Final Exam - Math 215 - Fall 2010

Do problems $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$. *Write all your proofs with care, using full sentences and correct reasoning.*

1. Give a proof by mathematical induction of the following statement:

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

for all $n = 1, 2, 3, \cdots$.

2. Suppose that there are n straight lines in the plane, positioned so that *each line intersects each of the other lines once.* Prove that the total number of intersection points among these n lines is equal to $n(n-1)/2$ for $n = 1, 2, 3, \cdots$. (Hint: You can proceed by induction on n and ask: If there are already n lines in the plane, how many new intersection points will occur when a new line is added to the set of n lines?)

3. Find integers r and s such that $30r + 43s = 1$.

4. Recall that a natural number p is said to be *prime* if it has no divisors other than 1 and itself. By convention, the number 1 is not taken to be a prime, so the prime numbers begin with $2, 3, 5, 7, 11, 13, \cdots$. *Prove that there are infinitely many distinct prime numbers.*

5. Prove that there exist irrational numbers a and b such that a^b is rational.

6. Prove that the following two statements are equivalent:

$$(A \Rightarrow B) \Rightarrow C$$

and

$$(A \vee C) \wedge (B \Rightarrow C).$$

In your proof, do *not* use truth tables. Use the facts that $A \Rightarrow B \; = (\sim A) \vee B$ and $\sim (A \wedge B) = \; (\sim A) \vee (\sim B)$, and give a completely algebraic proof.

7. (a) Give the definitions of the terms *injective* and *surjective* for a function $f : X \longrightarrow Y$ from a set X to a set Y.

(b) We define the composition of the function $f : X \longrightarrow Y$ and the function $g : Y \longrightarrow Z$ to be the function $g \circ f : X \longrightarrow Z$ with $g \circ f(x) = g(f(x))$ for all $x \in X$. A map $f : X \to Y$ between two sets is said to be *bijective* if it is both injective and surjective. Prove that if $f : X \to Y$ and $g : Y \longrightarrow Z$ are both bijective, then $g \circ f : X \longrightarrow Z$ is also bijective.

8. Let there be given an infinite list of sequences of 0's and 1's

$$s^1, s^2, s^3, \cdots.$$

That is, for each natural number n we have

$$s^n = (s^n_1, s^n_2, s^n_3, \cdots)$$

where each entry s^n_k is equal either to 0 or to 1. Construct a sequence s,

$$s = (s_1, s_2, s_3, \cdots)$$

of 0's and 1's such that $s \neq s^n$ for any $n = 1, 2, 3, \cdots$.

9. Let X be any set. Let $P(X)$ denote the set of subsets of X. Let

$$F : X \longrightarrow P(X)$$

be any well-defined mapping from X to its power set $P(X)$. Show that F is not surjective.

10. Recall that we say that two integers n and m are *congruent modulo p*

$$n \equiv m \ (mod\ p)$$

exactly when

$$n - m = kp$$

for some integer k.
(a) Prove that if $a \equiv b \ (mod\ p)$ and $b \equiv c \ (mod\ p)$, then $a \equiv c \ (mod\ p)$.
(b) Prove that for any integer x, $(x - p)^2 \equiv x^2 \ (mod\ p)$.