Gaussian Elimination and Row-Echelon Form

Problem 1: Use the Reduced Row-Echelon form to find all solutions of the equations

$$\begin{array}{rcrcrcr} x & + & 3y & + & z & = & 3 \\ 2x & + & 5y & + & z & = & 8 \\ 3x & + & 8y & + & 2z & = & 11 \end{array}$$

You must show all your steps and work for credit.

Problem 2: Find the general solution of $\begin{bmatrix} 1 & 3 & 3 \\ 2 & 6 & 9 \\ -1 & -3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$.

Problem 3: (a) Find the row-reduced echelon form of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

(b) What are the solutions of the system $Ax = 0$? (Check!)

Problem 4: Given the equations

$$\begin{aligned} x + 2y + 3z - 3w &= 1 \\ 4x + 5y + 6z - 6w &= 1 \\ 7x + 8y + 9z - 8w &= 1 \end{aligned}$$

a) Give the Reduced Row-Echelon form of the associated augmented matrix.

b) Which are the free variables? Which are the dependent variables?

c) Give the general solution of the system of equations.

Problem 5: Given the two equations

$$\begin{aligned} x + 2y + 3z - 4w &= 2 \\ 2x + 4y + 3z + w &= 5 \end{aligned}$$

Use the method of row reduction to solve the system. Indicate which are the free variables, which are the dependent variables. What is the geometric interpretation of the solution?

Problem 6: Let a, b, c be constants, and consider the system of equations

$$\begin{array}{rcrcrcr} 3x & + & 3y & + & z & = & a \\ x & + & y & + & 2z & = & b \\ 5x & + & 5y & & & = & c \end{array}$$

Find the equation that the constants a, b, c must satisfy so that these equations are consistent.

Matrix Algebra and Manipulating Matrices

Problem 1: In each case, give an example of a matrix which is

- <u>not the identity matrix</u>
- <u>not the zero matrix</u>,

and satisfies:

a) A is a 2×2 diagonal matrix with an inverse.

b) B is a 2×2 matrix with rank 1.

c) C is a 2×2 symmetric matrix with no inverse.

d) O is a 2×2 orthogonal matrix.

So, you must find four matrices A, B, C and O.

Matrix Determinants

Problem 1: Find the determinant of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$.

Problem 2: Use either the definition of determinant in terms of cofactors, or the method of row operations, to calculate the determinant of

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

Problem 3: Calculate the determinant of the matrix $B = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 6 & 2 & 0 \\ 1 & 1 & -2 & 3 \end{bmatrix}$.

Problem 4: Find the determinant of the matrix $\mathbf{A^3}$ where $A = \begin{bmatrix} 5 & 1 \\ 6 & 2 \end{bmatrix}$.

Problem 5: Given the matrices $A = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$,

calculate the following determinants:

a) $|A|$, $|B|$ and $|C|$

b) $|ABC^2|$

c) $|7 \cdot B|$

d) $|A^7 \cdot B|$

e) $|B \cdot C^{-1}|$

f) $|B^T C A^{-1}|$

g) $|A - B|$

Problem 6: a) Find the determinant of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 7 \end{bmatrix}$.

b) Use the solution to part a) to explain how many solutions the equation $A\vec{x} = \vec{b}$ has, where

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Matrix Inverses

Problem 1: a) Find the inverse (by any method) of $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$.

b) Use the above to express the solutions of $A\vec{x} = \vec{b}$ in terms of the constants b_1 and b_2.

Problem 2: Give the formula for the inverse of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Problem 3: Use the method of Gaussian Elimination to find the inverse for $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 9 & 12 \end{bmatrix}$.

Problem 4: Use the method of Cofactors to find the inverse for $A = \begin{bmatrix} -1 & 2 & 1 \\ 2 & -1 & 2 \\ 1 & -2 & 1 \end{bmatrix}$.

Problem 5: Find the inverse of the following matrices (and check your answers.)
Do not use a calculator – you will be required to show all your work and computations.

a) $C = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & -2 & -3 \end{bmatrix}$

b) $C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 1 & 1 \end{bmatrix}$

c) $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}$

d) $A = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

Problem 6: For what values of the variable λ does the matrix **D** below have an inverse? **Explain your answer!**

$$D = \begin{bmatrix} 3-\lambda & 3 & 1 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & \lambda+1 \end{bmatrix}$$

Problem 7: Let A be an $n \times n$ matrix. Suppose that the system of equations $AX = 0$ has a unique solution. Explain why the inverse A^{-1} has to exist.

Vector Spaces and Subspaces

Problem 1: Consider the subset of vectors in $\mathbf{R}^2$ given by

$$S = \{(x, x^2) \text{ where x is any real number } \}$$

Is S a vector subspace? Justify your answer carefully.

Problem 2: Is the set $\left\{ \begin{bmatrix} x \\ x^3 \end{bmatrix} \text{where } x \in \mathbb{R} \right\}$ a vector subspace of $\mathbb{R}^2$? Justify your answer.

Problem 3: Let V be the space of real-valued functions of x. Show the solution set S of the equation

$$f'(x) = xf(x)$$

is a subspace of V.

Problem 4: Let V be the space of all differentiable functions on the line. Let W be the subset of all functions f which are solutions of the differential equation $f'' + 5f = 0$. Show that the solution set W is a subspace of V.

Problem 5: Let $A^{m \times n}$ be a matrix with m rows and n columns. What are the four fundamental subspaces associated to A? Give the definition of each of the following:

- $\text{Col}(A)$ = the column space of A.
- $\text{Row}(A)$ = the row space of A
- $\text{Null}(A)$ = the null space of A
- $\text{Conull}(A)$ = conull space of A

Linear Independence, Spanning, Basis, and Dimension

Problem 1: Find a basis for the subspace V of $\mathbb{R}^3$ spanned by the vectors

$$\vec{u}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$$

Problem 2: In the space $\mathcal{P}_3$ of polynomials of degree 2 or less, are the "vectors" $\{1+x, 1-x, 1+x+x^2\}$ linearly dependent or independent?

Problem 3: a) For $A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 3 & 1 & 2 \\ 3 & 1 & -1 & 6 \end{bmatrix}$ find a basis for the row space and the column space.

b) Is $A\vec{x} = \vec{b}$ solvable for all $\vec{b}$?

Problem 4: For the vectors

$$\vec{w}_1 = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{w}_2 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} \text{ and } \vec{x} = \begin{bmatrix} -9 \\ -2 \\ 5 \end{bmatrix}$$

Is $\vec{x}$ in the span of $\{\vec{w}_1, \vec{w}_2\}$? If so, write $\vec{x}$ as a linear combination of $\{\vec{w}_1, \vec{w}_2\}$.

Problem 5: Is $[1,2,3]^T$ in the span of $[4,0,5]^T$ and $[6,0,7]^T$?

Problem 6: a) Find a basis for the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 5 \\ -3 \\ 2 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ 4 \\ -2 \\ 0 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 3 \\ 8 \\ -5 \\ 4 \end{bmatrix}$$

b) What is the dimension of the span of the vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$?

Problem 7: Do the "vectors" $1+x$, $1-x$, x^2 span the space $\mathcal{P}_3$ of polynomials of degree at most 2?

Problem 8: Find a basis for the subspace of 2×2 matrices $A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$ satisfying $a_{1,1} + a_{2,2} = 0$.

Column Space, Row Space, Null Space, Conull Space

Problem 1: A is an $m \times n$ matrix. Let

- Col(A) denote the column space of A
- Row(A) denote the row space of A
- Null(A) denote the null space of A
- Conull(A) the co-null space of A

For each of the following questions, your answer should be one of the above 4 spaces. Justify your answer by stating why you think it is correct.

a) The set of vectors perpendicular to the column space is what space?

b) The vector equation $A\vec{x} = \vec{b}$ has a solution if $\vec{b}$ belongs to what subspace?

c) The set of vectors perpendicular to the row space is what space?

d) The vector equation $A\vec{x} = \vec{b}$ has a unique solution if what space is $\{0\}$?

e) What number do you get if you add the dimensions of all 4 spaces?

Problem 2: Give a basis for the **column space**, **row space** and **null space** of the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 3 & 4 & 7 & 1 \\ 2 & 4 & 4 & 2 \end{bmatrix}$$

Problem 3: Find a basis for the null-space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 5 & 5 \\ 3 & 6 & 7 & 8 \end{bmatrix}$$

Problem 4: a) Find a basis for the column space of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 4 & 2 & 0 \end{bmatrix}$.

b) Find a basis for the perpendicular space Col$(A)^{\perp}$

c) Find a basis for Conull(A)

Problem 5: Let $B = \begin{bmatrix} 1 & 2 & -1 & 1 & 1 \\ 2 & 4 & 2 & 4 & 10 \\ 3 & 6 & -2 & 1 & 0 \end{bmatrix}$

Find a basis for the four fundamental spaces of B: the column space, the row space, the null space and the co-null space (the null space of the transpose B^T).

Problem 6: Given the system of equations

$$\begin{array}{ccccccc} x & + & y & + & z & = & c_1 \\ x & + & 2y & + & 2z & = & c_2 \\ x & + & 3y & + & 3z & = & c_3 \end{array}$$

a) For what values of $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ does the system have a solution?

b) If there exists a solution for a given $\vec{c}$, how many are there?

c) Find the basis for the co-null space of the matrix associated to the system of equations above.

d) What is the relation between your answers to part a) and c)?

Problem 7: A is a 3×5 matrix and $L\colon \mathbb{R}^5 \to \mathbb{R}^3$ is defined by $L(\vec{v}) = A \cdot \vec{v}$. Suppose that A has rank 3.

a) What is the dimension of the kernel of L?

b) What is the dimension of the range of L?

Explain your answers in terms of how you would find basis of these spaces if the matrix of A were given!

Problem 8: Let $A = \begin{bmatrix} 2 & 1 & 3 & 0 \\ 4 & 2 & 6 & 2 \\ 6 & 3 & 9 & 3 \end{bmatrix}$

a) Give the **Reduced Row Echelon form** of the matrix A

b) Find a basis for the **null-space** of the matrix A

c) Find a basis for the **column space** of the matrix A

d) What is the **dimension** of the null–space $N(A)$ and the column space $C(A)$?

e) Answer **True** or **False**, and explain your answer:

The equation $A\vec{x} = \vec{b}$ has a solution for every vector $\vec{b} \in \mathbb{R}^3$.

Change of Basis and Coordinates

Problem 1: Find the coordinates of $\vec{p} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ with respect to the basis $\vec{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Problem 2: Find the new coordinates $[a, b, c]^T$ of the point $\vec{x} = [7, 5, 6]^T$ with respect to the basis for $\mathbb{R}^3$ given by the vectors

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Problem 3: Given the vectors in $\mathbb{R}^2$

$$\vec{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

a) Find the transition matrix S corresponding to change of basis from $\{\vec{v}_1, \vec{v}_2\}$ to $\{\vec{u}_1, \vec{u}_2\}$.

b) Find the coordinate expression of $\vec{p} = 3\vec{v}_1 - \vec{v}_2$ with respect to the basis $\{\vec{u}_1, \vec{u}_2\}$.

Linear Transformations and Finding a Matrix Representation

Problem 1: Let $\mathcal{P}_3$ be the space of polynomials of degree 2. Show that the map $L\colon \mathcal{P}_3 \to \mathcal{P}_3$ given by

$$L(p(x)) = p(x) - x \cdot p'(x)$$

is linear. (Here, $p'(x)$ denotes the first derivative of the polynomial $p(x)$.)

Problem 2: Find the matrix, in the standard basis for $\mathbb{R}^3$, for the linear transformation

$$L \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x - y - z \\ x - 2y + z \\ -x + 3y + 2z \end{bmatrix}.$$

b) Find the kernel of L

Problem 3: Define the linear transformation $L\colon \mathcal{P}_3 \to \mathcal{P}_3$ by

$$L(p(x)) = x\ p''(x) - 2x\ p'(x) + p(x)$$

Find the matrix representing L with respect to the basis $\{1, x, x^2\}$ of $\mathcal{P}_3$.

Problem 4: Find the matrix representation for the linear transformation

$$L \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4x - y \\ -x + 4y \end{bmatrix}.$$

with respect to the basis $\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

Problem 5: Let V be the space of functions with basis $\{\sin(x), \cos(x), \sin(2x), \cos(2x)\}$.

Define the linear transformation $L\colon V \to V$ by

$$L(f) = f'' + f' - 4f$$

a) Find the matrix representing L with respect to the given basis.

b) Find the kernel of L

Problem 6: Let a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(v_1, v_2, v_3) = (3v_1 + 2v_2 + v_3, 2v_1 + v_2, v_2).$$

Give the matrix (in the *standard* basis) for T.

Problem 7: Let V be the vector space spanned by the functions $\{e^x, e^{2x}, e^{3x}\}$,

and let $L\colon V \to V$ be the linear transformation defined by $L(f) = f' - 2f$.

a) Find the matrix representing L with respect to the basis $\{e^x, e^{2x}, e^{3x}\}$ of V.

b) Find the kernel of L.

Problem 8: Define the linear transformation $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $L(\vec{v}) = A\vec{v}$ where $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$.

Find the matrix of L with respect to the new basis $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Change of Basis for Linear Transformations and Similarity

Problem 1: The linear transformation $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ has matrix $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$ with respect to the standard basis $\{\vec{e}_1, \vec{e}_2\}$ of $\mathbb{R}^2$. Find the matrix of L with respect to the new basis

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Problem 2: a) Find the matrix representation A with respect to the standard basis $\{\vec{e}_1, \vec{e}_2\}$ of $\mathbb{R}^2$ for the linear transformation

$$L \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4x - y \\ -x + 4y \end{bmatrix}.$$

b) Find the matrix representation B of L with respect to the basis $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Problem 3: Let $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by $\mathbf{L} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4y + 6z \\ -2x - 3y \\ x + 2y + z \end{bmatrix}$.

a) Find the matrix representing L with respect to the standard basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ of $\mathbb{R}^3$.

b) Use the answer to part a) to find the matrix representing L with respect to the new basis

$$\vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Problem 4: Given the vectors in $\mathbb{R}^2$

$$\vec{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

a) Find the transition matrix S corresponding to change of basis from $\{\vec{v}_1, \vec{v}_2\}$ to $\{\vec{u}_1, \vec{u}_2\}$.

b) The linear transformation $L\colon R^2 \to R^2$ has a matrix representation $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ with respect to the basis $\{\vec{u}_1, \vec{u}_2\}$. Find the matrix representation B of L with respect to the basis $\{\vec{v}_1, \vec{v}_2\}$.

Problem 5: For the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \ \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

a) Find the transition matrix S corresponding to the change of basis from the standard basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ of $\mathbb{R}^3$ to the new basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

b) Let $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$L(\vec{v}_1) = \vec{v}_1, \ L(\vec{v}_2) = 2 \cdot \vec{v}_2, \ L(\vec{v}_3) = 3 \cdot \vec{v}_3$$

Find the matrix representing L with respect to the standard basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ of $\mathbb{R}^3$.

Problem 6: a) Let $\{\vec{v}_1, \vec{v}_2\}$ be a basis for $\mathbb{R}^2$, and let L be a linear transformation of $\mathbb{R}^2$ so that

$$L(c_1\vec{v}_1 + c_2\vec{v}_2) = (c_1 + 3c_2)\vec{v}_1 + (2c_1 + 4c_2)\vec{v}_2$$

Find the matrix representing L with respect to the basis $\{\vec{v}_1, \vec{v}_2\}$.

b) Suppose that $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Find the matrix representing L with respect to the standard basis of $\mathbb{R}^2$.

Problem 7: Let $A = \begin{bmatrix} 7 & -2 \\ 15 & -4 \end{bmatrix}$. Define the linear map $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $L(\vec{x}) = A\vec{x}$

a) Find the matrix B for the linear map L with respect to the new basis $\vec{u}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $\vec{u}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

b) Suppose that $\vec{p}$ has coordinates $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ with respect to the basis $\{\vec{u}_1, \vec{u}_2\}$. Find $L(\vec{p})$ with respect to the basis $\{\vec{u}_1, \vec{u}_2\}$.

c) Suppose that $\vec{p} = \vec{u}_2$. Find $L^{101}(\vec{u}_2) = L(L(\ldots L(L(\vec{u}_2))\ldots))$.

Eigenvalues, Eigenvectors and Eigenspaces

Problem 1: Find the eigenvalues and corresponding eigenvectors for $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

Problem 2: Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = 3, \lambda_2 = 0, \lambda_3 = 0$.

a) Find the eigenvectors for these eigenvalues.

b) Note A is symmetric. Find an *orthogonal* matrix S with $S^{-1}AS = D$ diagonal.

Problem 3: Find the eigenvalues and eigenvectors for $A = \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$.

Systems of Differential Equations

Problem 1: Given the differential equations with initial conditions

$$\begin{array}{rclcrcl} x' & = & 3x & + & 4y & ; & x(0) = 1 \\ y' & = & -2x & - & 3y & ; & y(0) = 2 \end{array}$$

Find the functions $x(t)$ and $y(t)$.

Problem 2: a) Give the general solution of the differential system

$$\begin{array}{rcrcr} y_1' & = & y_1 & + & y_2 \\ y_2' & = & -2y_1 & + & 4y_2 \end{array}$$

b) Give the particular solution when $y_1(0) = 3$ and $y_2(0) = 1$.

Finding Powers, Square Roots and Exponentials of Matrices

Problem 1: For the matrix $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$

a) Find 2×2 matrices S and D such that $A = S \cdot D \cdot S^{-1}$

b) Use your answer to part a) to calculate A^5.

Problem 2: For $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$ find the matrix e^A. (Your answer should be a 2×2 matrix.)

Problem 3: For $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$, find the 2×2 matrix e^{tA}.

Problem 4: For $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

a) Calculate A^2, A^3, A^4, A^5.

b) Find the eigenvalues and eigenvectors for A.

c) Use your answer to part b) to calculate A^{10}.

d) Use your answer to part b) to get a formula for A^n when n is a positive integer.