CHAPTER II

LINKS AND O(n)-MANIFOLDS

The first two sections of this chapter deal with invariants of links in the three sphere. We show that the Seifert pairing may be computed from a projection of the link and relate this to the Murasugi signature of the link. Then quadratic forms for the link are discussed. The symmetry group of a link is defined and computed for torus links. Then we discuss the equivariant classification of O(n)-manifolds with orbit space D^4 and fixed point set corresponding to a link in $\partial D^4 \approx S^3$ ("link-manifolds"). The equivariant classification of link manifolds is related to the symmetry group of the link. Finally, we show how the diffeomorphism classification of a link manifold is determined by invariants of the corresponding link. The chapter ends with some computations and examples.

1. Invariants of Links in S^3

Here we discuss the signature and quadratic form of a link and show how these are related to the Seifert pairing and to invariants of the double branched cover of S^3 branching along the given link. These invariants will be used in Section 4 to classify O(n)-manifolds.

Let $L \subset S^3$ be a link. L will also denote a given projection of the link onto S^2 with only double-point intersections. Assume that each component of L is given an orientation.

Assuming that the link diagram (i.e., the projection on S^2) is connected, Seifert's algorithm to form a spanning surface for the link proceeds as follows. One divides S^2 into regions bounded by various circles obtained from the link diagram. Each circle is obtained by choosing a point on the projection which is not a crossing point and traveling along the projection in the direction of its orientation to a small neighborhood of a cross point. Suppose that the neighborhood of the cross point contains oriented line segments s_1 and s_2 crossing at $P = s_1 \cap s_2$. Then $s_1 = s_1' \cup s_1''$, $s_2 = s_2' \cup s_2''$, $s_i' \cap s_i'' = P$. If you are on, say, s_1', then it will be possible to cross over to either s_2' or s_2'' (not both) and continue your oriented journey avoiding the cross point (see Fig. 1). Continue in this fashion until returning to the starting point. This traces out one of the circles.

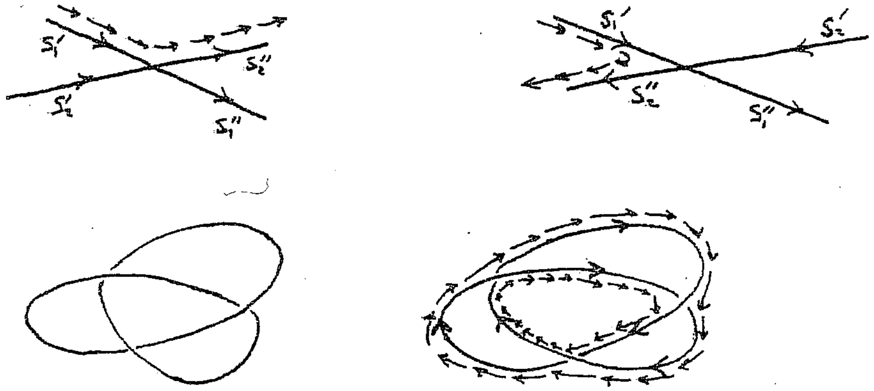


Figure 1

Call such a circle *black* if one of the two regions into which it divides S^2 does not contain any other circles. Non-black circles are *red*. If a region in S^2 has boundary a black circle and contains no other circles, *color it black*. S^2 is now divided into white (uncolored) and black disks and, less finely, into various regions bounded by red circles (see Fig. 2).

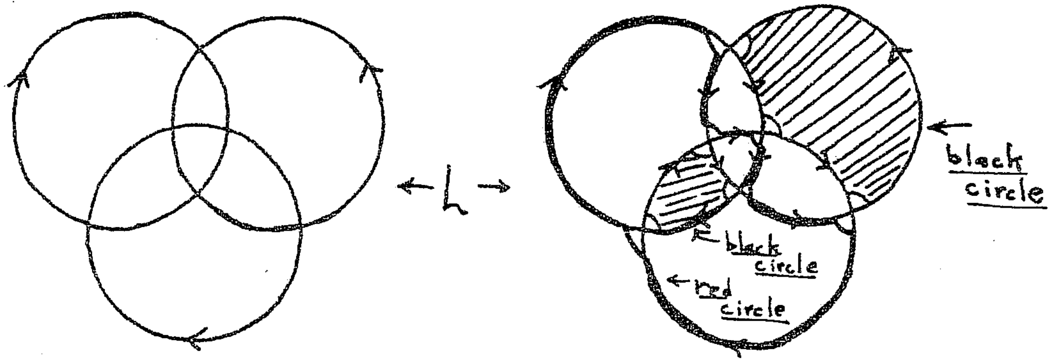


Figure 2

Now regard the black regions as disks. Attach disjoint disks to the boundaries of the red circles. Complete by filling in twists at the crossings to connect the disks (see Fig. 3).

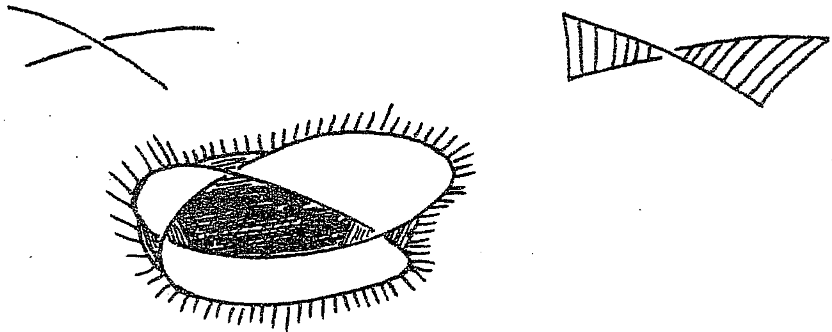


Figure 3

The result is a compact surface $F \subset S^3$ with boundary L. F is called the Seifert surface for L corresponding to the given projection.

Lemma 2.1. F is orientable.

Proof. This is clear. Orient each disk according to its already oriented boundary and note that on crossing a twist from one disk to another the orientation changes appropriately.

Orienting F as above, let

i_* ≡ push off F in direction of the positive normal.

i^* ≡ push off F in direction of the negative normal.

Then one has the Seifert pairing

$$\theta\colon H_1(F) \times H_1(F) \to \mathbb{Z},\ \theta(x,y) = \ell(x, i^*y).$$

We wish to compute θ in terms of information which can be read from a link projection. The first task is to indicate a convenient set of generators for $H_1(F)$.

A. Homology of F

1) First suppose that the projection of L contains no red circles (call such a link projection a special projection). Then, except for the twists, F lies on S^2. Hence, F has the homotopy type of S^2 - {union of white regions}. For each white region W there is a loop w on F encircling it (see Fig. 4). Hence, if the white regions are $W_1, W_2, \ldots, W_{n+1}$ with corresponding loops $w_1, \ldots, w_{n+1}$, then any n of these loops will form a basis for $H_1(F)$.

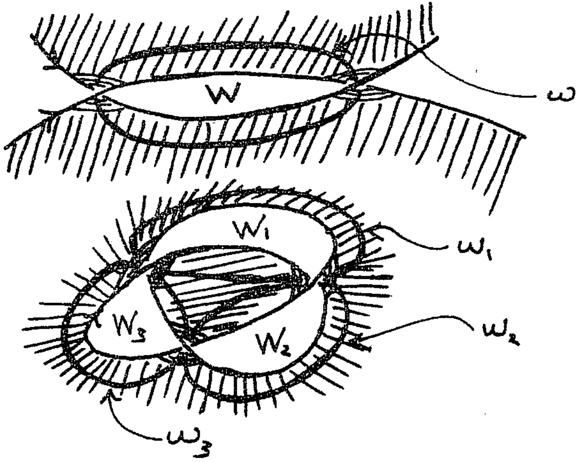


Figure 4

$\{w_1, w_2\}$ basis for $H_1(F)$

2) Now suppose that the link projection contains some red circles.

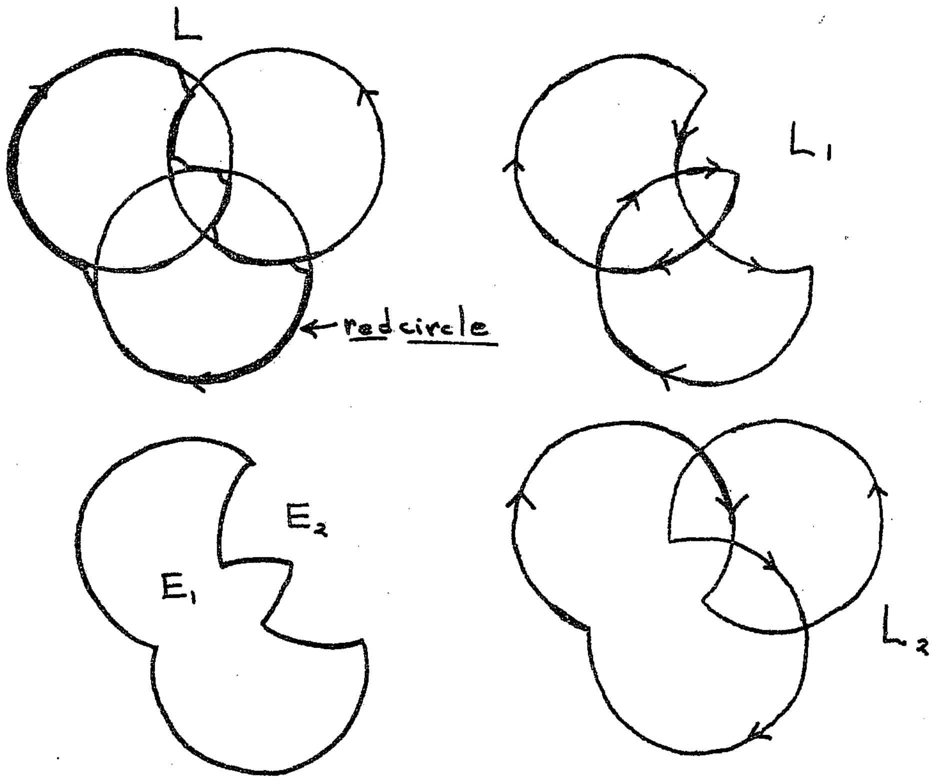


Figure 5

Let the red circles divide S^2 into regions $E_1, E_2, \ldots, E_m$. Thus, the boundary of E_i consists of one or more red circles and E_i contains no red circles in its interior.

It is clear that L may be written as the union of link projections $L_1, \ldots, L_m$ where $L_i = (L \cap E_i) \cup \partial E_i$ (see Fig. 5). Notation: $L = L_1 * L_2 * \ldots * L_m$. Each L_i is the projection of a well defined link. Furthermore, each L_i is a special projection (see Fig. 5 and 6).

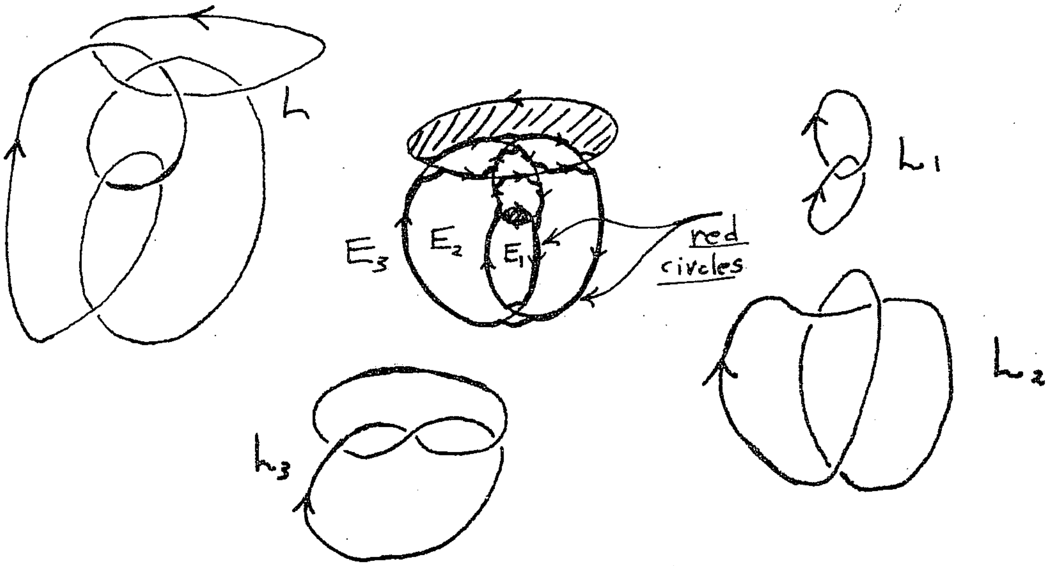


Figure 6

Number the white regions W_{ij} where the subscript j indicates that $W_{ij} \subset E_j$. Let w_{ij} be a loop on F encircling W_{ij}. Denote the white regions in E_j by $W_{1j},\dots,W_{n_j+1,j}$.

Proposition 2.2. $\{w_{ij} \mid 1 \le i \le n_j,\ 1 \le j \le m\}$ is a basis for $H_1(F)$.

Proof. Regard $L = L_1 * L_2 * \dots * L_m$. Let F_i = Seifert surface for F_i. By the above discussion, $F_i \overset{\alpha_i}{\hookrightarrow} F$ and $(\alpha_i)_*: H_1(F_i) \to H_1(F)$ is an injection. $F_i \cap F_j = L_i \cap L_j = \partial E_i \cap \partial E_j$ = union of red circles.

$H_1(F_i \cap F_j) \xrightarrow{0} H_1(F_i) \oplus H_1(F_j) \to H_1(F_i \cup F_j) \to 0$, $F = F_1 \cup \dots \cup F_m$.

Hence, by induction,

$$H_1(F) \cong H_1(F_1) \oplus H_1(F_2) \oplus \dots \oplus H_1(F_m).$$

This implies the proposition.

Thus, a basis for $H_1(F)$ is the result of choosing all but one white region from each E_i and taking the corresponding loops.

B. Seifert Pairing for F

1) Again suppose that the link has a special projection (no red circles).

Note that in order for the projection of L to contain no red circles, the segments on the boundary of each white region must alternate in orientation (see Fig. 7).

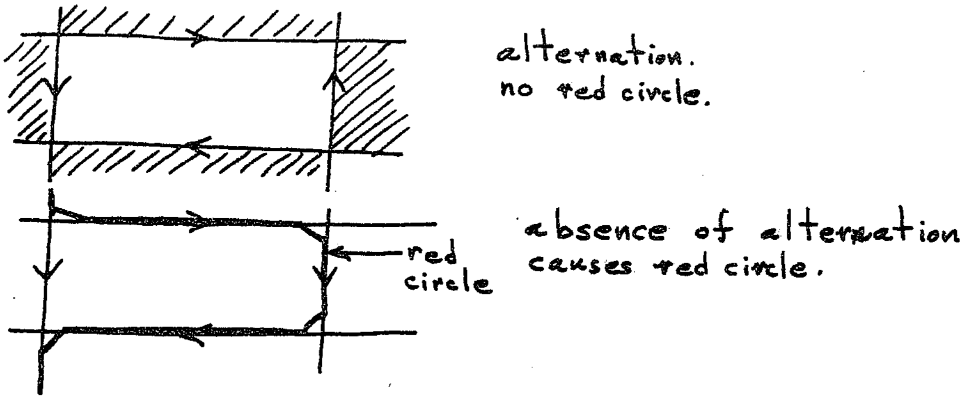


Figure 7

Thus, each crossing on the boundary of a white region is one of the two types illustrated in Figure 8.

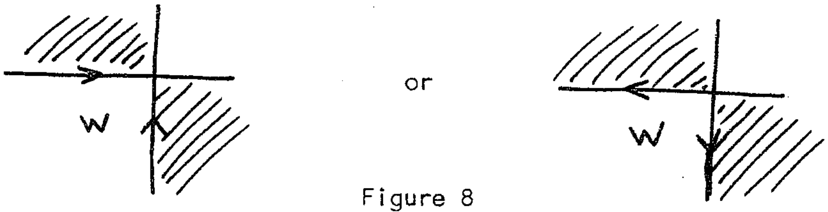


Figure 8

Choose an orientation for the Seifert surface as follows: Each black region lies on S^2 and has an oriented boundary corresponding to the orientation of the link. Let this boundary orientation determine the orientation of the black region. Determine a positive normal to the surface by the right-hand rule (see Fig. 9).

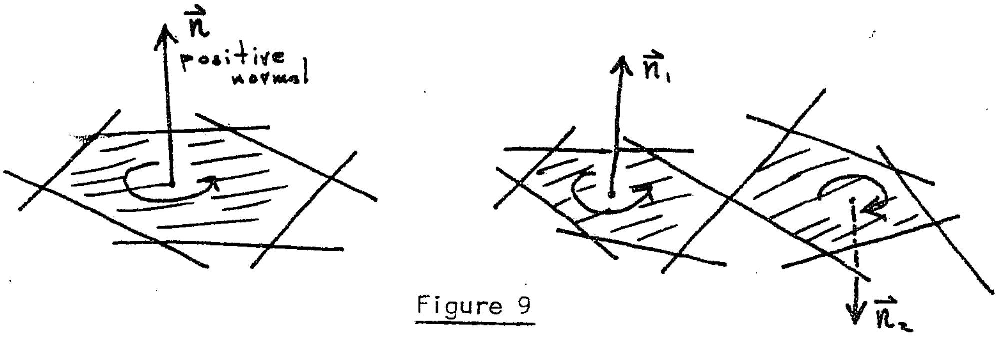


Figure 9

Now orient each of the loops w_i so that whenever w_i passes through a black region with "outward normal" (i.e., its normal agrees with the standard outward normal to S^2), it shares the orientation of the boundary of that region (see Fig. 10).

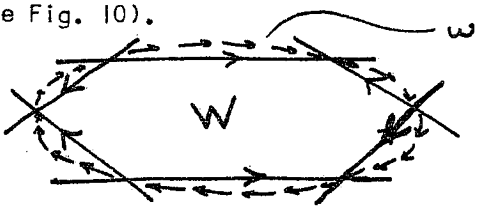


Figure 10

Notation.

Figure 11

Crossings will be denoted by the "double-dot" notation of Figure 11.

Calculation of θ now amounts to some case checking as illustrated in Figure 12. The result may be summarized as follows: Let c be a crossing common to two white regions W and W'.

Define $d_{WW'}(c) = +1, -1, 0, 0$, according as the crossing is of types i), ii), iii), iv) in Figure 12. Thus,

$$d_{WW'}(c) = \begin{cases} 0 & \text{if there is } \underline{\text{no dot}} \text{ at c } \underline{\text{in W}} \\ \pm 1 & \text{as in Figure 13.} \end{cases}$$

Figure 13

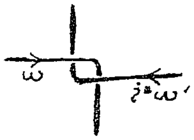
Hence, $$\theta(w,w') = \sum_{c\in\partial W\cap\partial W'} d_{WW'}(c)$$

$$\theta(w,w) = -\sum_{W'\neq W}\ \sum_{c\in\partial W\cap\partial W'} d_{WW'}(c)$$

Figure 12

i)

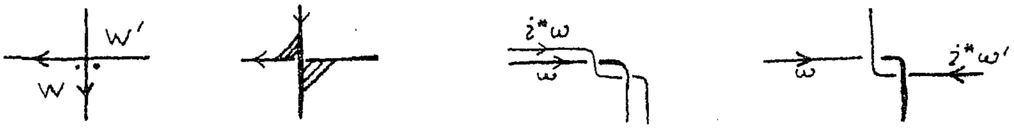
c contributes −1 to $\ell(w,i^*w)$



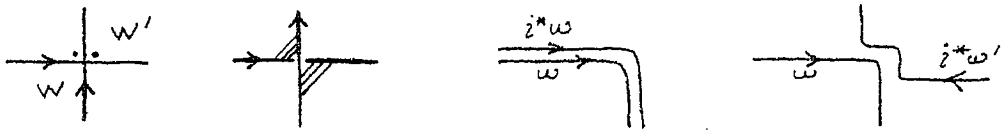
c contributes +1 to $\ell(w,i^*w')$

$d_{WW'}(c) = +1$

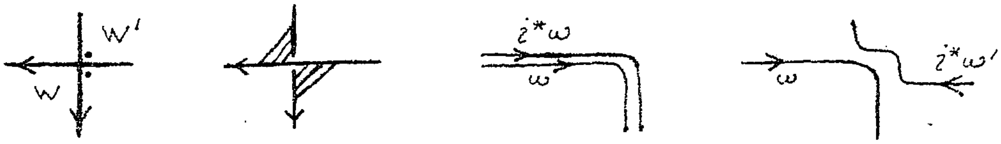
ii)



iii)



iv)



Examples. See Figures 13a, 13b, 13c.

Figure 13a

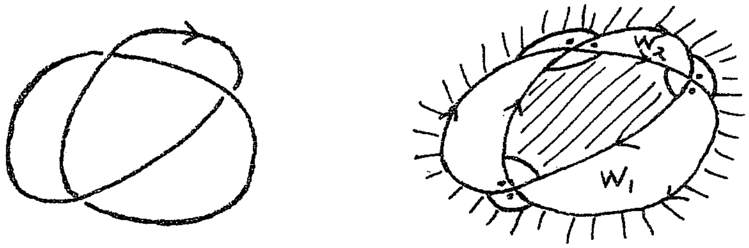


Figure 13b

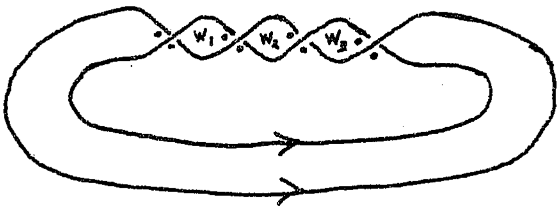
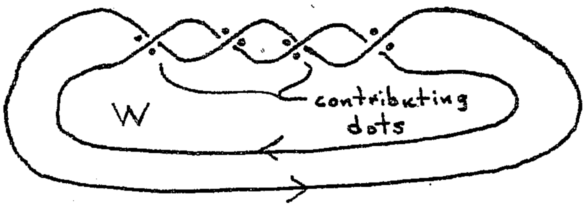


Figure 13c



a)

$\theta(W_i, W_j)$	W_1	W_2
W_1	1	-1
W_2	0	1

$V = (V_{ij}) = (\theta(w_i, w_j))$

Thus, $V = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ for the trefoil knot in 13a. Note that the intersection matrix for F is $\Delta = V - V' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The matrix Γ of Chapter 1 is $\Gamma = \Delta^{-1}V = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$. Since $\Gamma^k - (\Gamma - I)^k$ is a relation matrix for $H_1(M_k)$, it is easy to see the six-fold periodicity and calculate these groups.

b) $V = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ c) $V = [-2]$

2) Now suppose that the link projection contains some red circles. Using the notation of (2) of Section A write $L = L_1 * L_2 * \dots * L_m$, $L_i \subset E_i$, etc.

Lemma 2.3. Restricted to $\{w \mid W \subset E_i\}$ the Seifert pairing is the pairing for the link L_i.

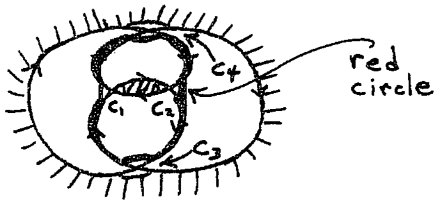
Proof. Suppose $W, W' \subset E_i$. Then the crossings which they have in common are crossings for the link L_i. Hence, if $c \in \partial W \cap \partial W'$ is such a crossing, then, since L_i is special, $d_{WW'}(c)$ is well defined and $\theta(w,w') = \sum_{c \in \partial W \cap \partial W'} d_{WW'}(c)$.

Note that if c is a crossing on ∂W, $W \subset E_i$ and c is not a crossing for any other $W' \subset E_i$, then c does not contribute to $\theta(w,w)$ since w will not pass through this twist. Hence,

$$\theta(w,w) = -\sum_{W' \subset E_i} \sum_{c \in \partial W \cap \partial W'} d_{WW'}(c).$$

This proves the lemma.

Definition. A crossing c belongs to a region E if it is common to two distinct white regions $\subset E$. See Figure 14.



E_1 = interior of red circle
E_2 = exterior of red circle
c_1, c_2 belong to E_1
c_3, c_4 belong to E_2

Figure 14

Remark. Always assume that the link projection contains no crossings of the type pictured in Figure 15. These can obviously always be eliminated.

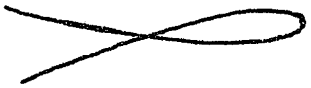


Figure 15

Next we must take care of the case $\theta(w,w')$ where $W \subset E_i$, $W' \subset E_j$, and $i \neq j$. First define an index $\eta(c) = \pm 1$ according as a crossing is a left or a right overpass (see Fig. 16).

Figure 16

Define $\varepsilon_{WW'}(c) = \begin{cases} 1 & \text{if } W \text{ is left of } W' \text{ with respect to the orientation of the red circle shared by } \partial W, \partial W'. \\ 0 & \text{otherwise.} \end{cases}$

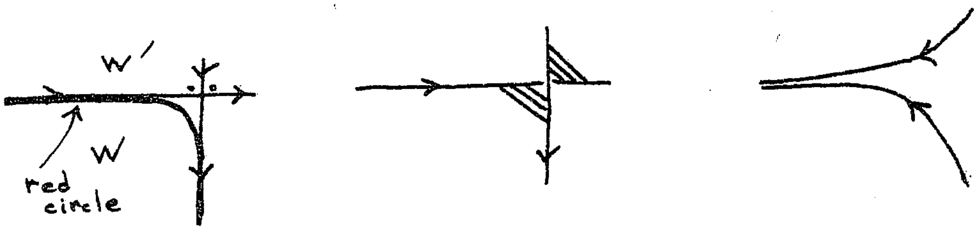
$$\phi_{WW'}(c) = \begin{cases} +1 & \text{if } W \text{ has a dot, } W' \text{ no dot at } c \\ -1 & \text{if } W \text{ and } W' \text{ have a dot} \\ 0 & \text{if } W \text{ has no dot at } c. \end{cases}$$

$$\Delta_{WW'}(c) = \eta(c)\varepsilon_{WW'}(c)\phi_{WW'}(c).$$

Claim. $\theta(w,w') = \sum_{\substack{c \varepsilon \partial W \cap \partial W' \\ c \notin E_i}} \Delta_{WW'}(c)$

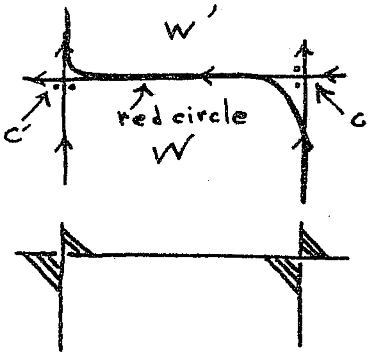
As in part 1), this is verified by case-checking the local contributions to linking numbers at the crossings. The relevant cases are illustrated in Figure 17. Figure 18 is a summary of the algorithm for finding θ.

i)

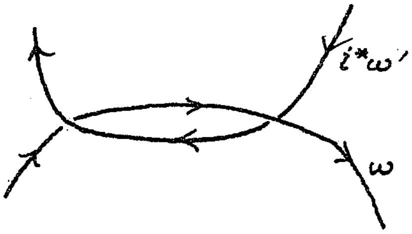


contribution = 0

ii)



Note: If $W \subset E$, $W' \subset E'$, then $c \in E'$, $c' \in E$.



contribution = -1

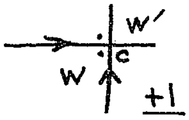
$\eta(c) = +1$, $\varepsilon_{WW'}(c) = +1$, $\phi_{WW'}(c) = -1$

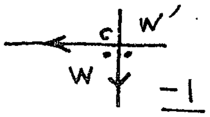
$\Delta_{WW'}(c) = -1$

Figure 17

Figure 18--Recipe for Calculating Seifert Pairing

I) White Regions in Same Domain E



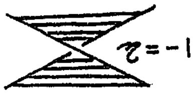


$d_{WW'}(c) = \pm 1$
or 0 if no dot in W.

$$\theta(w,w') = \sum_{c\in(\partial W\cap\partial W')\cap E} d_{WW'}(c)$$

$$\theta(w,w) = -\sum_{W'\neq W}\ \sum_{c\in(\partial W\cap\partial W')\cap E} d_{WW'}(c)$$

II) White Regions in Different Domains



Index Crossing

$$\varepsilon_{WW'}(c) = \begin{cases} 1 & \text{W left of W' with respect to red circle} \\ 0 & \text{otherwise.} \end{cases}$$

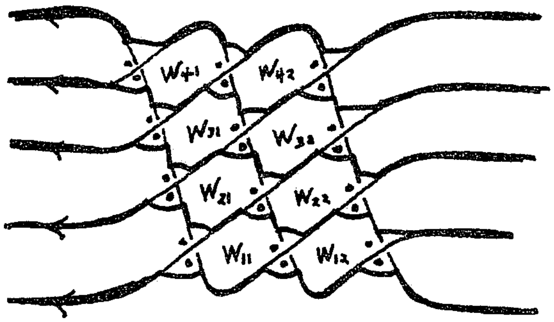
$$\phi_{WW'}(c) = \begin{cases} +1 & \text{W has dot, W' has no dot} \\ -1 & \text{both W and W' have dots} \\ 0 & \text{W has no dot.} \end{cases}$$

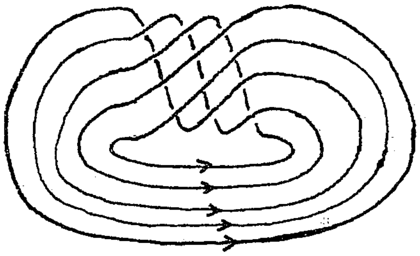
$$\Delta_{WW'}(c) = \eta(c)\varepsilon_{WW'}(c)\phi_{WW'}(c)$$

$$\theta(w,w') = \sum_{\substack{c\in(\partial W\cap\partial W')\cap E' \\ (W\subset E,\, W'\subset E')}} \Delta_{WW'}(c)$$

Torus Link Example

((3,5) torus knot)





	W_{11}	W_{12}	W_{21}	W_{22}	W_{31}	W_{32}	W_{41}	W_{42}
W_{11}	1	-1	-1	1				
W_{12}	0	1	0	-1				
W_{21}			1	-1	-1	1		
W_{22}			0	1	0	-1		
W_{31}					1	-1	-1	1
W_{32}					0	1	0	-1
W_{41}							1	-1
W_{42}							0	1

Thus, $$\mathbb{V} = [\theta] = \begin{bmatrix} V & -V & & \\ & V & -V & \\ & & V & -V \\ & & & V \end{bmatrix}$$

$$V = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Seifert Matrix
for a (3,2)

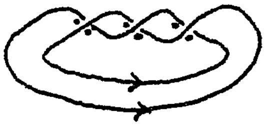


Figure 19

Remarks. Thus, we have an algorithm which computes the Seifert matrix V from a link projection.

Comparison with Murasugi [27] shows that our procedure obtaining V gives a matrix identical to his "Link-Matrix." Although he does not mention the Seifert pairing in his paper, it does seem likely that this was his method for arriving at the matrix.

Murasugi lets $M = V + V'$ and asks how M changes under elementary deformations of the link projection [see 30, page 7]. He finds that M undergoes only transformations of the following type:

$$M \longleftrightarrow QMQ', \quad Q \text{ unimodular}$$

$$M \leftrightarrow \begin{bmatrix} M & 0 \\ 0 & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{bmatrix}$$

Since the projections of any two equivalent links may be related by a sequence of elementary transformations, this shows that the bilinear form determined by M is an invariant of link type as long as two forms are considered equivalent up to direct sums with $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Since the signature $\sigma(U) = 0$, we see that $\sigma(M)$ is an invariant of link type. This leads to the definition:

Definition. Let $L \subset S^3$ be a link. Let V be the Seifert matrix of L computed from a connected projection of L. Letting $M = V + V'$, define the signature of L as

$$\sigma(L) \equiv \sigma(M).$$

Corollary. If F is the Seifert surface used in the above definition and N_2, M_2 are constructed as in Chapter I, (M_2 = 2-fold branched cover of S^3 along L) then

$$\sigma(L) = \sigma(M_2).$$

Proof. $\sigma(M_2) = \sigma(V + V')$ since $V + V'$ = intersection form for M_2.