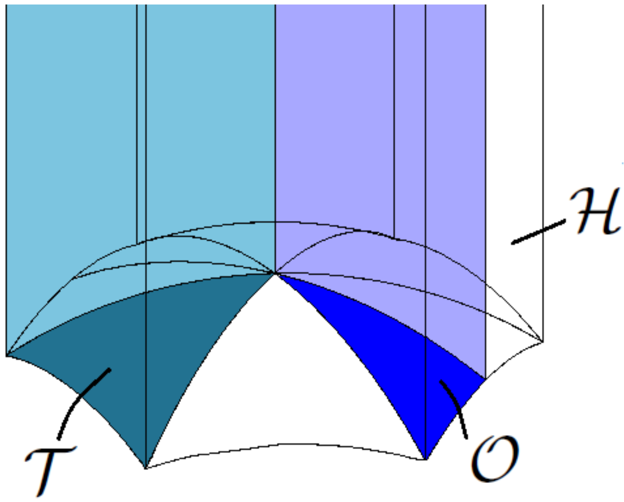

Figure 3.4: The polyhedra $\mathcal{H}$, $\mathcal{T}$ and $\mathcal{O}$

In particular, the dihedral angles α, β and γ of $\mathcal{O}$ are immediately obtained from the angle computation in $\mathcal{D}$ in section 2.2. We obtain

$$\alpha = \frac{\pi}{6},\ \beta = \frac{\pi}{3},\ \text{and}\ \gamma = \frac{\pi}{3}.$$

Visualizing $\mathcal{O}$ only as a combinatorial object, we obtain the following situation.

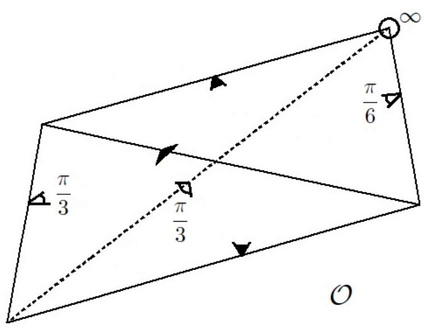

Figure 3.5: The orthoscheme $\mathcal{O}$

Furthermore, the principal parameter δ is given by

$$\begin{aligned}\delta &= arctan\left(\frac{\sqrt{-sin^2\left(\frac{\pi}{6}\right)\,sin^2\left(\frac{\pi}{3}\right)+cos^2\left(\frac{\pi}{3}\right)}}{cos\left(\frac{\pi}{6}\right)\,cos\left(\frac{\pi}{3}\right)}\right)\\ &= arctan\left(\frac{\sqrt{-\frac{1}{4}\cdot\frac{3}{4}+\frac{1}{4}}}{\frac{\sqrt{3}}{2}\cdot\frac{1}{2}}\right)=\ldots=arctan\left(\frac{\sqrt{3}}{3}\right)=\frac{\pi}{6}.\end{aligned}$$

Using the formula of Theorem 3.1, the properties of the function Л and the values of α, β, γ and δ above, we obtain

$$\begin{aligned}vol(\mathcal{O}) &= \frac{1}{4}\left(\text{л}\left(\frac{\pi}{3}\right)-\text{л}\,(0)-\text{л}\left(\frac{\pi}{3}\right)+\text{л}\,(0)+\text{л}\left(\frac{\pi}{2}\right)-\text{л}\left(\frac{\pi}{6}\right)+2\,\text{л}\left(\frac{\pi}{3}\right)\right)\\ &= \frac{1}{4}\left(-\text{Л}\left(\frac{\pi}{6}\right)+2\,\text{Л}\left(\frac{\pi}{3}\right)\right)=\frac{1}{4}\left(-\frac{3}{2}\,\text{Л}\left(\frac{\pi}{3}\right)+2\,\text{Л}\left(\frac{\pi}{3}\right)\right)\\ &= \frac{1}{8}\,\text{Л}\left(\frac{\pi}{3}\right).\end{aligned}$$

Finally, we deduce for the volume of $\mathcal{D}$:

$$vol(\mathcal{D})=48\,vol(\mathcal{O})=6\,\text{Л}\left(\frac{\pi}{3}\right).$$

On the other side, τ_2 is the gluing of two copies of an ideal regular hyperbolic tetrahedron τ. The regularity of τ implies for its dihedral angles that

$$\alpha^{(1)}=\alpha^{(2)}=\alpha^{(3)}=\frac{\pi}{3}.$$

Thus, the volume of τ_2 is given by Theorem 3.2 as follows.

$$vol(\tau_2)=2\,vol(\tau)=2\,\left(\text{Л}\left(\frac{\pi}{3}\right)+\text{Л}\left(\frac{\pi}{3}\right)+\text{Л}\left(\frac{\pi}{3}\right)\right)=6\,\text{Л}\left(\frac{\pi}{3}\right).$$

Finally, one observes that

$$vol(\mathcal{D})=6\,\text{Л}\left(\frac{\pi}{3}\right)=vol(\tau_2),$$

which proves that $vol(\mathcal{U}^3/\Gamma_{\mathcal{D}})=vol(\tau_2)$ as required.

3.2 The Hyperbolization Theorem for Knots

In this section, we give the general criterion to decide whether a knot complement is hyperbolizable or not. This is a corollary of Thurston's Hyperbolization Theorem (giving conditions for the interior of Haken manifolds to be hyperbolic). References for this sections are [CR] and [Th2].

We begin with some definitions.

Definition. A *torus knot* is a knot that can be embedded as a simple closed curve in an unknotted torus in $\mathbb{R}^3$.
More precisely, the torus knot $t_{p,q}$ of *type* p, q is the knot which wraps around the standard solid torus p times in the longitudinal direction, and q times in the meridian direction.

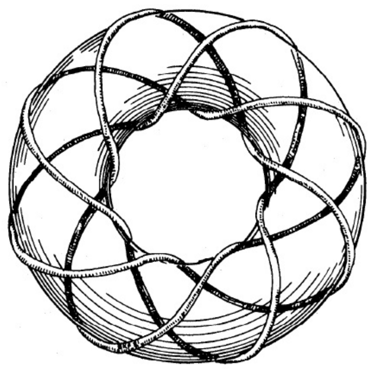

Figure 3.6: The torus knot $t_{3,8}$ (Picture: [Th2], p.358)

Definition. Let T_1 be an unknotted solid torus in $\mathbb{R}^3$, and $k_1 \subset T_1$ a knot. Let k_2 be a non-trivial knot, and T_2 a tubular neighbourhood of k_2 in $\mathbb{R}^3$. Let $h : T_1 \to T_2$ be a homeomorphism of T_1 onto T_2. The image $k := h(k_1)$ is called *satellite knot* of *pattern* k_1 and *companion* k_2.

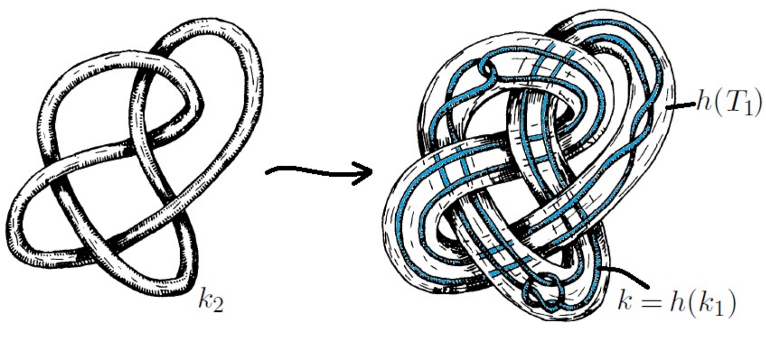

Figure 3.7: A satellite knot and its companion (Picture: [Th2], p.358)

Furthermore, we recall the definition of a hyperbolic structure on a knot complement.

Definition. Let $k \subset \mathbb{R}^3$ be a knot. A *hyperbolic structure* on the manifold $\mathbb{R}^3 \setminus k$ is a Riemannian metric on it such that every point of $\mathbb{R}^3 \setminus k$ has a

neighbourhood isometric to an open subset of $\mathbb{H}^3$.
Knots admitting a hyperbolic structure on their complement are called *hyperbolic knots.*

We now are able to state the criterion.

Theorem. *(Thurston's Hyperbolization Theorem for Knots) A knot $k \subset \mathbb{R}^3$ is hyperbolic if and only if k is neither a torus knot nor a satellite knot.*

The difficulty which arises now is to decide which knots are torus knots or satellite knots. For knot diagrams with many crossings, this problem is non-trivial. However, several classes of knots are known to be hyperbolic. Some of them can be found in [Ad2].

Conclusion

We have seen in details the path taken by Riley to find the first explicit hyperbolic structure on a knot complement. We have made use of Poincaré's Polyhedron Theorem to produce a discrete torsion-free group $\Gamma_{\mathcal{D}}$ such that $\mathbb{R}^3 \setminus K$, $K = 4_1$, is homeomorphic to the complete non-compact oriented hyperbolic 3-manifold $\mathcal{U}^3/\Gamma_{\mathcal{D}}$. We have even produced a glued polyhedron $\mathcal{D}^*$ which can help to vizualise the combinatorial structure of the manifold.

Thurston's method doesn't lead to such an explicit structure, but is easier to deal with for different reasons. For example, his approach gives a triangulation of the knot complement with ideal regular tetrahedra. This can be translated into algorithms in order to produce a computer program which is able to give directly several informations about the manifold. Jeffrey Weeks, one of Thurston's students, made use of this idea to write the program *SnapPea* (*http://www.geometrygames.org/SnapPea/*) which is still used nowadays to work with knots, links and their complements.
Furthermore, by Mostow-Prasad rigidity, we know that Thurston's manifold obtained by gluing two ideal regular tetrahedra has to be isometric to $\mathcal{D}^*$. We have computed as an illustration the explicit volume of $\mathcal{D}^*$ and showed that it equals the volume of two ideal regular tetrahedra.

The difficulty which arises with Riley's approach is that there doesn't seem to be any generalization of the process. Given any knot k, how can we obtain an explicit discrete torsion-free group $\Gamma < Isom(\mathbb{H}^3)$ such that $\mathbb{H}^3/\Gamma$ is homeomorphic to $\mathbb{R}^3 \setminus k$?
However, working out [Ri1] shows how various mathematical concepts such as knots, hyperbolic isometries, discrete groups and fundamental polyhedra can be used to investigate the world of hyperbolic 3-manifolds.

Finally, we have mentioned that hyperbolic knots (i.e. knots whose complement can be hyperbolized) turn out to represent the huge majority of knots. Riley's first intuition was that the figure-eight was a particular knot, which could be seen as counter-example. It is only after meeting Thurston that he realized that his construction was in fact the first explicit example.

Acknowledgements

I would like to thank Prof. Ruth Kellerhals for having suggested this topic, for her communicative enthusiasm, her sensible advice and comments, and her availability.

My thanks also go to my family, which has always supported me (and still does so) in my studies. *Fleur* has always listened to my wails and helped me to get a fix; thank to her for having often renewed my motivation.

My office mate Chrystel has had to bear my stress and bad days, and also deserves to be thanked here. Thank to Annick for the discussions and the parallel work.

Finally, “my” first-year students, by their questions and interest, have forced me to always try to be clear and simple in my approach. Thank to them, too.

Appendix: An unpublished article by Riley

A personal account of the discovery of hyperbolic
structure on some knot complements

Robert Riley

I discovered, quite unexpectedly, the phenomenon of hyperbolic structure on three knot complements early in 1974, and managed to get two papers on the topic published in 1975. At some moment between the dates of publication of these papers, William Thurston independently discovered the phenomenon and ran away with the idea. In late June or early July 1976 he learned of my work, and so when we met later in July he immediately told me that he had been trying for about a year to prove the hyperbolization conjecture for Haken 3–manifolds.

Colin Adams published a semipopular account of knot theory in "The Knot Book" [1], and a copy of this came into my hands recently. On page 119 he gives an account of the hyperbolic structure discovery which is just plain wrong. He does get the names of the two people concerned and the priority right, but nothing else. The present paper is an attempt to set the record straight. I shall relate what I did, why, and when. There will be too much detail about small matters, but this will convey the spirit of my projects. Indeed, I think my old papers were very open about my project, and a close look at them and their dates of submission should have made the present history unnecessary. Furthermore, Bill Thurston's account of my work in [13] is entirely fair, except for being too generous about my influence on his thinking.

So below I give the history of my project from its beginning to the moment I met Professor Thurston. The story is told as I saw it, and the emphasis is on motivation and dates. Many dates are only approximate because most entries in my notebooks are undated, but the uncertainties are never more than about a month. I include an intermediate example, worked out between the discoveries of the hyperbolic structures for the figure–eight knot (4_1) and for 5_2. This example ought not be on the main line of development, but in fact it was, and it served to undermine my initial expectation that the figure–eight is the only knot which could possibly be hyperbolic. I close with some comments on the early work of H. Gieseking & Max Dehn, and on the article [15] of W. Thurston.

1. The early years

On settling in Amsterdam in October 1966 I wrote off to virtually everyone publishing in knot theory for their reprints and preprints. I recall with gratitude that R.H. Fox and H. Seifert were especially generous. An unassuming little paper by Fox [2], written in Utrecht some 20 miles away, took my fancy. Here Fox advertised the notion of longitude in a knot group by using it, together with representations on the alternating group A_5, to distinguish the square and granny knots. I was intrigued by the success of A_5, and took the first steps toward writing out explicit procedures to find all A_5-representations of a knot group in 1967–68. When I got my first temporary appointment at Southampton (England) in 1968 this became my main project, with results summarized in [6, 7]. So by 1970 I was after the parabolic representations (p-reps) of a knot group, initially because they were easier to manage than the general non-abelian representations (nab-reps). The 2-bridge case is especially tractable, because the representations are governed by a simple polynomial whose rule of formation is easily programmed in Fortran. This tractability extends to all r-bridge knots which are symmetric about an r-fold axis of rotation that cyclically permutes the bridges, but most knots of bridge number > 2 are not so symmetric. The explicit algebraic description of the equivalence classes of p-reps of an unsymmetric knot is so difficult that only a few examples have been worked explicitly, and I have found the full curve of all nab-representations of only one unsymmetric 3-bridge knot, 8_{20}. Around 1971 I wrote some primitive Fortran programs to find the p-reps of a few 3-bridge knots and used the output to discover the commuting trick of [7II], but at the time this topic was mainly pure frustration.

In 1971 a plea for help from me was passed on to Professor G.E. Collins, the instigator of the SAC-1 file of Fortran routines for doing the kind of algebraic calculations I needed. He sent me a pile of very poorly printed manuals containing the program listings, lots of errata slips, and the advice that the 24 bit word size of the Southampton university computer would require some doubly recursive programming in assembly language. (The reference count field in a SAC atom would be too small without this recursion, and hence impose a strict limit on the allowed complexity of calculations). He also mentioned that I would need to get someone to punch up the 6000 or so cards of the 1971 SAC. Well, that someone had to be me, but fortunately only some 4000 cards, plus the assembly language parts, were needed for my application. It took about eight months to do all this, and I never did get the double recursion for the reference count right. So my more ambitious calculations were killed as soon as the reference count tried to reach 128, but I still managed to do most of what I wanted. By 1 October 1972, the day my fourth temporary appointment at Southampton ceased, I had done the elimination–of–variables part of the solving for an algebraic description of the set of p–reps for several 3–bridge knots, including 9_{35}. Each SAC run required several hours of CPU time, and could not have been attempted during term time. Perhaps some distorted memory of this story is the source of the "immense computer program that was designed to attempt to show that some knots are hyperbolic" bit in Adams' account. In fact, the PNCRE package which does just this was developed from 1976, and it was always fast enough for term time, even during the day on a grossly overloaded 1960's computer.

2. The preparation

In October 1972 I had a large pile of SAC output which needed more computer analysis to become meaningful, and no prospect of further employment. So I spent the next three months walking the Pennine Way and walking in Wales until the prospect of a six month appointment in Strasbourg opened up. While I was walking in the Vosges this materialized, and I was able to complete the algebraic description of the equivalence classes of p-reps for several knots, including 9_{35}, cf. [11]. (I recall a puzzling difficulty with 9_{32} that was explained a decade later as the consequence of dropping the deck of data cards, perhaps in 1971, and reassembling it almost exactly right).

The knot 9_{35} has a large symmetry group (dihedral of order 12, [11]), and also an unusually large number of algebraic equivalence classes of p-reps, facts which I believe are related. The SAC calculations had given me a polynomial $p(x) \in \mathbb{Z}[x]$ of degree 25 which I had to factor as the first step. When one has no symbolic manipulation package available this is done by finding the roots of $p(x) = 0$ and examining them for clues. The polynomial $p(x)$ (and its relative for 9_{48} which was even worse) defeated several commercially produced root-finding routines, but a final resort routine succeeded, sort of, and I was able to infer factors

$$p_1 = 1 + x, \quad p_2 = 1 + 2x + 7x^2 + 5x^3 + x^4, \quad p_3 = \cdots ,$$

and soon

$$p(x) = (1+x)^{10} p_2(x)^2 \cdots .$$

Only the cubic factor remained unguessed, and of course it turned out to be the one giving the hyperbolic structure four years later. Each factor $p_k(x)$ of $p(x)$ had to be tested to see if it gave an equivalence class of p-reps or was spurious, and I expected $1 + x$ to be spurious. To my surprise it gave p-reps on

$$G_i = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, A_i \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} A_i^{-1} \right\rangle \subset SL_2(\mathbb{Z}[i]), \quad A_i = \begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix},$$

where $i = \sqrt{-1}$. This was in June 1973, and I probably did not understand what a Kleinian group is at the time, but I could see G_i is discrete and wondered what its presentation was. Also, as I watched the printout emerge from the line printer I guessed that these p–reps must be an instance of an undiscovered theorem, and the same evening stated and proved the theorem. (Writing it up for publication is taking longer. In December 1991 I used Maple to extend the theorem to algebraic varieties of nab–reps and add some new material. In 1993 I told Tomotada Ohtsuki about this, giving no detail, and he promptly found a better proof and more new material. I hope to proceed to a joint paper soon.)

After the summer vacation of 1973 when I returned to Southampton, the professors of the mathematics department granted me the use of an office and all university facilities, except the computer which was *heavily* overloaded. By then I had learned by some osmosis what a Kleinian group is and read Maskit's paper [4] on Poincaré's Theorem on Fundamental Polyhedra. This made progress on G_i above possible, and I soon had its presentation. (I also found that Fricke and Klein had considered G_i, or something very like it, cf. Fig. 151 on page 452 of [3]). Success with G_i led to success with the image $\pi K\theta$ of a p–rep of the figure–eight knot group in November 1973. Recall that

$$\pi K\theta = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -\omega & 1 \end{bmatrix} \right\rangle, \quad \omega = \frac{-1+\sqrt{-3}}{2},$$

so the group is obviously discrete and only its presentation was in doubt. I remember my surprise at finding this p–rep is faithful. The first version of my account [8] of this was received by the Editors on 30 November 1973, and it didn't mention the orbit space $\mathbb{H}^3/\pi K\theta$ because I had not even thought of it.

Why not?! Well, the result was perhaps a fortnight old, and I didn't have a premonition of hyperbolic structure on knot complements. Years later I learned that it had not only been thought of, but attempted and discussed privately by the Kleinian groupies since 1968. Nothing had been written and none of this had reached me. The key to seeing that the orbit space of $\pi K\theta$ had to be the figure–eight complement was seeing the peripheral torus in the orbit space. This torus occurs as the image of Euclidian plane $\Pi(t) = \{(z,t) : z \in \mathbb{C}\} \subset \mathbb{H}^3$ for any $t > 1$. In my diagram $\Pi(t)$ meets the fundamental domain not in a parallelogram but in a zigzag shape (four hexagonal discs), and perhaps the zigzag temporarily prevented me from seeing the torus. This is silly, because the stabilizer of the torus is the free abelian group $(\pi K\theta)_\infty$ generated by $z \mapsto z+1$, $z \mapsto z+2\sqrt{-3}$, and $(\pi K\theta)_\infty$ has to be considered explicitly during the verification that Poincaré's theorem applies to my supposed Ford fundamental domain. But silly or not, it took perhaps seven weeks, till January 1974, for me to see the torus. Verification that $\mathbb{H}^3/\pi K\theta = S^3 -$ fig–eight took perhaps a day, and consisted of looking at my reprint of Waldhausen's paper [16]. It seems unfortunate that this was too easy, and that I should have been forced to develop a direct geometrical argument, but once the pressure was off I didn't want to do it. I expect that a direct geometrical construction works for all non–torus two bridge knots, and that it would prove the conjectures of [12 §4], so the matter will not be a waste of effort.

The figure–eight discovery was not decisive for me as it was for Thurston. I expected that Shimizu's lemma, viz.

$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\rangle \text{ is not discrete when } ad - bc = 1, \quad 0 < |c| < 1,$$

would preclude the discreteness of the images $\pi K\theta$ of the potentially faithful p–reps θ for all other knots. (In particular, I predicted Alan Reid's theorem [5] that the figure–eight is the only arithmetic hyperbolic knot). However, by the time I mailed off the revised version of [8] that was actually accepted I had recognized the true situation, but, I suppose out of laziness, I didn't revise [8] again to make an announcement.

R.H. Fox died within a few days of the figure–eight discovery.

3. The intermediate example

I now had a beautiful discovery, and a certain fear of testing whether something similar was true for the obvious next case, the knot 5_2 of two–bridge types $(7,3)$, $(7,5)$. Instead of going for 5_2 directly I temporized by taking up a different kind of example, the groups $\pi K\theta$ associated to a cubic factor $f(u)$ of the p–rep polynomial for the knot 8_{11} of two bridge types $(27,17)$, $(27,19)$. To give an account of this we need to recall the basics of two bridge knot groups and their p–reps.

A two bridge knot normal form corresponds to a pair (α,β) of integers, where $\alpha > 1$ is odd, β is odd, $gcd(\alpha,\beta) = 1$, and $-\alpha < \beta < \alpha$. The knot group πK for (α,β) depends not on β itself but on $|\beta|$, so we may as well assume $0 < \beta < \alpha$. Then

$$\pi K = |x_1, x_2 : wx_1 = x_2 w|, \quad w = x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_2^{\epsilon_{\alpha-1}}, \tag{3.1}$$

where $\epsilon_j = \epsilon_{\alpha-j} = \pm 1$, and the exponent sequence $\vec{\epsilon} = (\epsilon_1, \cdots, \epsilon_{\alpha-1})$ is determined by a simple rule, cf [7, 12]. A longitude γ_1 in the peripheral subgroup $\langle x_1, \gamma_1 \rangle$ of x_1 is a certain word $\tilde{w}^{-1} w x_1^{2\sigma}$ on x_1, x_2. A normalized p–rep $\theta = \theta(\omega)$ of πK is a homomorphism such that

$$x_1\theta = A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad x_2\theta = B = B_\omega = \begin{bmatrix} 1 & 0 \\ -\omega & 1 \end{bmatrix}, \tag{3.2}$$

where $\omega \in \mathbb{C}$. Indeed, ω is a root of the p–rep polynomial $\Lambda[u] \in \mathbb{Z}[u]$ which may be reducible but which has no repeated roots. Then the longitude entry $g(\theta)$ or $g(\omega)$ for $\theta(\omega)$ is found by

$$\gamma_1\theta = \begin{bmatrix} -1 & g(\theta) \\ 0 & -1 \end{bmatrix},$$

and is readily computable once ω is known. To factor $\Lambda(u)$ without a system like SAC, Macsyma, or Maple but when a polynomial root finding package is available find the roots and list the pairs $(\omega, g(\omega))$. Factors stand out as having pairs where $g(\omega)$ evidently belongs to a proper subfield of $\mathbb{Q}(\omega)$. In the case of 8_{11} we found the factor

$$f(u) = -1 + u(1+u)^2$$

by $g(\theta) = -6$ for its roots. The roots of $f(u)$ are

$$\omega_1 = -1.23278 + 0.79255i, \quad \omega_2 = \bar{\omega}_1, \quad \omega_3 = 0.46557, \tag{3.3}$$

(rounded to 5 decimal accuracy). Today this factor is explained as an instance of Theorem B of [12], and it clearly had something to do with the discovery of the theorem. I had $f(u)$ by 1971.

By February 1974 my worries about the figure–eight knot brought me to consider the group $\Gamma = \langle A, B \rangle$, $B = B_\omega$, where ω is the ω_1 of (3.3). I simply went for a Ford domain $\mathcal{D}$ of Γ using graph paper, compass and ruler, and the first programmable calculator available at Southampton. (That would have cost about two months gross salary if I had still been employed). It didn't take long to get the diagram of Fig. 1, and when the time came to think about proof the closing trick and angle sum trick of [10] came to mind automatically. As far as I know this group Γ is the first group proved discrete by Poincaré's theorem where these tricks are necessary. Perhaps the first people to wonder about using Poincaré's theorem for computation with potentially discrete groups didn't see these simple tricks in advance, didn't have a specific example they really needed, and shied away from getting too involved.

We give a little more detail on Γ and its Ford domain $\mathcal{D}$ illustrated in Fig. 1. This is taken from an unpublished paper CPG, written in late 1974 and early 1975, doing all the discrete non Fuchsian cases where the group $\pi K\theta$ corresponds to a root of a cubic polynomial, viz. 5_2, 7_4, and 8_{11}. The case 5_2 is worked in [11], and 7_4 is similar to but easier than 7_7 also worked in [11]. Our group Γ is somewhat like the modest example of [10] but much easier.

Let πK be the group of $(27, 17)$ presented as in (3.1), so $\Gamma = \pi K\theta$ as in (3.2). We have words

$$u := x_1^{-1}x_2x_1, \quad v_1 := ux_2^{-1}x_1x_2^{-1}, \quad w_1 := v_1x_1^{-1}v_1^{-2}x_2^{-1}v_1x_1^{-1}.$$

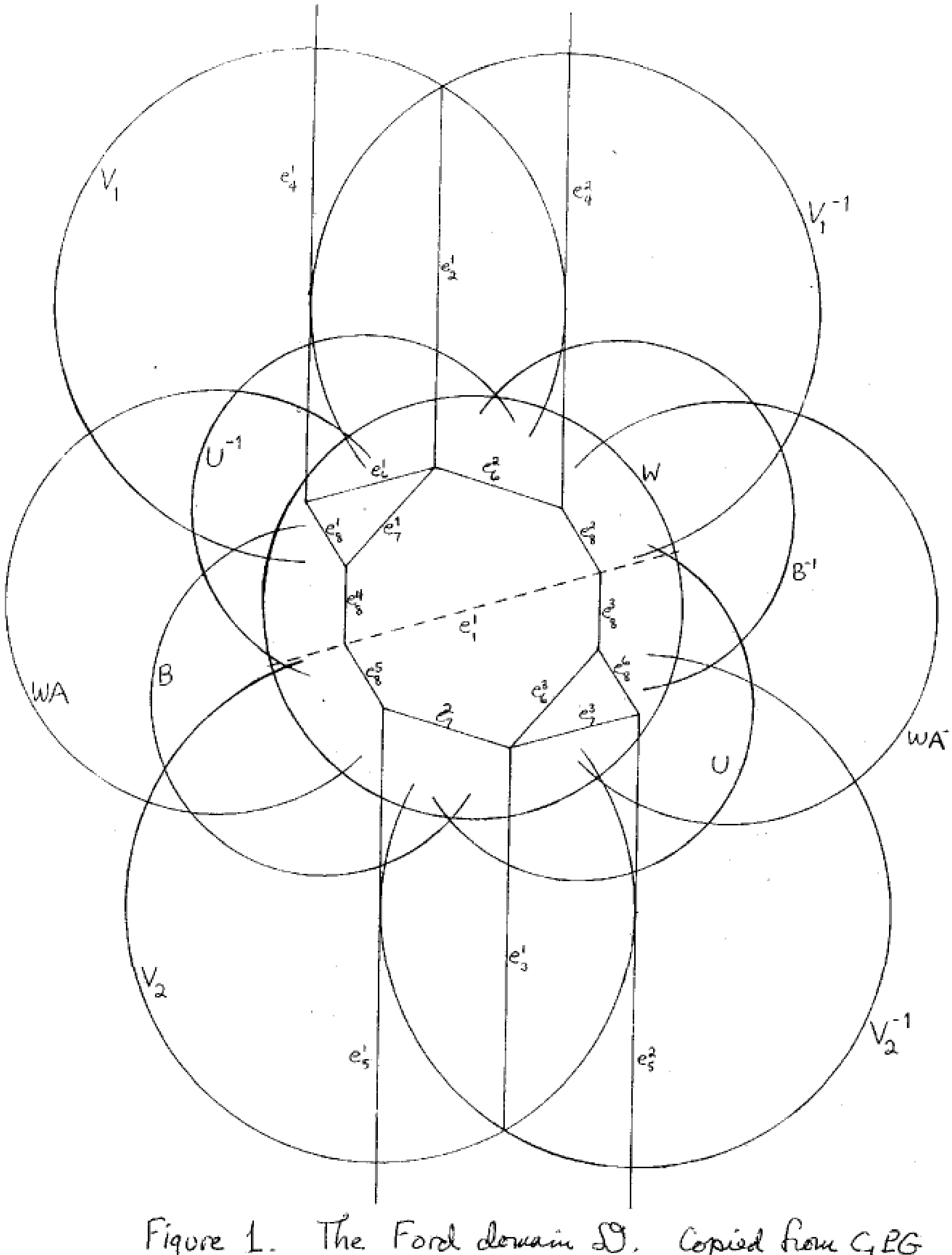

Figure 1. The Ford domain $\mathcal{D}$. Copied from GLPG (I get a new plotter soon).

The word w of (3.1) is $w_1 x_1$, so $w_1 x_1 = x_2 w_1$ holds in πK. These words u, v_1 were found by straightforward search of subsegments of w to correspond to spheres carrying sides of tentative Ford domains. The search for the sides of a fundamental domain has to be guided by some principle, since a Cantorian exhaustion is too slow, and segments of w worked well, both here and later for all two bridge knots.

We found easily that the elements

$$A = x_1\theta, \quad U = u\theta, \quad V_1 = v_1\theta, \quad W_1 = w_1\theta, \quad V_2 = U^{-1}W_1$$

seem to be the side pairing transformations of the tentative Ford domain $\mathcal{D}$ of Fig. 1. Thus we read off from Fig. 1 a proposed presentation for Γ: generators A, U, V_1, V_2, W_1. relations

$$W_1^2 = V_1^3 = V_2^3 = (A^{-1}V_1)^2 = (A^{-1}V_2)^2 = E,$$

$$V_1 = W_1 U^{-1}, \quad V_2 = U^{-1}W_1, \quad U = A^{-1}W_1 A W_1 A^{-1}.$$

To use the closing trick and angle sum tricks of [10] it is necessary to verify directly that these relations hold in Γ. For this it helps to see copies of the modular group $SL_2(\mathbb{Z})$ in Γ. Let

$$A_* := \begin{bmatrix} 1 & u+u^2 \\ 0 & 1 \end{bmatrix},$$

then

$$V_1 \equiv A_*^{-1} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} A_*, \quad V_2 \equiv A_* \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} A_*^{-1} \pmod{f(u)}.$$

So $\langle A, V_1 \rangle$ and $\langle A, V_2 \rangle$ are conjugate to $SL_2(\mathbb{Z})$ in $SL_2(\mathbb{Z}[\omega])$. All the proposed relations now can be verified by straightforward computation in $SL_2(\mathbb{Z}[u])$ modulo $f(u)$. Then the arguments of [10] show that Γ is discrete, $\mathcal{D}$ is a fundamental domain for it, and that these relations present the group. This made a good confidence–building exercise for me, and might do the same for other people. Note that this $\mathcal{D}$ is simpler than the Ford domain for 5_2 discussed in [11], so Γ really is an intermediate example.

4. Completion of the discovery

This procrastination had now given me a bigger worry which can be put thus: Why should the Great Lord have performed a unique miracle to make Γ discrete, for no visible reason at all?! The answer is compelling: He didn't! If Γ is discrete then many other groups have to be discrete, in direct defiance of Shimizu's lemma, and, since each case of discreteness requires a good reason, there must be general theorems explaining this discreteness. It is a little ironic that this prediction was amply vindicated for (suitable) 3-manifold groups, but, at this writing, the general theorem explaining the discreteness of Γ has not been stated, let alone proved.

During a few weeks further procrastination the above considerations compelled me to predict that a knot in S^3 is hyperbolic unless it clearly was not. Early in March 1974, I think, I finally went to work on 5_2 and in a few hours had confirmed my prediction. This completed the essential part of my discovery, and all later cases, such as 7_4 and several links, were just routine examples at most illustrating matters of secondary importance, such as the symmetries of a knot. In fact, for a while I was confused by the symmetries and thought that a too-rich symmetry group would preclude the hyperbolic structure, but I eventually found my mistake. So by late 1974 I had gotten it right: a knot is hyperbolic unless its group contains a noncyclic abelian subgroup which is not peripheral. Making bold sweeping conjectures is unnatural for me, and I didn't venture to predict anything about arbitrary 3-manifolds. I suppose that I might have predicted which 3-manifolds were hyperbolic had someone pressed me on the issue in conversation, but I was too isolated and unknown for that to happen. The locals at Southampton were rather cool about the whole project, except for David Singerman. He liked it enough to propose that we try to get the Science Research Council (of Great Britain) to support me on a hyperbolic project at Southampton University while I got my Ph.D. and looked for a permanent job. His plan was to time the submission of the proposal so that the referee would be at the summer 1975 conference on Kleinian groups at Cambridge where I would publicize hyperbolic structure. Whether or not the plan worked, the Kleinian groupies liked my examples, especially because these examples pointed up the importance of their own work. The SRC did fund the project generously, ultimately for four years 1976–1979.

The first two years of the project were devoted to the development of the system PNCRE [10], a file of Fortran subroutines to compute with explicit subgroups of $SL_2(\mathbb{C})$. PNCRE was not easy to develop and its first output came early in 1977. Meanwhile, about March 1976, a colleague gave me a preprint of Thurston's lecture [13] on folliations of surfaces. This was the first I heard of him, and I recall that on reading it I became certain that he and I would never share any common mathematical interest. In late June 1976 a friend drove me up to the University of Warwick to hear a lecture by J. Milnor on topics like Sarkovskii's theorem. Directly he was finished I very nervously (read: scared stiff) introduced myself to him and told him about examples of hyperbolic knots/links. He was interested, and asked a number of direct questions, so that in a minute he understood the status of my project (examples only). I did not guess that he already knew something about the matter. I was so scared that when he asked me to repeat my name I simply ran away. But perhaps even before we got back to Southampton that evening, Milnor had asked the locals who in Britain was interested in hyperbolic structure on knot complements, and directly afterwards Thurston had his hands on my two papers. If not, he did when I sent my papers to Milnor the next week (early July 1976).

Later that month I was invited, to put it mildly, to spend a week in David Fowler's home in Warwick. His wife is French, and she felt that that year she simply had to bring the children to France to meet their relatives. She naturally had the house and garden filled with beautiful plants which need constant watering. The summer of '76 was a famous drought in which the water shortage was so severe that the only legal water for plants was used bath water. Hence the urgent need to have the Fowler's home occupied every night, and David Rand, who had been a student at Southampton and was taking up a lectureship at Warwick, put me down for one week.

On my arrival in the common room of the Warwick Mathematics Department, David Epstein sprang up and asked me who I was. He had seen my face on numerous occasions over the years, most recently when I sat directly behind him in Milnor's lecture, and wanted to know. On hearing my name, a tall man sprawled over three chairs sprang up. He said he was Bill Thurston, that he wanted to meet me, and that for about a year he had been working on a general conjecture which included everything I was doing. The shock was immense. I am afraid that I react badly to surprises, and I became quite unpleasant for the rest of the week. Fortunately Bill didn't hold it against me later. His later statement (page 177 of [15]),

> "...; and I have not actively or effectively promoted the field or the careers of the excellent people in it."

was either not written with me in mind or he judges me not to satisfy the qualification. He certainly did advance my career actively: strong letters of recommendation, several thousands of dollars from his Waterman Fellowship, inclusion in the 1980–81 Thurston–Sullivan NSF project at Boulder, and a trip to Binghamton at my request. I owe everything to the people who have so generously supported me over the years when I needed help most: H.B. Griffiths, David Singerman, and Bill Thurston, and I am deeply grateful to them all.

5. Hindsight

The question is: Why did the explicit discovery of hyperbolic structure on at least some knot complements wait until 1974? Wilhelm Magnus told us that H. Gieseking, in a thesis written in 1912 under the direction of Max Dehn, considered a group G_1 of hyperbolic isometries of a ball model B^3 of $\mathbb{H}^3$ and certain of its subgroups. The fundamental domain for G_1 is a regular ideal tetrahedron T_1, and G_1 contains orientation reversing elements. Gieseking considered the orientation preserving subgroup G_2 of index two whose fundamental domain is two tetrahedra glued together along a face, without recognizing that G_2 is isomorphic to the figure–eight knot group and its orbit space is the figure–eight complement. Magnus told me that Dehn considered these groups only as exercises in geometric symmetry: the geometric description of G_1, T_1 is so simple that Poincaré's theorem simply has to apply directly. If Dehn had known that the figure–eight knot was involved he certainly would have had Gieseking publish, he would have given the matter the greatest publicity, and the development of 3–manifold theory would have gone very differently. So why did they not recognize the figure–eight complement?

I propose an answer to this question analogous to my own experience: I didn't see the peripheral torus for several weeks but when I did I knew what I had to have. I began with the figure–eight and had it in mind. They began with an exercise in symmetry and had nothing further in mind. Furthermore they would have to ask the question: for $\epsilon > 0$ let S_ϵ be the 2–sphere in B^3 with centre $\vec{0}$ and radius $1 - \epsilon$. Then G_1 maps S_ϵ to itself, so what is the orbit space S_ϵ/G_1? With hindsight the answer is obvious: a Klein bottle. Dehn would have answered this question easily once it had been raised, and I feel certain the Klein bottle would have disturbed him deeply. The result would have burned within him until he was driven to get to the bottom of the matter, and somehow he would have found the figure–eight. They had about two years to do this before the Great War of 1914–18 swept Gieseking to his doom. Dehn's good students were probably all destroyed, and most likely Dehn was so distraught at their loss that he couldn't bear to think about his joint projects with them any longer.

As far as I know, the next time critical examples of hyperbolic structure on 3–manifolds should have been found was in the late 1950's, during the period of euphoria caused by Papakyrikopoulos' breakthrough with his proofs of Dehn's lemma etc. The topic was definitely thought of, but nothing happened, perhaps because the man concerned did not have anything specific to work on, and he certainly had a lot of other important projects to pursue. In 1968 a Kleinian groupie wondered whether a knot complement could be hyperbolic, and chose as example to test this idea the trefoil knot. He soon found it didn't work and was discouraged. (Actually, the trefoil complement does carry hyperbolic orbifold structures of infinite volume, but nobody wanted that). In the early 1970's he actually visited Southampton University and met me, but somehow the crucial topic didn't come up in the discussion. If it had I would have put him onto the figure–eight and even given him the exact matrices to use. I could not have done the calculation with Poincaré's theorem at the time (he could), but I did have Waldhausen's paper to help with identifying the orbit space.

I would like to close by quoting a paragraph from page 175 of Thurston's essay [15].

> "Neither the geometrization conjecture nor its proof for Haken manifolds was in the path of any group of mathematicians at the time — it went against the trends in topology for the preceding 30 years, and it took people by surprise. To most topologists at the time, hyperbolic geometry was an arcane side branch of mathematics, although there were other groups of mathematicians such as differential geometers who did understand it from certain points of view. It took topologists a while just to understand what the geometrization conjecture meant, what it was good for, and why it was relevant."

Well, this is not quite right. For one thing, it is too strongly put. When I meet them in 1975 the Kleinian groupies had been knowledgeable about the hyperbolization conjecture for Haken manifolds for at least a couple of years, but they saw it as too much for themselves. For another, what really took people aback was the speed with which the task was completed (excepting the write–up). Thurston simply didn't give anyone starting from my examples the time to get involved. And few serious mathematicians would look at one modest example of something pretty and immediately formulate the most sweeping conjecture for 3–manifolds which could possibly be true, and then plunge in. Thurston's success at doing this is his own personal triumph, and not a closing out of a golden opportunity that the rest of us were fool enough to lose.

I had thought of saying something about the history of Bill Thurston's thinking about hyperbolic structure in the two years before we met, but I am afraid to repeat Colin Adam's mistake. There are rumours that he initially thought the hyperbolic structure for the figure–eight was impossible, because of difficulties with the lift of a Seifert surface to $\mathbb{H}^3$, and that he discussed these matters with William Jaco at a conference. The story continues that when Bill got back to Princeton he found his supposed contradiction disappear (the lift of the Seifert surface meets the sphere at infinity in a Peano curve), that this completely reversed his expectations, and that he first got the figure–eight out of an example of Troels Jørgensen. I cannot vouch for any of this.

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