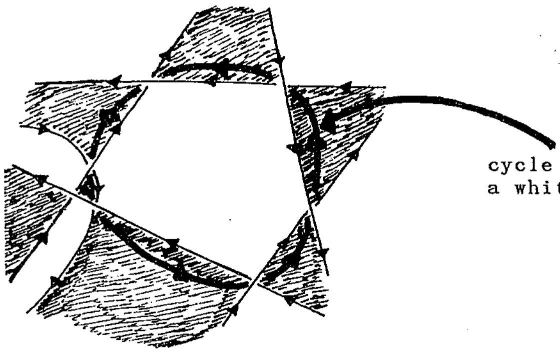
{c|c encircles a white region in the diagram}.



cycle c encircling a white region

orient c compatibly with the planar orientation ↺

Since $F_{K'}$ has the homotopy type of the plane punc-
·ed by the white regions, we see that rank $H_1(F_{K'})$
:(white regions)-1. In counting, count all the bounded
te regions. Then to obtain $H_1(F_K)$, note that
.k $H_1(F_K)$ = rank $H_1(F_{K'})-k$ where k is the number of
.cer circles. For example, in the figure on p. 185, we
·e by this account $\rho(F_K) = \rho(F_{K'})-1$ and $\rho(F_{K'}) = 7$.
refore $\rho(F_K) = 6$. Note that this is in accord with the
mula of Proposition 7.2. In fact, $H_1(F_K)$ has as basis
cycles $\{c_1,c_2,c_3,c_4,c_5,c_6,c_7\}$. We have added c_4
ce $c_1+c_2+c_3+c_4 \approx \alpha$ ($\approx$ denotes homology of cycles) and
bounds a disk in F_K.

rcise. Explain how to obtain a basis for $H_1(F_K)$ in
general case of k tracer circuits $\alpha_1,\alpha_2,\cdots,\alpha_k$.

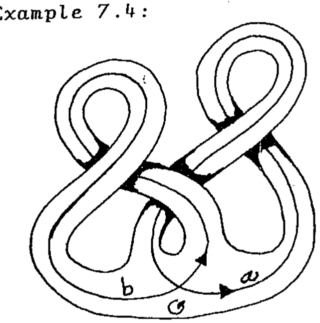
Give a procedure for deciding which white cycles to retain or throw away.

SEIFERT PAIRING

We now define an algebraic method for measuring the embedding of an oriented surface $F \subset S^3$. Given $F \subset S^3$, and a cycle a on F, let a^* denote the result of pushing a a very small amount into S^3-F along the positive normal direction to F. Using this, we define the Seifert pairing $\theta : H_1(F) \times H_1(F) \longrightarrow Z$ by the formula $\theta(a,b) = lk(a^*,b)$. This is a well-defined, bilinear pairing. It is an invariant of the ambient isotopy class of the embedding $F \subset S^3$.

Seifert invented a version of this pairing in [S]. He used it to investigate branched covering spaces. It has since proved to be extraordinarily useful in both classical and higher-dimensional knot theory.

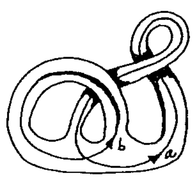
Example 7.4:



θ	a	b
a	-1	1
b	0	-1

The surface F is oriented so that the positive normal points out of the page, toward the reader. For the self-linking $\theta(a,a) = lk(a^*,a)$, a^* may be represented by a parallel copy of a along the surface. Thus $\theta(a,a)$ can be computed from a disk with bands, by counting curls with sign.

Example 7.5:

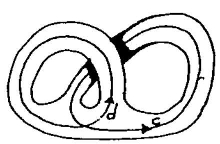


θ	a	b
a	−1	1
b	0	0

Note:

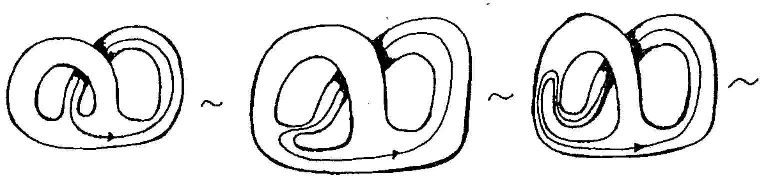
θ	a+b	b
a+b	0	1
b	0	0

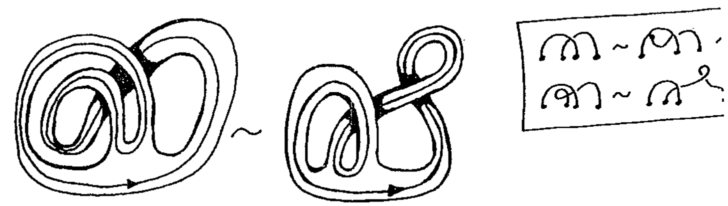
$\theta(a+b,a+b) = \theta(a,a)+\theta(b,a)+\theta(b,b) = 0$



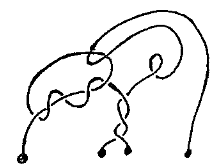
θ′	c	d
c	0	1
d	0	0

Thus these pairings are isomorphic. In fact, these two embeddings are isotopic:

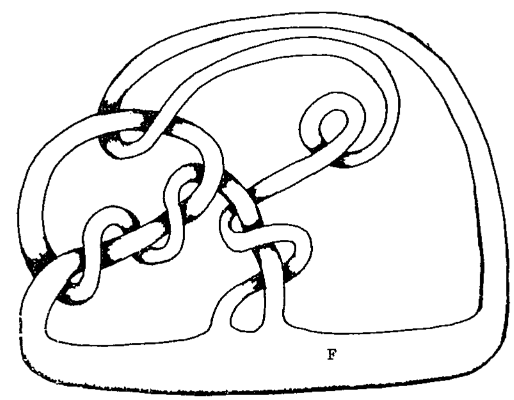




We can, if we want to do it, indicate a banded su entirely in topological script. Thus



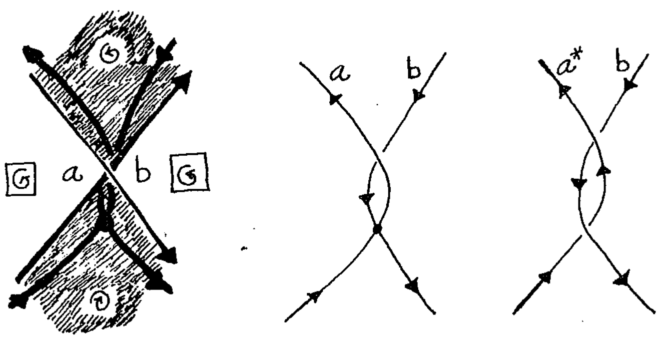
represents the surf



Exercise. Determine the Seifert pairing for this surface F.

SEIFERT PAIRING FOR THE SEIFERT SURFACE

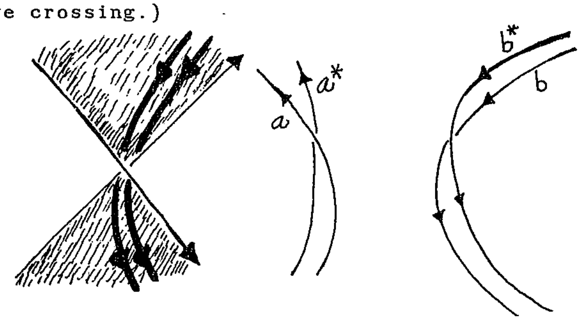
Now let's work out an algorithm for computing the Seifert pairing from a Seifert surface (without pushing it into band-form). Recall that $H_1(F_K)$ is generated by the white cycles. (These are circles encircling white regions in $F_{K'}$.) Thus we must determine how each crossing in the diagram contributes to the Seifert linking number $\theta(a,b)$.



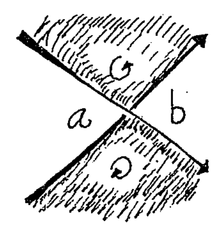
Here is a positive crossing, with Seifert surface shaded, and white regions a and b labelled. The cycles corresponding to these regions are labelled and drawn. Note that the cycles must intersect in order to continue following their courses around the white regions. Let's write $\theta(a,b)$ and $\theta(b,a)$ for the local contribution of this crossing. Then

$$\theta(a,b) = +1$$
$$\theta(b,a) = 0.$$

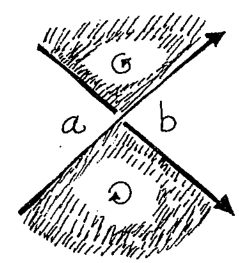
Note that $a \cdot b = +1$ also, where $x \cdot y$ denotes interse number of cycles on the surface. (The signs reverse f negative crossing.)



The self-linking contribution is $\theta(a,b) = -\frac{1}{2} = \theta(b,b)$. (*Note:* The cycles bounding white regions are all orie compatibly with an orientation for the white region itself.)

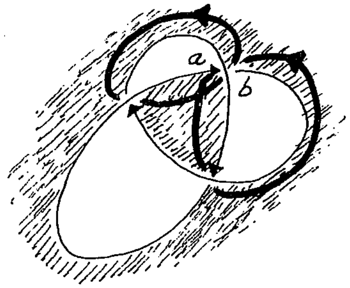


$$\begin{cases} \theta(a,b) = +1 \\ \theta(b,a) = 0 \\ \theta(a,a) = \theta(b,b) = -1/2 \end{cases}$$



$$\begin{cases} \theta(a,b) = 0 \\ \theta(b,a) = -1 \\ \theta(a,a) = \theta(b,b) = +1/2 \end{cases}$$

For example:



θ	1	b
a	-1	1
b	0	-1

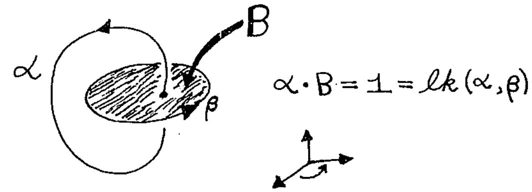
Here a and b interact at only one crossing. But we look at two crossings to compute $\theta(a,b)$ and $\theta(b,b)$.

Exercise. Compute the Seifert pairing for F_K of Figure 7.1.

Exercise. Let $x \cdot y$ denote intersection number on the surface F. Show that for all $x,y \in H_1(F)$,

$$\theta(x,y)-\theta(y,x) = x \cdot y.$$

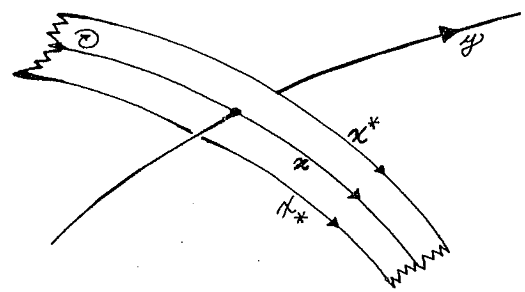
Hint: Do it for Seifert surface first. Then try the general case. To do the general case it helps to have the following description of linking numbers: Let $\alpha,\beta \subset S^3$ be two disjoint oriented curves. Let B be an oriented surface bounding β. Isotope α so that α intersects B transversally. Then $lk(\alpha,\beta) = \alpha \cdot B$.



[Why is this independent of the choice of B?]

Exercise. Prove, using Seifert (or spanning) surfaces, that this description of linking implies our original description.

Now return to the formula $\theta(x,y)-\theta(y,x) = x \cdot y$, contemplate



∂B = boundary of $B = x^* - x_*$

$$\begin{aligned}\theta(x,y)-\theta(y,x) &= lk(x^*,y)-lk(y^*,x)\\ &= lk(y,x^*)-lk(y^*,x)\\ &= lk(y,x^*)-lk(y,x_*)\\ &= lk(y,x^*-x_*)\\ &= y \cdot B\\ &= x \cdot y.\end{aligned}$$

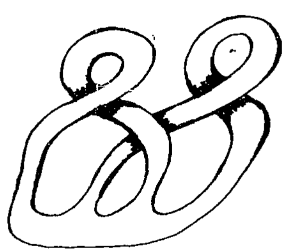
DIFFERENT SURFACES FOR ISOTOPIC KNOTS

A given knot or link can have many different spanning surfaces. For example, two isotopic diagrams will have rather different Seifert surfaces. How are all the different surfaces spanning a knot related to one another?

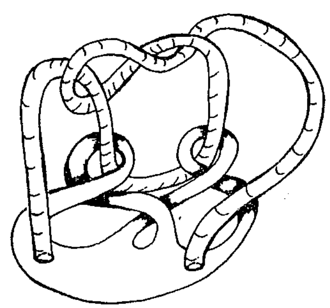
The answer is, in principle, surprisingly simple. Consider the following way to complicate a spanning surface:

1) Cut out two discs, D_1, D_2.
2) Take a tube $S^1 \times I$ and embed it in S^3 disjointly from the surface, but with the tube boundary attached to ∂D_1 and ∂D_2.

This is called doing a 1-surgery to the surface.



F



F after surgery

The reverse operation consists in finding a curve α on F such that α bounds a disk S^3-F. Then cut out $\alpha \times I$ from F and cap off with two D^2's.

before after

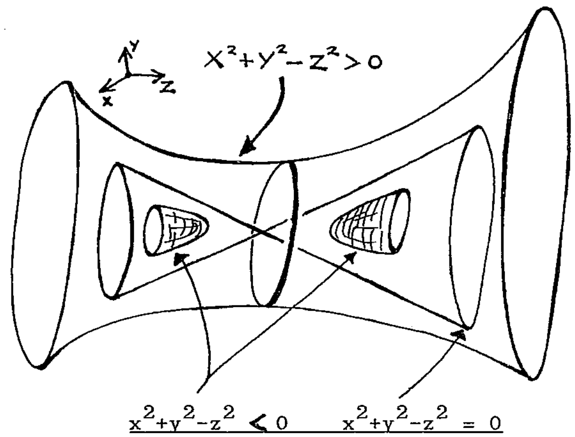
This is a 0-surgery. It simplifies the surface (i.e., reduces genus).

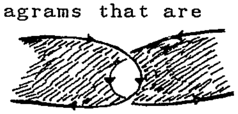
These two surgery operations give us different surfaces with the same boundary.

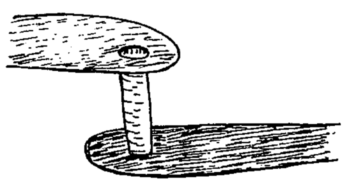
DEFINITION 7.6. *Let* F *and* F′ *be oriented surfaces u boundary that are embedded in* S^3. *We say that* F *and are* S-equivalent ($F \underset{S}{\sim} F'$) *if* F′ *may be obtained fro* F *by a combinations of* 0-*surgery*, 1-*surgery and ambie isotopy.*

THEOREM 7.7 [L1]. *Let* F *and* F′ *be connected, orient spanning surfaces for ambient isotopic links* $L, L' \subset S^3$ *Then* F *and* F′ *are* S-*equivalent.*

Proof sketch: Let $X = S^3 \times I$ and suppose that $\alpha: S^1 \times$ $\longrightarrow S^3$ is the ambient isotopy from $L = \alpha(S^1 \times 0)$ to $L' = \alpha(S^1 \times 1)$. Then we get an embedding of an annulus in via $\hat{\alpha} : S^1 \times I \longrightarrow X$, $\hat{\alpha}(\lambda,t) = (\alpha(\lambda,t),t)$. If we form $M = (F \times 0) \cup \hat{\alpha}(S^1 \times I) \cup (F^1 \times 1)$, then this is a closed sur face embedded in $S^3 \times I$. One then shows that $M = \partial W$ where W is a 3-manifold embedded in $S^3 \times I$. W can b arranged so that $(S^3 \times t) \cap W$ has only Morse critical points of type $x^2+y^2-z^2$ or $-z^2-y^2+z^2$. These correspond to the 0-surgeries and 1-surgeries we described earlier.

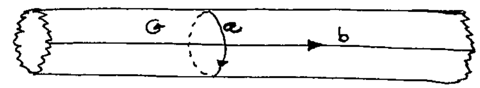


Remark: It may be of interest to look directly at the S-equivalences between Seifert surfaces for diagrams that are related by Reidemeister moves. For example,  is obtained from  by the surgery

 .

Now consider the Seifert pairings for S-equivalent surfaces. Suppose that F' is obtained from F by adding a tube. Then $H_1(F') \cong H_1(F) \oplus Z \oplus Z$ where these two

extra factors are generated by a <u>meridian for the tube</u> and an element <u>b</u> that passes once along the tube orier so that $a \cdot b = 1$.



We then have $\theta(a,a) = 0$, $\theta(a,b) = 1$, $\theta(b,a) = 0$ and $\theta(a,x) = \theta(x,a) = 0$ for all $x \in H_1(F)$. Let θ_0 denot the Seifert pairing for F. Then we have

$$\theta = \begin{array}{c|c|c|} & \theta_0 & 0 \\ a & 0 & 0 \\ b & \beta & 0 \end{array}$$

where β is a row vector, and α is a column vector. Because of the row $(\overline{0},0,1)$, θ becomes on change of ba

$$\left[\begin{array}{c|cc} \theta_0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ \beta & 0 & 0 \end{array}\right].$$

An enlargement of this kind is called an S-equivalence. More generally, two matrices θ and ψ are said to be S-equivalent if ψ can be obtained from θ by a combin: tion of congruence ($\theta \longrightarrow P\,\theta\,P'$ where P' is the tran. pose of P, P invertible over Z. This corresponds to basis change.) and enlargements and contractions (revers of enlargement) as above. If θ and ψ are S-equivalent, we write $\theta \underset{S}{\sim} \psi$.

COROLLARY 7.8. *Let K and K' be ambient isotopic knots or links with connected spanning surfaces F (for K) and F' (for K'). Let θ be the Seifert pairing for F and ψ be the Seifert pairing for F'. Then θ and ψ are S-equivalent.*

INVARIANTS OF S-EQUIVALENCE

DEFINITION 7.9. *Let F be a connected spanning surface for the knot or link K and θ the Seifert pairing for F. Define*

(i) *The* <u>*determinant of K*</u>, $D(K) = D(\theta+\theta')$ *where* D *denotes determinant.*

(ii) *The* <u>*potential function of K*</u>, $\Omega_K(t) \in Z[t^{-1},t]$ *by the formula* $\Omega_K(t) = D(t^{-1}\theta - t\theta')$.

(iii) *The* <u>*signature of K*</u>, $\sigma(K) \in Z$, *by* $\sigma(K) = \text{Sign}(\theta+\theta')$ *where Sign denotes the signature of this matrix.* (See definition below.)

Of course the gadgets produced in this definition are not going to change under S-equivalence! Hence they will be invariants of K.

For example, if $\theta = \left[\begin{array}{c|cc} \theta_0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ \alpha & 0 & 0 \end{array}\right]$ then $\theta+\theta' =$

$\left[\begin{array}{c|cc} \theta_0+\theta_0' & 0 & \alpha' \\ \hline 0 & 0 & 1 \\ \alpha & 1 & 0 \end{array}\right]$ and $D(t^{-1}\theta - t\theta') = D(t^{-1}\theta_0 - t\theta_0')$ because

$$D\begin{bmatrix} 0 & t^{-1} \\ -t & 0 \end{bmatrix} = 1.$$

For the signature, recall that a symmetric matrix over Z can be diagonalized through congruence over Q (the rationals) or over $\mathbb{R}$. Let e_+ denote the number positive diagonal entries, and e_- the number of negat diagonal entries. The signature, Sign(M), is defined the formula $\text{Sign}(M) = e_+ - e_-$. It is an invariant of th congruence class of M. (See [HNK].) Note in particul that $\text{Sign}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 0$. From this it follows that Sign(θ is an invariant of its S-equivalence class, hence an invariant of K. We shall also show that $\sigma(K)$ is an invariant of concordance.

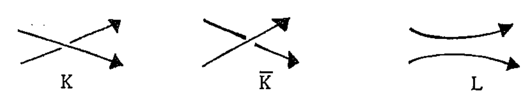
<u>The potential function provides a model for the Co polynomial</u>:

THEOREM 7.10.

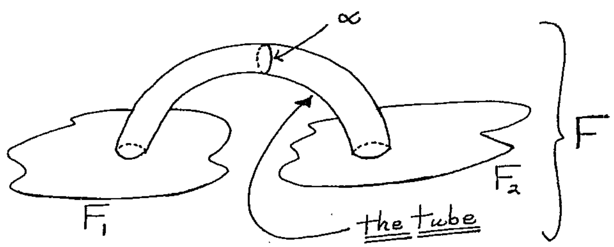
(i) *If K and K' are ambient isotopic orient links, then* $\Omega_K(t) = \Omega_{K'}(t)$.

(ii) *If* $K \sim 0$, *then* $\Omega_K(t) = 1$.

(iii) *If links* $K, \overline{K}$ *and* L *are related as belo then* $\Omega_K - \Omega_{\overline{K}} = (t-t^{-1})\Omega_L$.



Proof: We have already proved (i) and (ii). Note that $\Omega_K = 0$ if K is a split link. To see this, choose disjoint spanning surfaces for two pieces of the link, and connect these by a tube to form a connected spanning surface F.

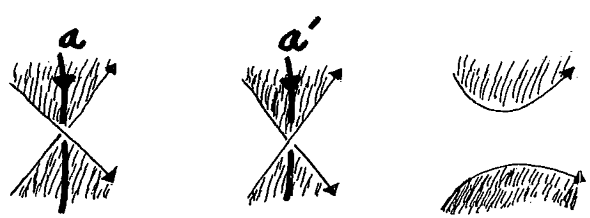


If α is a meridian of this type, then

$$H_1(F) \cong H_1(F_1) \oplus H_1(F_2) \oplus Z$$

where α generates the extra copy of Z. Since $\theta(\alpha,x) = \theta(x,\alpha) = 0 \ \forall\, x \in H_1(F)$, it follows that $\Omega_K(t) = 0$.

We use this discussion as follows. Consider Seifert surfaces for K, $\overline{K}$ and L. Locally, they appear as



F_K $\qquad$ $F_{\overline{K}}$ $\qquad$ F_L

We see that $H_1(F_K)$ and $H_1(F_{\overline{K}})$ will have one more homology generator than F_L, unless it should happen L is a split diagram. But in this case $\Omega_L = 0$ whil and $F_{\overline{K}}$ are isotopic by a 2π twist. Thus $\Omega_K - \Omega_{\overline{K}} =$ ($= \Omega_L$, proving (iii).

If L is not a split diagram, then the extra gen ator may be represented as $\underline{a}$ on F_K and $\underline{a'}$ on $F_{\overline{K}}$. see that $\theta(a',a') = \theta(a,a)+1$. Hence $\theta_K = \left[\begin{array}{c|c} n & \beta \\ \alpha & \theta_L \end{array}\right]$, $\theta_{\overline{K}} = \left[\begin{array}{c|c} n+1 & \beta \\ \alpha & \theta_L \end{array}\right]$, with appropriate choice of bases. I now a straightforward determinant calculation to show $\Omega_K - \Omega_{\overline{K}} = (t-t^{-1})\Omega_L$. ∎

Remark: By our axiomatics, it follows that the Conway polynomial and our potential function are related by th substitution $z = t-1/t$. Thus $\Omega_K(t) = \nabla_K(t-1/t)$. It amusing to solve the reverse. Then

$$t = z+1/t.$$

Hence

$$t = z+\cfrac{1}{z+\cfrac{1}{z+\cdots}}.$$

Using the notation $[z+1\circlearrowleft]$ for the continued fraction $z+\cfrac{1}{z+\cfrac{1}{z+\cdots}}$, we have $\nabla_K(z) = \Omega_K([z+1\circlearrowleft])$. In particula

$$\nabla_K(1) = \Omega_K\left[\frac{1+\sqrt{5}}{2}\right].$$

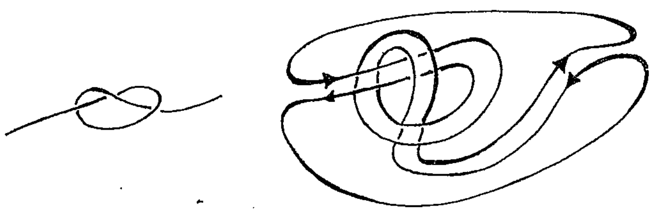
We shall return to this subject!

Example: Let T be a trefoil with $\theta = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$. Then

$$\Omega_T = D\begin{bmatrix} -t^{-1}+t & t^{-1} \\ -t & -t^{-1}+t \end{bmatrix} = (t-t^{-1})^2+1 = z^2+1.$$

This agrees with our previous calculations.

Example: Given a knot K, let K denote the numerator of the tangle obtained by running a parallel copy of K with opposite orientation. K is a link of two components.



K $\qquad\qquad\qquad\qquad$ $\hat{K}$

Since K has a spanning surface that is an annulus, we see that $\theta = [-lk(\hat{K})]$ is a Seifert matrix for $\hat{K}$. Therefore $\Omega_K = (t^{-1}-t)(-lk(\hat{K}))$ and hence $\underline{\nabla_K = lk(\hat{K})z}$. Apparently, in this case the Conway polynomial is much easier to compute using the Seifert pairing. (Compare this discussion with the last exercise of Chapter IV of these notes.)

TRANSLATING ∇ *AND* Ω.

Note that $\Omega_K(t) = D(t^{-1}\theta - t\theta')$. Therefore

$$\Omega_K(t^{-1}) = D(t\theta - t^{-1}\theta')$$
$$= D(t\theta' - t^{-1}\theta)$$
$$\therefore \quad \Omega_K(t^{-1}) = D(-(t^{-1}\theta - t\theta')).$$

Since θ is $2g\times 2g$ for knots, $(2g+1)\times(2g+1)$ for 2-c ponent links, we conclude that $\Omega_K(t^{-1}) = (-1)^{\mu+1}\Omega_K(t)$ where μ is the number of components of K.

To obtain a practical method of translation betwe Ω_K and ∇_K, we need to write $t^n+(-1)^n t^{-n} = T_n$ in te of $z = t-t^{-1}$. Look at the pattern:

$$t^2+t^{-2} = (t-t^{-1})^2+2 = z^2+2$$
$$t^3-t^{-3} = (t-t^{-1})^3+3t-3t^{-1} = z^3+3z.$$

__Exercise__. Let $T_n = t^n+(-1)^n t^{-n}$ and $z = t-t^{-1}$. __Shov__ that $T_{n+2} = zT_{n+1}+T_n$ for $n \geq 0$.

$$t-t^{-1} = z$$
$$t^2+t^{-2} = z^2+2$$
$$t^3-t^{-3} = z^3+3z$$
$$t^4+t^{-4} = z^4+4z^2+2$$
$$t^5-t^{-5} = z^5+5z^3+5z$$
$$t^6+t^{-6} = z^6+6z^4+9z^2+2.$$

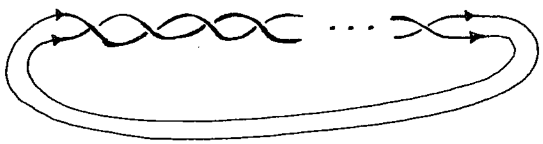
Show that the coefficient of z^2 in $t^{2n}+t^{-2n}$ is n^2.

We can use this exercise to obtain a curious formu for the second Conway coefficient $a_2(K)$. For let K knot. Then K has potential function in the form

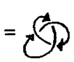
$\Omega_K(t) = b_0+b_1(t^2+t^{-2})+b_2(t^4-t^{-4})+\cdots+b_n(t^{2n}+t^{-2n})$. It follows from our exercise that

$$a_2(K) = b_1+4b_2+9b_3+16b_4+\cdots+n^2b_n.$$

<u>Exercise</u>. Compute Seifert pairing, determinant, potential function and signature for the torus knots and links of type (2,n).



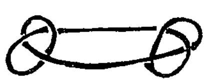
<u>n Crossings</u>

<u>Exercise</u>. Prove that $\sigma(K^!) = -\sigma(K)$ when K is a knot and $K^!$ is its mirror image. Calculate $\sigma(T)$ and thereby show that T =  and $T^!$ =  are not ambient isotopic.

<u>Exercise</u>. Prove that for knots K, K′,

$$\sigma(K\#K') = \sigma(K)+\sigma(K').$$

Use this exercise and the previous exercise to distinguish the granny and the square knot.



<u>square</u>

<u>granny</u>

<u>Exercise</u>. Choose a knot or link and compute everything can.

<u>Exercise</u>. Let K be a knot. Show that $\nabla_K(2i)/|\nabla_K(2i$ $= i^{\sigma(K)}$. Use this in conjunction with the (easily prov fact $\sigma_{K-} \leq \sigma_{K+} \leq 2 + \sigma_{K-}$ [ K+  K−] to show ho inductively calculate knot signatures using a skein dec osition (see [C1], [G1]).

Apply this method to the knot 9_{42} (see the end o Section 19 of Chapter VI in these notes) to show that has signature 2. This completes our earlier assertion 9_{42} is not amphicheiral.

X

THE ALEXANDER POLYNOMIAL AND THE ARF INVARIANT

Recall that we have defined, for a knot K, the invariant $A(K) \in Z_2$ via $A(K) \equiv a_2(K)$ (modulo-2) where $a_2(K)$ is the second Conway coefficient. And we showed (Chapter V) that $A(K) = 0$ for ribbon knots. In this chapter we will show that $A(K)$ is identical with the Arf invariant, $ARF(K)$, which is the Arf invariant of a mod-2 quadratic form related to K.

MOD-2 QUADRATIC FORMS

First recall that a mod-2 quadratic form q is a mapping $q : V \longrightarrow Z_2$ where V is a Z_2-vector space such that V has a bilinear symmetric pairing $\langle\ ,\ \rangle : V \times V \longrightarrow Z_2$. The mapping q must satisfy the following property:

$$(*)\qquad q(x+y) = q(x)+q(y)+\langle x,y\rangle \quad \text{for all } x,y \in V.$$

Remark: Over a field of characteristic $\neq 2$ quadratic forms and symmetric bilinear forms are in 1-1 correspondence. Thus if $[\ ,\ ] : W \times W \longrightarrow F$ is a symmetric

bilinear form, and char $F \neq 2$, then we can define $Q(x) = [x,x]/2$ and obtain:

$$Q(x,y) = \tfrac{1}{2}([x+y,\ x+y])$$
$$= \tfrac{1}{2}([x,x]+2[x,y]+[y,y])$$
$$\therefore\ Q(x,y) = Q(x) + Q(y) + [x,y].$$

In characteristic 2 the situation is subtler, and more tha one quadratic form may correspond to a given bilinear forn

Classically, a quadratic form in two variables looks like a quadratic polynomial,

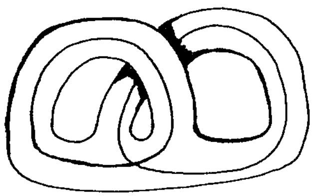
Beware the change of variables.

$Q(x,y) = ax^2+bxy+cy^2$ and if char $\neq 2$ then we can write $ax^2+bxy+cy^2 = (x,y)\begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ and classify the form $ax^2+bxy+cy^2$ by analyzing the congruence class of the matrix $\begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$.

In characteristic = 2, there is still a symmetric bilinear form associated with a quadratic polynomial, but now it occurs because $2 = 0$: If $Q(x,y) = ax^2+bxy+cy^2$, let $v = (x,y)$, $v_1 = (x_1,y_1)$, $v_2 = (x_2,y_2)$. Then

$$Q(v_1+v_2) = a(x_1+x_2)^2 + b(x_1+x_2)(y_1+y_2) + c(y_1+y_2)^2$$
$$= ax_1^2 + ax_2^2 + b(x_1y_1+x_2y_2+x_1y_2+x_2y_1) + cy_1^2 + cy_2^2$$
$$= Q(v_1) + Q(v_2) + b(x_1y_2+x_2y_1)$$
$$= Q(v_1) + Q(v_2) + v_1\begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix}v_2'.$$

The associated symmetric bilinear form has matrix $\begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} = b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. This should remind us of the mod-2 intersection form on the (punctured) torus:



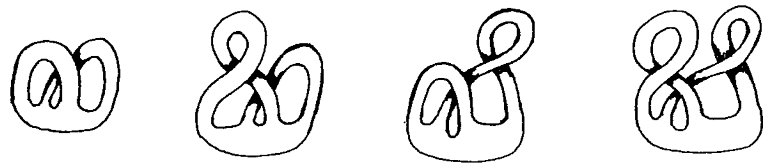
DEFINITION 10.1. *Let* $K \subset S^3$ *be a knot and* F *a connected oriented spanning surface for* K *with Seifert pairing* $\theta : H_1(F) \times H_1(F) \longrightarrow Z$. *Let* $V = H_1(F) \otimes Z_2$, $\overline{\theta} = \theta$ *on* V, *and let* $\langle\ ,\ \rangle$ *denote the mod-2 reduction of the intersection form* S *on* $H_1(F)$. *The* *__mod-2 quadratic form of__* F *is then defined by* $q(x) = \overline{\theta}(x,x)$ *for all* $x \in V$.

Note that

$$\begin{aligned} q(x+y) &= \overline{\theta}(x+y,\ x+y) \\ &= \overline{\theta}(x,x) + \overline{\theta}(y,y) + \overline{\theta}(x,y) + \overline{\theta}(y,x) \\ &\equiv q(x) + q(y) + (\theta(x,y)-\theta(y,x)) \pmod 2 \\ &\equiv q(x) + q(y) + S(x,y) \pmod 2 \\ \therefore\ q(x+y) &= q(x) + q(y) + \langle x,y\rangle . \end{aligned}$$

Thus the Seifert pairing produces a mod-2 quadratic form that is naturally associated with any spanning surface. We

see that with respect to the standard basis (symplectic basis) for the surface it is easy to write the quadratic polynomial that corresponds to the form. Thus:



$q_0 = xy$ $\quad x^2+xy$ $\quad xy+y^2$ $\quad x^2+xy+y^2 = q$

We know that the first three surfaces are isotopic, hence the forms xy, x^2+xy and $xy+y^2$ must be isomorphic! Indeed, this is the case. For example $x^2+xy = x(x+y)$ a so is isomorphic to xy via the change of basis $x' = x$, $y' = x+y$.

These four forms are nondegenerate in the sense that the associated bilinear form is nondegenerate. Here it i in matrix form $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Nondegeneracy of $\langle\ ,\ \rangle$ means that the matrix of $\langle\ ,\ \rangle$ is nonsingular.

In fact, we have just shown that there are at most t isomorphism classes of nondegenerate dimension-two mod-2 forms: $q_0 = xy$ and $q_1 = x^2+xy+y^2$. It is easy to see that q_0 and q_1 are not isomorphic. For, if $V = Z_2 \times Z$ then q_0 takes a majority of elements to 0, while q_1 takes a majority of elements to 1. Thus we have classifi rank-2 forms over Z_2.

DEFINITION 10.2. *Let* V *be a finite dimensional vector space over* Z_2 *and* $q : V \longrightarrow Z_2$ *a nondegenerate quadratic form. The* *Arf invariant* $ARF(q) \in Z_2$ *is defined by the formula*

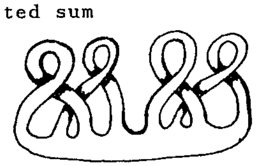
$$ARF(q) = \begin{cases} 0 & \textit{if } q \textit{ takes a majority of elements to } 0 \\ 1 & \textit{if } q \textit{ takes a majority of elements to } 1. \end{cases}$$

Certainly ARF is an invariant so long as it is well-defined. Indeed it is well-defined, and this comes about as follows:

(i) Symmetric bilinear forms over Z_2 are all (when nondegenerate) sums of forms of type $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. That is, there is a symplectic basis $\{a_1, \cdots, a_g, b_1, \cdots, b_g\}$ for V such that $\langle a_i, b_j \rangle = \delta_{ij}$, $\langle a_i, a_j \rangle = \langle b_i, b_j \rangle = 0$ for all i and j. This, of course, is given geometrically in our Seifert form case.

(ii) It follows from (i) that any nondegenerate mod-2 quadratic form is a direct sum of the two-dimensional forms. Hence it is a direct sum involving q_0 and q_1.

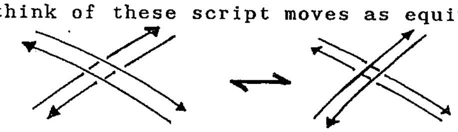
(iii) $q_1 \oplus q_1 \cong q_0 \oplus q_0$. This is the basic fact. You can prove it by a basis-change, or you can see it geometrically by taking the connected sum



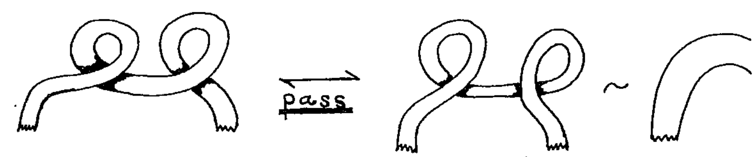
which has the form $q_1 \oplus q_1$ and find the basis change by topological script! Here we can use mod-2 script in the plane so that

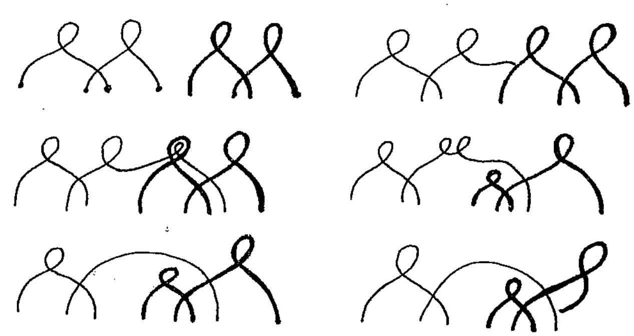
These modifications do not change the mod-2 quadratic form of the corresponding surface.

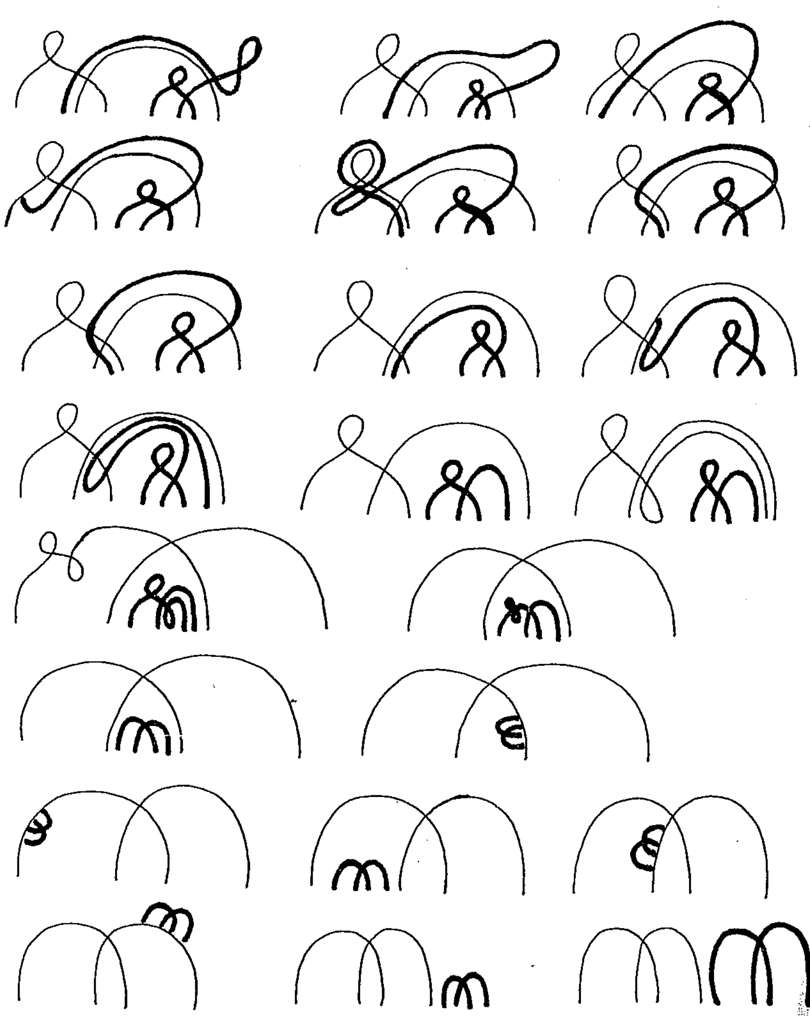
You may also think of these script moves as equival to consequences of



performed on the bands (compare with pass-equivalence, Chapter V). For then







Therefore $q_1 \oplus q_1 \cong q_0 \oplus q_0$. [Some of us will go to great lengths to avoid a little algebra.]

As a result, any mod-2 form is of the form

$$q_0 \oplus \cdots \oplus q_0 = \emptyset_0 \quad \text{or}$$

$$q_1 \oplus q_0 \oplus \cdots \oplus q_0 = \emptyset_1 .$$

It is then a counting matter to see that $\mathrm{ARF}(\emptyset_0) = 0$ and $\mathrm{ARF}(\emptyset_1) = 1$. Thus, we have classified all nondegenerate mod-2 quadratic forms, and shown the utility of the ARF invariant in the process.

(iv) It follows from what we have said, that $q \oplus q'$ ha an Arf invariant whenever q and q' have Arf invariants. Furthermore,

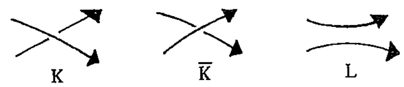
$$\mathrm{ARF}(q \oplus q') = \mathrm{ARF}(q) \oplus \mathrm{Arf}(q').$$

(v) It can be shown that (do it!) if $\{a_1, \cdots, a_g, b_1, \cdots, b_g\}$ is a symplectic basis for V $q : V \longrightarrow Z_2$ a mod-2 quadratic form, then $\mathrm{ARF}(q) = \sum_{k=1}^{g} q(a_k)q(b_k)$. This gives an explicit formula for ARF.

Let $K \subset S^3$ be a knot. We now define $\mathrm{ARF}(K) \in Z_2$ by the formula $\mathrm{ARF}(K) = \mathrm{ARF}(q)$ where q is the mod-2 quadratic form of any spanning surface for K. We leave i as an exercise in s-equivalence to see that this is an invariant of K.

THEOREM 10.3. *If knots* K *and* $\overline{K}$ *are related by one crossing change, and* L *is the 2-component link obtained* *by splicing this crossing, then*

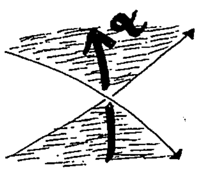
$$\mathrm{ARF}(K) - \mathrm{ARF}(\overline{K}) = \ell k(L).$$



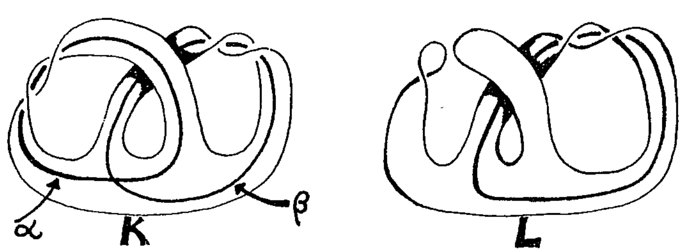
COROLLARY. *Let* A(K) *be the* mod-2 *reduction of the second Conway coefficient* $a_2(K)$. *Then* $A(K) = \mathrm{ARF}(K)$.

Proof: Exercise.

Proof of Theorem. Also an exercise. Compare on a spanning surface with the curve α depicted to the left as part of a symplectic basis. Note that you can assume that this appears as part of a band, and that the the dual curve β is on another band so



that the simplest picture gives:



THEOREM 10.4 (Levine [L2]). *Let* $K \subset S^3$ *be a knot. Let* $\Delta_K(t)$ *be the Alexander polynomial for* K. *Then*

$$\mathrm{ARF}(K) = 0 \Longleftrightarrow \Delta_K(-1) \equiv \pm 1 \pmod 8$$

$$\mathrm{ARF}(K) = 1 \Longleftrightarrow \Delta_K(-1) \equiv \pm 3 \pmod 8.$$

Proof: $\Delta_K(z)$ denotes the Conway polynomial. We know (Proposition 9.3) that

$$\Delta_K(\sqrt{t} - 1/\sqrt{t}) \doteq \Delta_K(t).$$

Hence $\Delta_K(2i) \doteq \Delta_K(-1)$ where $i = \sqrt{-1}$. Now, for a knot, $\Delta_K(z) = 1 + a_2z^2 + a_4z^4 + \cdots$. Hence $\Delta_K(2i) \equiv 1 - 4a_2(K)$ (mod 8 Since $a_2(K) \equiv \mathrm{ARF}(K)$ (mod-2), the theorem follows immed ately from this. ■

Remark: $3^2 \equiv 1$ (modulo 8).

In order to get a taste of the power of Levine's result for calculating Arf invariants, we now give a brief introduction to Fox's Free Differential Calculus, a its use in computing Alexander polynomials, hence, deriva tively, in computing ARF.