Exam2 - Math 313 **- Fall** 2014

1. Prove that $1 = .9999\cdots$ by using the ϵ-N definition of limit directly. Note that you will have to explain what is the meaning of an expression like $.999\cdots$ as a limit of a certain sequence.

2. (a) Give the definition of convergence for a series $S = \sum_{n=1}^{\infty} b_n$ where S is a real number and b_n is a real number for each natural number n. Note that you should be making use of the quantities $S_n = \sum_{k=1}^{n} b_k$ in your definition.

   (b) Prove $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$ for all $n = 1, 2, 3, \cdots$.

   (c) Prove that the series $\sum_{k=1}^{\infty} \frac{k}{2^k}$ converges and determine its value.

3. True or False: The set of decimals $\{\, .d_1 d_2 \cdots d_n \,|n = 1, 2, 3, \cdots\}$ where each d_k is a digit between 0 and 9, is uncountable. Justify your answer.

4. (a) Let
$$\rho = .d_1 d_2 d_3 \cdots$$
   where $d_n = 1$ if n is a prime number and $d_n = 0$ if n is not a prime number. Thus
$$\rho = .0110101000101000101000100000101000001000101 00\cdots.$$
   Regarding ρ as an infinite decimal, prove that ρ is an irrational number.

   (b) Consider the statement "In the decimal expansion of ρ there are infinitely many occurrences of the pattern 101.". Translate this statement into a statement about prime numbers.