KONIGSBERG BRIDGES PROBLEM. Graph theory was born when a Swiss mathematician named Leonhard Euler (pronounced "oiler") solved the problem of the Konigsberg Bridges. It is said that the people of Konigsberg amused themselves by trying to devise a walking path around their city which would cross each of their seven bridges once and only once and return them to their starting point. As it happened no one ever found such a path and so people naturally suspected that no such path existed. This problem came to the attention of Euler and he subsequently published his solution in *Commentarii Academiae Scientarum Imperialis Petropolitanae*. In short, he was able to prove that it was impossible to devise a path that crossed each of the bridges only once. First he drew a sketch of the town labeling the four land masses as A, B, C, and D and the seven bridges a, b, c, d, e, f, and g. This is shown below.

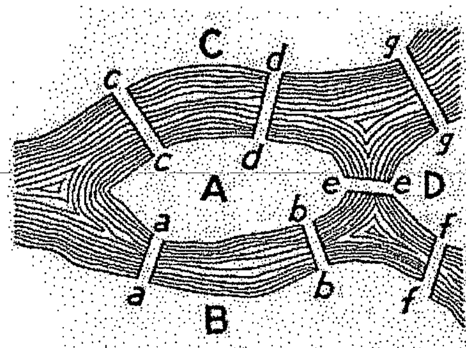
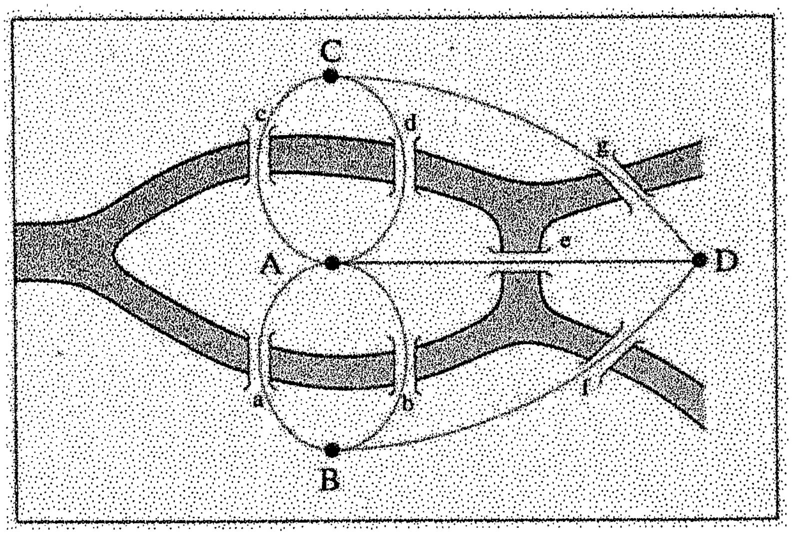


FIGURE 98. *Geographic Map: The Königsberg Bridges.*

Euler then simplified this picture by drawing a graph with the four land masses represented by points and the seven bridges by lines which connect the points. This produced the following graph.



At this point the Konigsberg Bridges problem is "set up" and ready for analysis. Euler started by representing the walking trip as a sequence of the land mass letters A, B, C, and D. Between adjacent letters, of course, a bridge is crossed. He then noted that land mass A has 5 bridges crossing into it or out of it and the other land masses 3 such bridges. The final question he asked himself was how many appearances of each land mass letter did there have to be in the sequence of letters. With this beginning you may now be able to complete a proof that the Konigsberg Bridges problem has no solution.

Adapted from "Problem Solving Across the Disciplines" by R. R. Kadesch, Prentice Hall, 1997.

Return to Honors PS1500 home page

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Theorem. For n a natural number ≥ 1,

$$1+2+3+\dots+n = \frac{n(n+1)}{2}.$$

Proof #1.

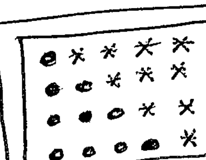
Let $x = 1+2+\dots+n$

Then $x = n+(n-1)+\dots+1$

Hence $2x = \underbrace{(n+1)+(n+1)+\dots+(n+1)}_{n \text{ times}}$

$\therefore\ 2x = n(n+1)$

$\therefore\ x = n(n+1)/2.$ //



$4\times 5 = 2\times(1+2+3+4)$

Proof #2. We prove the theorem by induction on $n \in \{1,2,3,4,\dots\} = N$.

I. Let P_n denote the proposition
" $1+2+\dots+n = n(n+1)/2$ ".

Thus $P_1 : 1 = 1\cdot(1+1)/2$

$P_2 : 1+2 = 2\cdot(2+1)/2$

$P_3 : 1+2+3 = 3\cdot(3+1)/2$

Each of these is true. Thus P_n is true for $n=1$ (and $n=2$, $n=3$).

II. Now assume that P_K is true for some K.

then $(1+2+\dots+K)+(K+1) = \dfrac{K(K+1)}{2} + (K+1)$

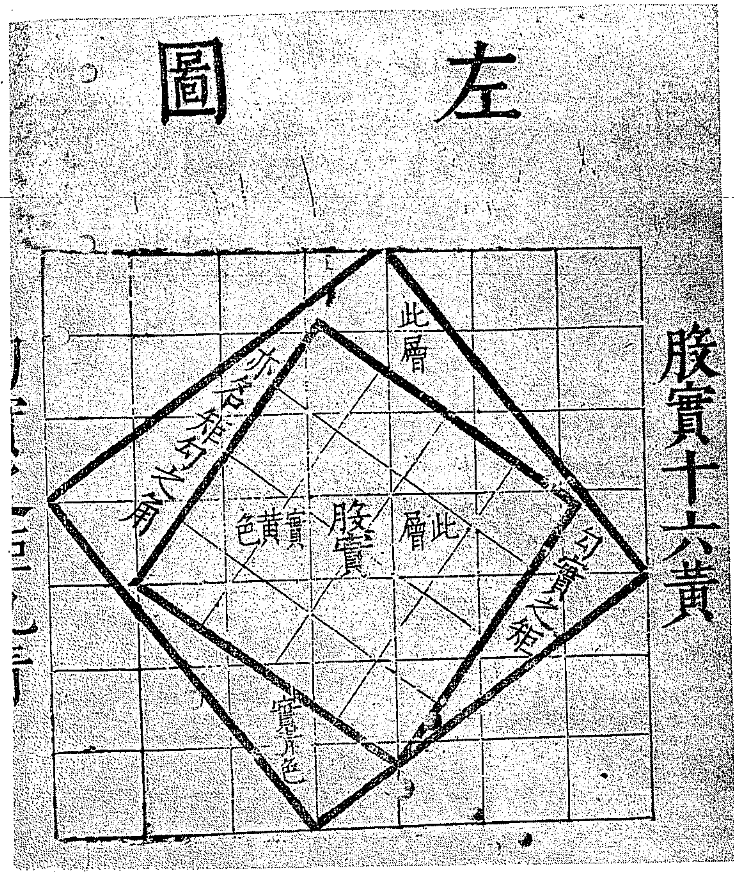
$= (K+1)\left(\dfrac{K}{2}+1\right) = (K+1)\dfrac{(K+2)}{2}$

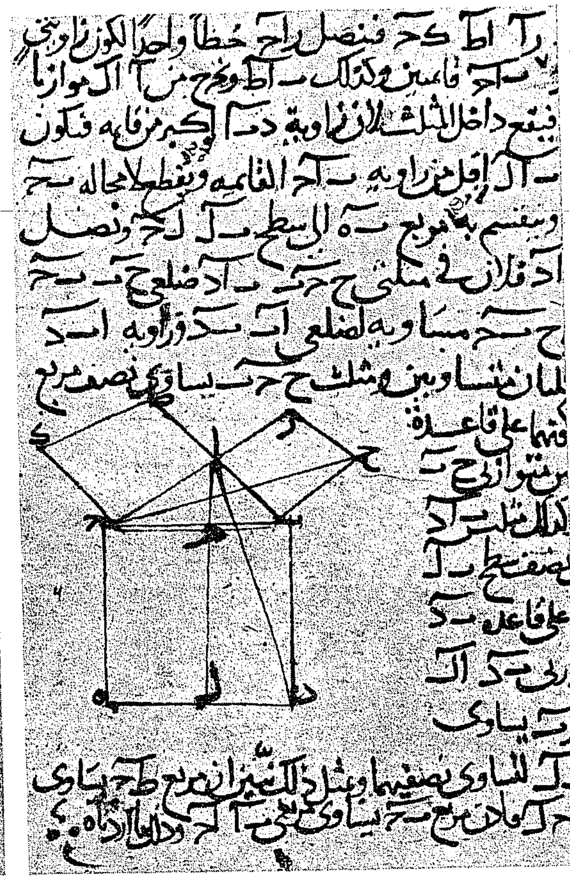
$= (K+1)((K+1)+1)/2.$

Thus $(P_K \Rightarrow P_{K+1})$ is true for all $K \in N$.

Since we have proved that P_1 is true, this completes the inductive proof of the Theorem. //

Discussion Problem: Compare $\boxed{\text{Proof \#1}}$ and $\boxed{\text{Proof \#2}}$.





From:
"The Ascent of Man"
by
Jacob Bronowski
(1973)

seems a bold and extraordinary thing to say, yet it is not extravagant; because what Pythagoras established is a fundamental characterisation of the space in which we move, and it is the first time that is translated into numbers. And the exact fit of the numbers describes the exact laws that bind the universe. In fact, the numbers that compose right-angled triangles have been proposed as messages which we might send out to planets in other star systems as a test for the existence of rational life there.

The point is that the theorem of Pythagoras in the form in which I have proved it is an elucidation of the symmetry of plane space; the right angle is the element of symmetry that divides the plane four ways. If plane space had a different kind of symmetry, the theorem would not be true; some other relation between the sides of special triangles would be true. And space is just as crucial a part of nature as matter is, even if (like the air) it is invisible; that is what the science of geometry is about. Symmetry is not merely a descriptive nicety; like other thoughts in Pythagoras, it penetrates to the harmony in nature.