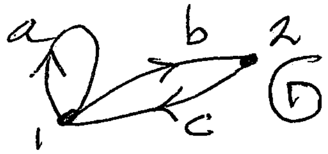
Week Three Notes — Math 310



$$\mathbb{A} = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$$ edge-labeled adjacency matrix

We will multiply to find $\mathbb{A}^n$ and keep track of the order of the products of a's and b's so they can represent walks on Ⓓ. For example

$\underline{a^2bcab}$ is a walk of length 6 from $\underline{1}$ to 2.

$$\mathbb{A} = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$$

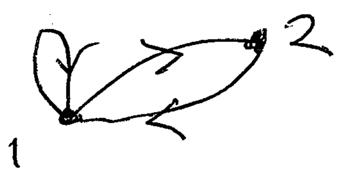
$$\mathbb{A}^2 = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} = \begin{pmatrix} a^2+bc & ab \\ ca & cb \end{pmatrix}$$

$$\mathbb{A}^3 = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}\begin{pmatrix} a^2+bc & ab \\ ca & cb \end{pmatrix} = \begin{pmatrix} a^3+abc+bca & a^2b+bcb \\ ca^2+cbc & cab \end{pmatrix}$$

$\underline{\text{Exercise}}$: Compute $\mathbb{A}^4$ and A^5.

$$\mathbb{A}^4 = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}\begin{pmatrix} a^3+abc+bca & a^2b+bcb \\ ca^2+cbc & cab \end{pmatrix}$$

See next page.



$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

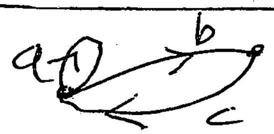
$$A^3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$$

$$A^5 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 5 \\ 5 & 3 \end{pmatrix}$$

$(A^4)_{12}$ = 3 walks of length 4 from node 1 to node 2

$$\begin{pmatrix} a & b \\ c & \phi \end{pmatrix}\begin{pmatrix} a^3+abc+bca & a^2b+bcb \\ ca^2+cbc & cab \end{pmatrix}$$



$$= \begin{pmatrix} a^4+a^2bc+abca+bca^2+bcbc & a^3b+abcb+bcab \\ ca^3+cabc+cbca & ca^2b+cbcb \end{pmatrix}$$

Week Three Notes — Math 310

Teaching Mathematica to List Walks on a Graph

x**y is non-commutative multiplication in Mathematica

$x**y \neq y**x$

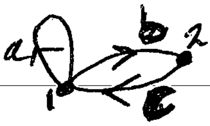
```
In[41]:=
    rule1 = {x_ ** (y_ + z_) :> x ** y + x ** z};
    rule2 = {x_ ** 0 :> 0};
    rule3 = {x_ ** 1 :> x};
    NC[t_] := Expand[t /. rule1 //. rule2 //. rule3 //. rule1 //. rule2 //. rule3]
```

← teaching Mathematica some rules of algebra.

```
In[45]:=
    NC[a ** (b + c + 0)]
    NC[a ** (b + 1)]

Out[45]= a ** b + a ** c

Out[46]= a + a ** b
```



$$\textcircled{G},\ A = A(\textcircled{G}) = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$$

$$A(\textcircled{G})\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} ax+bz & ay+bw \\ cx & cy \end{pmatrix}$$

$$\shortparallel$$

"$\textcircled{G}[\{\{x, y\}, \{z, w\}\}]$"

```
In[47]:=
    G[{{x_, y_}, {z_, w_}}] := {
      {NC[a ** x + b ** z], NC[a ** y + b ** w]},
      {NC[c ** x], NC[c ** y]}
     }

In[48]:=
    MatrixForm[G[{{x, y}, {z, w}}]]
    MatrixForm[G[{{1, 0}, {0, 1}}]]

Out[48]//MatrixForm=
    ( a ** x + b ** z   a ** y + b ** w )
    (     c ** x             c ** y     )

Out[49]//MatrixForm=
    ( a  b )
    ( c  0 )
```

(This is a recursive function definition.)

Mathematica computes the 12 entry of $(A(\textcircled{G}))^n$, getting a list of all the walks of length n from node 1 to node 2.

```
In[50]:= GI[1] = {{a, b}, {c, 0}}
         GI[n_Integer] := (GI[n] = G[GI[n - 1]])
         GG[n_Integer] := GI[n][[1]][[2]]
         Do[Print[k, "        ", GG[k]], {k, 1, 8, 1}]

Out[50]= {{a, b}, {c, 0}}

1      b

2      a ** b

3      a ** a ** b + b ** c ** b

4      a ** a ** a ** b + a ** b ** c ** b + b ** c ** a ** b

5      a ** a ** a ** a ** b + a ** a ** b ** c ** b + a ** b ** c ** a ** b + b ** c ** a ** a ** b + b ** c ** b ** c ** b

6      a ** a ** a ** a ** a ** b + a ** a ** a ** b ** c ** b + a ** a ** b ** c ** a ** b + a ** b ** c ** a ** a ** b +
   a ** b ** c ** b ** c ** b + b ** c ** a ** a ** a ** b + b ** c ** a ** b ** c ** b + b ** c ** b ** c ** a ** b

7      a ** a ** a ** a ** a ** a ** b +
   a ** a ** a ** a ** b ** c ** b + a ** a ** a ** b ** c ** a ** b + a ** a ** b ** c ** a ** a ** b +
   a ** a ** b ** c ** b ** c ** b + a ** b ** c ** a ** a ** a ** b + a ** b ** c ** a ** b ** c ** b +
   a ** b ** c ** b ** c ** a ** b + b ** c ** a ** a ** a ** a ** b + b ** c ** a ** a ** b ** c ** b +
   b ** c ** a ** b ** c ** a ** b + b ** c ** b ** c ** a ** a ** b + b ** c ** b ** c ** b ** c ** b

8      a ** a ** a ** a ** a ** a ** a ** b + a ** a ** a ** a ** a ** b ** c ** b + a ** a ** a ** a ** b ** c ** a ** b +
   a ** a ** a ** b ** c ** a ** a ** b + a ** a ** a ** b ** c ** b ** c ** b + a ** a ** b ** c ** a ** a ** a ** b +
   a ** a ** b ** c ** a ** b ** c ** b + a ** a ** b ** c ** b ** c ** a ** b + a ** b ** c ** a ** a ** a ** a ** b +
   a ** b ** c ** a ** a ** b ** c ** b + a ** b ** c ** a ** b ** c ** a ** b + a ** b ** c ** b ** c ** a ** a ** b +
   a ** b ** c ** b ** c ** b ** c ** b + b ** c ** a ** a ** a ** a ** a ** b + b ** c ** a ** a ** a ** b ** c ** b +
   b ** c ** a ** a ** b ** c ** a ** b + b ** c ** a ** b ** c ** a ** a ** b + b ** c ** a ** b ** c ** b ** c ** b +
   b ** c ** b ** c ** a ** a ** a ** b + b ** c ** b ** c ** a ** b ** c ** b + b ** c ** b ** c ** b ** c ** a ** b
```

Sample Solution

13(a) p. 43

$$\left[\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ 1 & 2 & 0 & 3 & 1 & -2 \\ 0 & 0 & 1 & 2 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \uparrow & & \uparrow & & & \end{array}\right]$$

Row Reduced

Let $x_2 = \alpha$, $x_4 = \beta$, $x_5 = \gamma$.

Then
$$x_1 + 2\alpha + 3\beta + \gamma = -2$$
$$x_3 + 2\beta + 4\gamma = 5$$

So
$$\boxed{\begin{aligned} x_1 &= -2\alpha - 3\beta - \gamma - 2 \\ x_3 &= -2\beta - 4\gamma + 5 \end{aligned}}$$

α, β and γ can be any real numbers.

$$\left.\begin{aligned} x_1 &= -2\alpha - 3\beta - \gamma - 2 \\ x_2 &= \alpha \\ x_3 &= -2\beta - 4\gamma + 5 \\ x_4 &= \beta \\ x_5 &= \gamma \end{aligned}\right\} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ -4 \\ 0 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} -1 \\ 0 \\ -4 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \\ 5 \\ 0 \\ 0 \end{pmatrix}$$

⑤

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \mathrm{Det}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad-bc.$$

This is the <u>determinant of a 2×2 matrix.</u>

<u>Note:</u> $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$$= \begin{pmatrix} ad-bc & -ab+ba \\ cd-dc & -cb+da \end{pmatrix}$$

$$= \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

$$= \Delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Delta = ad-bc.$$

Thus, if $\Delta = ad-bc \neq 0$,
then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$

5.1

$$M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \Delta = |M| = 4 - 6 = -2$$

$$M^{-1} = \frac{1}{\Delta} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \frac{-1}{2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -4/2 & --2/-2 \\ -\frac{-3}{2} & -1/2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3/2 & -\frac{1}{2} \end{pmatrix}$$

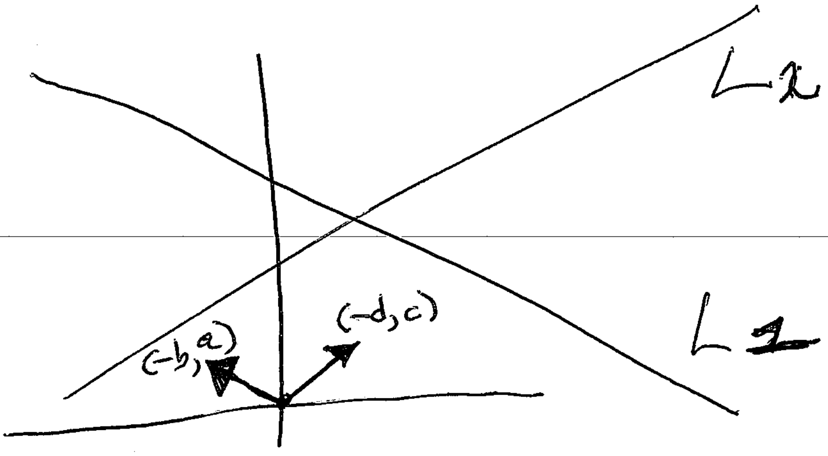
$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

✓

Lines and Slopes

$ax+by=e$: L_1 line

$cx+dy=f$: L_2 line



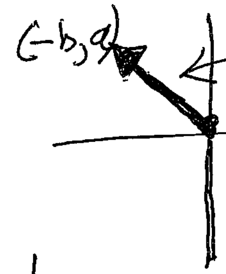
assume $b \neq 0$

$ax+by=e$: L_1

$by = -ax+e$

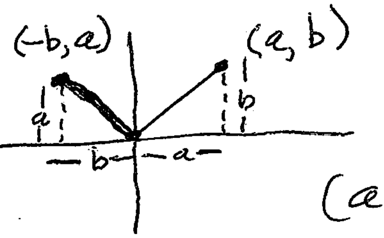
$y = \left(-\frac{a}{b}\right)x + \left(e/b\right)$

slope $-\frac{a}{b} = \frac{\Delta y}{\Delta x}$



direction vector for L_1.

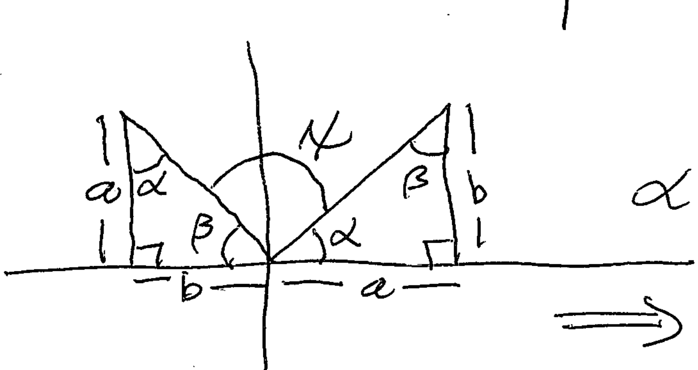
Note



(a,b) is perpendicular to $(-b,a)$.

$(a,b)\cdot(-b,a)$

$= -ab+ab = 0$.



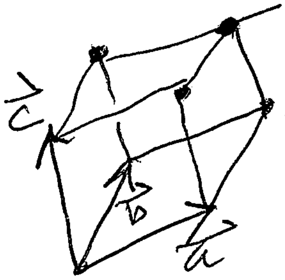
$\alpha+\beta=90^\circ$

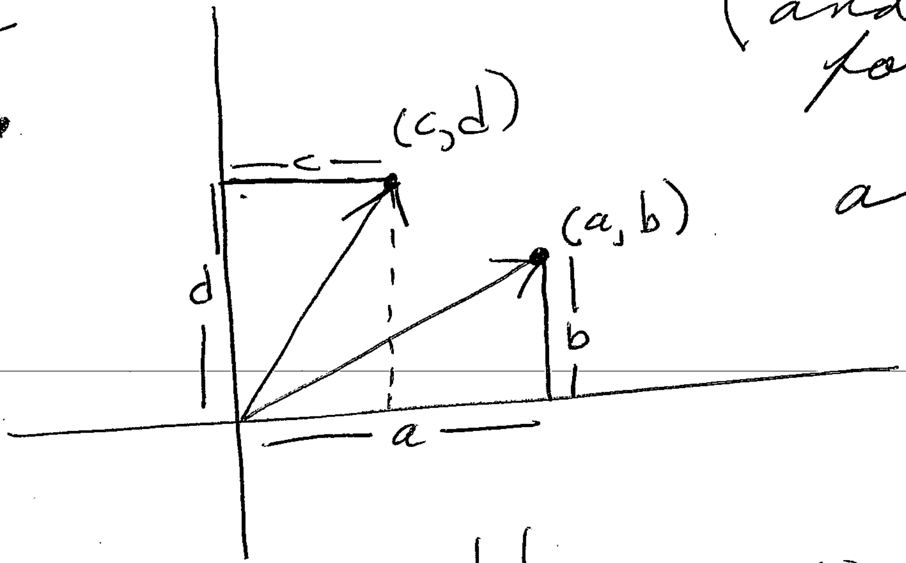
$x+\alpha+\beta=180^\circ$

$\Longrightarrow x=90^\circ$

Determinants Measure Area

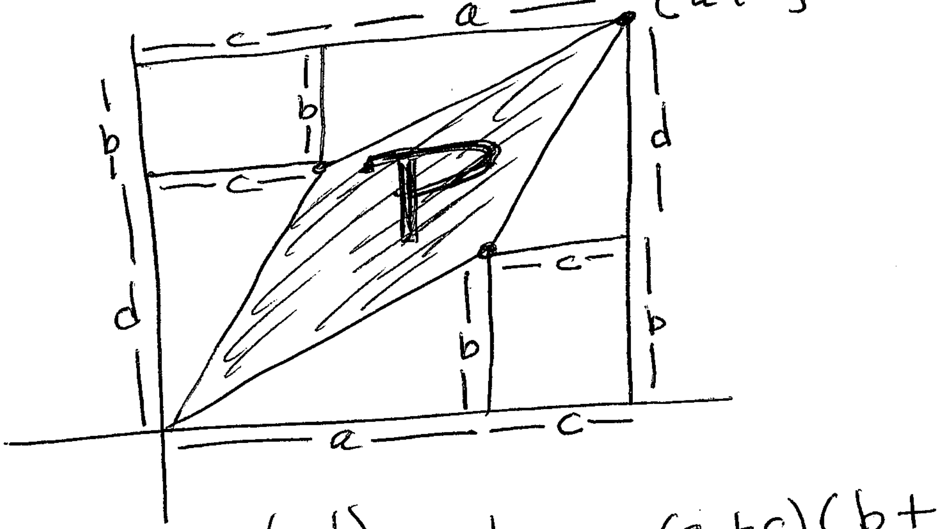
(and volume for 3x3 dets and ...)





This means that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$ when direction vectors of
$ax + by = e$
$cx + dy = f$
make an $\measuredangle \neq 0$ or $180°$.

Examine the area of **P** spanned by the parallelogram spanned by (a, b) and (c, d).



$$P + 2\left(\frac{ab}{2}\right) + 2\left(\frac{cd}{2}\right) + 2bc = (a+c)(b+d)$$

$$P + \cancel{ab} + \cancel{cd} + 2bc = \cancel{ab} + ad + bc + \cancel{cd}$$

$$\Rightarrow P = ad - bc.$$

Thus P = area of parallelogram **P**
$= \left|\text{Det}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right|.$

Fact: A, B n×n matrices
then

$$|AB| = |A||B|.$$

(The determinant of a product is the product of the determinants.)

---

e.g. $A = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$ $\quad |A| = 6-2 = 4$

$B = \begin{pmatrix} 6 & 1 \\ 1 & 1 \end{pmatrix}$ $\quad |B| = 6-1 = 5$

$$AB = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 6 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 12+1 & 2+1 \\ 12+3 & 2+3 \end{pmatrix} = \begin{pmatrix} 13 & 3 \\ 15 & 5 \end{pmatrix}$$

$$\begin{aligned} |AB| &= 13 \cdot 5 - 3 \cdot 15 \\ &= 65 - 45 \\ &= 20 \\ &= 4 \cdot 5 \end{aligned}$$

$$|AB| = |A||B|$$

Example: Using $|AB|=|A||B|$ to prove a formula about Fibonacci Numbers.

$$M=\begin{pmatrix}1&1\\1&0\end{pmatrix}$$

$$M^2=\begin{pmatrix}1&1\\1&0\end{pmatrix}\begin{pmatrix}1&1\\1&0\end{pmatrix}=\begin{pmatrix}2&1\\1&1\end{pmatrix}$$

$$M^3=\begin{pmatrix}1&1\\1&0\end{pmatrix}\begin{pmatrix}2&1\\1&1\end{pmatrix}=\begin{pmatrix}3&2\\2&1\end{pmatrix}$$

$$M^4=\begin{pmatrix}1&1\\1&0\end{pmatrix}\begin{pmatrix}3&2\\2&1\end{pmatrix}=\begin{pmatrix}5&3\\3&2\end{pmatrix}$$

$$M^5=\begin{pmatrix}1&1\\1&0\end{pmatrix}\begin{pmatrix}5&3\\3&2\end{pmatrix}=\begin{pmatrix}8&5\\5&3\end{pmatrix}$$

$\cdots$

$$\begin{array}{cccccccccccc}1, & 1, & 2, & 3, & 5, & 8, & 13, & 21, & 34, & 55, & 89, & 144, \ldots\\ \| & \| & \| & \| & \| & \| & \| & \| & \| & \| & \| & \|\\ f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 & f_9 & f_{10} & f_{11} & f_{12}\end{array}$$

$$\boxed{f_{n+1}=f_n+f_{n-1}} \qquad \underline{(f_0=0)}$$

$$M^n=\begin{pmatrix}f_{n+1}&f_n\\f_n&f_{n-1}\end{pmatrix}$$

$$|M|=\begin{vmatrix}1&1\\1&0\end{vmatrix}=-1$$

$$\therefore\ \boxed{|M^n|=|M|^n=(-1)^n} \quad \otimes$$

$$\therefore\ \boxed{f_{n+1}f_{n-1}-f_n^2=(-1)^n}$$

$$\begin{array}{r}34\\ \underline{13}\\ 102\\ \underline{34\ }\\ 442\end{array} \qquad \begin{array}{r}21\\ \underline{21}\\ 21\\ \underline{42\ }\\ 441\end{array}$$

e.g. $\left\{\begin{array}{l}\ldots\ 3,\ 5,\ 8,\ \ldots\\ 3\times 8-25=-1=(-1)^5\ \checkmark\\ \ldots\ 13,\ 21,\ 34,\ \ldots\\ 13\times 34-21^2=442-441=+1=(-1)^8\ \checkmark\end{array}\right.$

3X3 Det

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}$$

$$\underset{def}{=} a\begin{vmatrix} e & f \\ h & k \end{vmatrix} - b\begin{vmatrix} d & f \\ g & k \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ek-fh) - b(dk-fg) + c(dh-eg)$$

$$= aek - afh - bdk + bfg + cdh - ceg$$