This is a collection of problems taken from Math 310 quizzes, hour exams, and final exams during the last 5 years or so, primarily those written by Professor Steve Hurder, and also including some from exams written by Professor Stephen Smith.

They are "sample" problems only. But they are the best kind – most were actually given on previous exams. You should know how to work all of the problems! No solutions are provided unfortunately.

Use the problems as study guides - work them in a group and compare answers; if you don't understand what a question is asking, or how to do it, then open the book and find where it discusses the material, read that page or section, and then try the problem again.

Remember: Good Test Technique is to spend 5 minutes to look over the Final Exam first, and identify those problems which are "easy", or which you know how to work for sure. Do these problems first, but keep an eye on the time. If there are 10 problems on the test, then you should try to spend an average of 10 minutes per problem. This leaves you with 15 minutes at the end to check your answers, mark your answers clearly, and try to work any stubborn problems you didn't solve yet.

AND, the very best advice for taking a Final Exam – GET A GOOD NIGHT'S SLEEP!!! Math is always easier when you are alert and not too tired. All of your studying is wasted if you snooze through the test.

Topics for Math 310

Each section may cover only a subset of these topics, and/or cover additional material not listed here.

1. How to solve a system of linear equations.
2. How to row reduce a matrix.
3. How to manipulate matrices (multiply, transpose, add and subtract.)
4. How to find inverse and determinant of a matrix.
5. Know properties of the determinant and how to use.
6. Interpretations of the row and column spaces of a matrix.
7. How to find the rank and nullity of a matrix.
8. Know relationship of rank, nullity, and size of the matrix.
9. Know facts about vector spaces and subspaces, and how to use.
10. How to find a basis of a vector space.
11. How to find the matrix giving the change of coordinates associated to a change of basis, and its relationship with the different matrices representing a linear transformation.
12. How to prove a subset of a vector space is a subspace.
13. How to represent a linear operator by a matrix.
14. How to compute the image and kernel of a linear transformation.
15. How to compute the inner product of two vectors, the length of a vector, the angle between two vectors, the projection of one vector onto another.
16. The Cauchy-Schwartz inequality.
17. How to find a basis of a subspace.
18. The relationship among the various subspaces associated to an m by n matrix.
19. The relationship between a subspace and its orthogonal complement.
20. How to solve least squares problems.
21. How to work with arbitrary inner product spaces.
22. How to do the Gram-Schmidt process.
23. How to find eigenvalues, eigenvectors, and eigenspaces.
24. Know when a matrix is diagonalizable.
25. How to diagonalize a matrix.
26. How to solve a system of linear differential equations.
27. How to compute A^n, and e^{tA} ($\hat{}$means "to the exponent".)
28. How to analyze a Markov process.

Gaussian Elimination and Row-Echelon Form

Problem 1: Use the Reduced Row-Echelon form to find all solutions of the equations

$$\begin{array}{rcrcrcr} x & + & 3y & + & z & = & 3 \\ 2x & + & 5y & + & z & = & 8 \\ 3x & + & 8y & + & 2z & = & 11 \end{array}$$

You must show all your steps and work for credit.

Problem 2: Find the general solution of $\begin{bmatrix} 1 & 3 & 3 \\ 2 & 6 & 9 \\ -1 & -3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$.

Problem 3: (a) Find the row-reduced echelon form of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

(b) What are the solutions of the system $Ax = 0$? (Check!)

Problem 4: Given the equations

$$\begin{aligned} x + 2y + 3z - 3w &= 1 \\ 4x + 5y + 6z - 6w &= 1 \\ 7x + 8y + 9z - 8w &= 1 \end{aligned}$$

a) Give the Reduced Row-Echelon form of the associated augmented matrix.

b) Which are the free variables? Which are the dependent variables?

c) Give the general solution of the system of equations.

Problem 5: Given the two equations

$$\begin{aligned} x + 2y + 3z - 4w &= 2 \\ 2x + 4y + 3z + w &= 5 \end{aligned}$$

Use the method of row reduction to solve the system. Indicate which are the free variables, which are the dependent variables. What is the geometric interpretation of the solution?

Problem 6: Let a, b, c be constants, and consider the system of equations

$$\begin{array}{rcrcrcr} 3x & + & 3y & + & z & = & a \\ x & + & y & + & 2z & = & b \\ 5x & + & 5y & & & = & c \end{array}$$

Find the equation that the constants a, b, c must satisfy so that these equations are consistent.

Matrix Algebra and Manipulating Matrices

Problem 1: In each case, give an example of a matrix which is

- <u>not the identity matrix</u>
- <u>not the zero matrix</u>,

and satisfies:

a) A is a 2×2 diagonal matrix with an inverse.

b) B is a 2×2 matrix with rank 1.

c) C is a 2×2 symmetric matrix with no inverse.

d) O is a 2×2 orthogonal matrix.

So, you must find four matrices A, B, C and O.

LU Decomposition and Elementary Matrices

Problem 1: Give the LU-decomposition of $A = \begin{bmatrix} 1 & 1 \\ 5 & 3 \end{bmatrix}$.

That is, find lower-triangular L and upper-triangular U so that $A = LU$.

Problem 2: Give the LU decomposition of $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix}$.

Problem 3: Compute the "LU" factorization of the matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}$.

Problem 4: Compute the "LU" factorization of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Matrix Determinants

Problem 1: Find the determinant of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$.

Problem 2: Use either the definition of determinant in terms of cofactors, or the method of row operations, to calculate the determinant of

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

Problem 3: Calculate the determinant of the matrix $B = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 6 & 2 & 0 \\ 1 & 1 & -2 & 3 \end{bmatrix}$.

Problem 4: Find the determinant of the matrix $\mathbf{A^3}$ where $A = \begin{bmatrix} 5 & 1 \\ 6 & 2 \end{bmatrix}$.

Problem 5: Given the matrices $A = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$,

calculate the following determinants:

a) $|A|$, $|B|$ and $|C|$

b) $|ABC^2|$

c) $|7 \cdot B|$

d) $|A^7 \cdot B|$

e) $|B \cdot C^{-1}|$

f) $|B^T C A^{-1}|$

g) $|A - B|$

Problem 6: a) Find the determinant of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 7 \end{bmatrix}$.

b) Use the solution to part a) to explain how many solutions the equation $A\vec{x} = \vec{b}$ has, where

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Cramer's Rule

Problem 1: Use Cramer's Rule to solve for x and y which satisfy the system of equations

$$\begin{aligned} x+y &= 5 \\ x+3y &= 7 \end{aligned}$$

Check your answer!

Problem 2: Use Cramer's Rule to solve for the variable y in the system of equations

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 1 \\ x & + & -y & + & z & = & 0 \\ 2x & + & 3y & - & z & = & 0 \end{array}$$

Problem 3: Use Cramer's Rule to solve for the variable z in the system of equations

$$\begin{array}{rcrcrcr} x & + & 3y & + & z & = & 1 \\ 2x & + & y & + & z & = & 5 \\ -2x & + & 2y & - & z & = & -8 \end{array}$$

Problem 4: Use Cramer's Rule to solve for the variable x in the system of equations

$$\begin{array}{rcrcrcr} 2x & + & y & - & 3z & = & 0 \\ 3x & - & 2y & + & z & = & 4 \\ x & - & y & + & 2z & = & 2 \end{array}$$

Problem 5: Use Cramer's Rule to solve for the variable x in the system of equations

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & a \\ x & + & -y & + & z & = & b \\ 2x & + & 3y & - & z & = & c \end{array}$$

Your answer should be a polynomial in the variables a, b, c.

Matrix Inverses

Problem 1: a) Find the inverse (by any method) of $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$.

b) Use the above to express the solutions of $A\vec{x} = \vec{b}$ in terms of the constants b_1 and b_2.

Problem 2: Give the formula for the inverse of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Problem 3: Use the method of Gaussian Elimination to find the inverse for $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 9 & 12 \end{bmatrix}$.

Problem 4: Use the method of Cofactors to find the inverse for $A = \begin{bmatrix} -1 & 2 & 1 \\ 2 & -1 & 2 \\ 1 & -2 & 1 \end{bmatrix}$.

Problem 5: Find the inverse of the following matrices (and check your answers.)
Do not use a calculator – you will be required to show all your work and computations.

a) $C = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & -2 & -3 \end{bmatrix}$

b) $C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 1 & 1 \end{bmatrix}$

c) $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}$

d) $A = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

Problem 6: For what values of the variable λ does the matrix **D** below have an inverse? **Explain your answer!**

$$D = \begin{bmatrix} 3-\lambda & 3 & 1 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & \lambda+1 \end{bmatrix}$$

Problem 7: Let A be an $n \times n$ matrix. Suppose that the system of equations $AX = 0$ has a unique solution. Explain why the inverse A^{-1} has to exist.

Vector Spaces and Subspaces

Problem 1: Consider the subset of vectors in $\mathbf{R}^2$ given by

$$S = \{(x, x^2) \text{ where x is any real number } \}$$

Is S a vector subspace? Justify your answer carefully.

Problem 2: Is the set $\left\{ \begin{bmatrix} x \\ x^3 \end{bmatrix} \text{where } x \in \mathbb{R} \right\}$ a vector subspace of $\mathbb{R}^2$? Justify your answer.

Problem 3: Let V be the space of real-valued functions of x. Show the solution set S of the equation

$$f'(x) = xf(x)$$

is a subspace of V.

Problem 4: Let V be the space of all differentiable functions on the line. Let W be the subset of all functions f which are solutions of the differential equation $f'' + 5f = 0$. Show that the solution set W is a subspace of V.

Problem 5: Let $A^{m \times n}$ be a matrix with m rows and n columns. What are the four fundamental subspaces associated to A? Give the definition of each of the following:

- $\text{Col}(A)$ = the column space of A.
- $\text{Row}(A)$ = the row space of A
- $\text{Null}(A)$ = the null space of A
- $\text{Conull}(A)$ = conull space of A

Linear Independence, Spanning, Basis, and Dimension

Problem 1: Find a basis for the subspace V of $\mathbb{R}^3$ spanned by the vectors

$$\vec{u}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$$

Problem 2: In the space $\mathcal{P}_3$ of polynomials of degree 2 or less, are the "vectors" $\{1+x, 1-x, 1+x+x^2\}$ linearly dependent or independent?

Problem 3: a) For $A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 3 & 1 & 2 \\ 3 & 1 & -1 & 6 \end{bmatrix}$ find a basis for the row space and the column space.

b) Is $A\vec{x} = \vec{b}$ solvable for all $\vec{b}$?

Problem 4: For the vectors

$$\vec{w}_1 = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{w}_2 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} \quad \text{and } \vec{x} = \begin{bmatrix} -9 \\ -2 \\ 5 \end{bmatrix}$$

Is $\vec{x}$ in the span of $\{\vec{w}_1, \vec{w}_2\}$? If so, write $\vec{x}$ as a linear combination of $\{\vec{w}_1, \vec{w}_2\}$.

Problem 5: Is $[1, 2, 3]^T$ in the span of $[4, 0, 5]^T$ and $[6, 0, 7]^T$?

Problem 6: a) Find a basis for the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 5 \\ -3 \\ 2 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 2 \\ 4 \\ -2 \\ 0 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 3 \\ 8 \\ -5 \\ 4 \end{bmatrix}$$

b) What is the dimension of the span of the vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$?

Problem 7: Do the "vectors" $1+x$, $1-x$, x^2 span the space $\mathcal{P}_3$ of polynomials of degree at most 2?

Problem 8: Find a basis for the subspace of 2×2 matrices $A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$ satisfying $a_{1,1} + a_{2,2} = 0$.

Column Space, Row Space, Null Space, Conull Space

Problem 1: A is an $m \times n$ matrix. Let

- $\text{Col}(A)$ denote the column space of A
- $\text{Row}(A)$ denote the row space of A
- $\text{Null}(A)$ denote the null space of A
- $\text{Conull}(A)$ the co-null space of A

For each of the following questions, your answer should be one of the above 4 spaces. Justify your answer by stating why you think it is correct.

a) The set of vectors perpendicular to the column space is what space?

b) The vector equation $A\vec{x} = \vec{b}$ has a solution if $\vec{b}$ belongs to what subspace?

c) The set of vectors perpendicular to the row space is what space?

d) The vector equation $A\vec{x} = \vec{b}$ has a unique solution if what space is $\{0\}$?

e) What number do you get if you add the dimensions of all 4 spaces?

Problem 2: Give a basis for the **column space**, **row space** and **null space** of the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 3 & 4 & 7 & 1 \\ 2 & 4 & 4 & 2 \end{bmatrix}$$

Problem 3: Find a basis for the null-space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 5 & 5 \\ 3 & 6 & 7 & 8 \end{bmatrix}$$

Problem 4: a) Find a basis for the column space of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 4 & 2 & 0 \end{bmatrix}$.

b) Find a basis for the perpendicular space $\text{Col}(A)^{\perp}$

c) Find a basis for $\text{Conull}(A)$

Problem 5: Let $B = \begin{bmatrix} 1 & 2 & -1 & 1 & 1 \\ 2 & 4 & 2 & 4 & 10 \\ 3 & 6 & -2 & 1 & 0 \end{bmatrix}$

Find a basis for the four fundamental spaces of B: the column space, the row space, the null space and the co-null space (the null space of the transpose B^T).

Problem 6: Given the system of equations

$$\begin{array}{ccccccc} x & + & y & + & z & = & c_1 \\ x & + & 2y & + & 2z & = & c_2 \\ x & + & 3y & + & 3z & = & c_3 \end{array}$$

a) For what values of $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ does the system have a solution?

b) If there exists a solution for a given $\vec{c}$, how many are there?

c) Find the basis for the co-null space of the matrix associated to the system of equations above.

d) What is the relation between your answers to part a) and c)?

Problem 7: A is a 3×5 matrix and $L\colon \mathbb{R}^5 \to \mathbb{R}^3$ is defined by $L(\vec{v}) = A \cdot \vec{v}$. Suppose that A has rank 3.

a) What is the dimension of the kernel of L?

b) What is the dimension of the range of L?

Explain your answers in terms of how you would find basis of these spaces if the matrix of A were given!

Problem 8: Let $A = \begin{bmatrix} 2 & 1 & 3 & 0 \\ 4 & 2 & 6 & 2 \\ 6 & 3 & 9 & 3 \end{bmatrix}$

a) Give the **Reduced Row Echelon form** of the matrix A

b) Find a basis for the **null-space** of the matrix A

c) Find a basis for the **column space** of the matrix A

d) What is the **dimension** of the null–space $N(A)$ and the column space $C(A)$?

e) Answer **True** or **False**, and explain your answer:

The equation $A\vec{x} = \vec{b}$ has a solution for every vector $\vec{b} \in \mathbb{R}^3$.

Change of Basis and Coordinates

Problem 1: Find the coordinates of $\vec{p} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ with respect to the basis $\vec{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Problem 2: Find the new coordinates $[a, b, c]^T$ of the point $\vec{x} = [7, 5, 6]^T$ with respect to the basis for $\mathbb{R}^3$ given by the vectors

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ \vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Problem 3: Given the vectors in $\mathbb{R}^2$

$$\vec{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

a) Find the transition matrix S corresponding to change of basis from $\{\vec{v}_1, \vec{v}_2\}$ to $\{\vec{u}_1, \vec{u}_2\}$.

b) Find the coordinate expression of $\vec{p} = 3\vec{v}_1 - \vec{v}_2$ with respect to the basis $\{\vec{u}_1, \vec{u}_2\}$.

Linear Transformations and Finding a Matrix Representation

Problem 1: Let $\mathcal{P}_3$ be the space of polynomials of degree 2. Show that the map $L\colon \mathcal{P}_3 \to \mathcal{P}_3$ given by

$$L(p(x)) = p(x) - x \cdot p'(x)$$

is linear. (Here, $p'(x)$ denotes the first derivative of the polynomial $p(x)$.)

Problem 2: Find the matrix, in the standard basis for $\mathbb{R}^3$, for the linear transformation

$$L\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x - y - z \\ x - 2y + z \\ -x + 3y + 2z \end{bmatrix}.$$

b) Find the kernel of L

Problem 3: Define the linear transformation $L\colon \mathcal{P}_3 \to \mathcal{P}_3$ by

$$L(p(x)) = x\ p''(x) - 2x\ p'(x) + p(x)$$

Find the matrix representing L with respect to the basis $\{1, x, x^2\}$ of $\mathcal{P}_3$.

Problem 4: Find the matrix representation for the linear transformation

$$L\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4x - y \\ -x + 4y \end{bmatrix}.$$

with respect to the basis $\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

Problem 5: Let V be the space of functions with basis $\{\sin(x), \cos(x), \sin(2x), \cos(2x)\}$.

Define the linear transformation $L\colon V \to V$ by

$$L(f) = f'' + f' - 4f$$

a) Find the matrix representing L with respect to the given basis.

b) Find the kernel of L

Problem 6: Let a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(v_1, v_2, v_3) = (3v_1 + 2v_2 + v_3, 2v_1 + v_2, v_2).$$

Give the matrix (in the *standard* basis) for T.

Problem 7: Let V be the vector space spanned by the functions $\{e^x, e^{2x}, e^{3x}\}$,

and let $L\colon V \to V$ be the linear transformation defined by $L(f) = f' - 2f$.

a) Find the matrix representing L with respect to the basis $\{e^x, e^{2x}, e^{3x}\}$ of V.

b) Find the kernel of L.

Problem 8: Define the linear transformation $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $L(\vec{v}) = A\vec{v}$ where $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$.

Find the matrix of L with respect to the new basis $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Change of Basis for Linear Transformations and Similarity

Problem 1: The linear transformation $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ has matrix $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$ with respect to the standard basis $\{\vec{e}_1, \vec{e}_2\}$ of $\mathbb{R}^2$. Find the matrix of L with respect to the new basis

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Problem 2: a) Find the matrix representation A with respect to the standard basis $\{\vec{e}_1, \vec{e}_2\}$ of $\mathbb{R}^2$ for the linear transformation

$$L\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4x - y \\ -x + 4y \end{bmatrix}.$$

b) Find the matrix representation B of L with respect to the basis $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Problem 3: Let $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by $\mathbf{L}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4y + 6z \\ -2x - 3y \\ x + 2y + z \end{bmatrix}$.

a) Find the matrix representing L with respect to the standard basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ of $\mathbb{R}^3$.

b) Use the answer to part a) to find the matrix representing L with respect to the new basis

$$\vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Problem 4: Given the vectors in $\mathbb{R}^2$

$$\vec{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

a) Find the transition matrix S corresponding to change of basis from $\{\vec{v}_1, \vec{v}_2\}$ to $\{\vec{u}_1, \vec{u}_2\}$.

b) The linear transformation $L\colon R^2 \to R^2$ has a matrix representation $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ with respect to the basis $\{\vec{u}_1, \vec{u}_2\}$. Find the matrix representation B of L with respect to the basis $\{\vec{v}_1, \vec{v}_2\}$.

Problem 5: For the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

a) Find the transition matrix S corresponding to the change of basis from the standard basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ of $\mathbb{R}^3$ to the new basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

b) Let $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$L(\vec{v}_1) = \vec{v}_1, \ L(\vec{v}_2) = 2 \cdot \vec{v}_2, \ L(\vec{v}_3) = 3 \cdot \vec{v}_3$$

Find the matrix representing L with respect to the standard basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ of $\mathbb{R}^3$.

Problem 6: a) Let $\{\vec{v}_1, \vec{v}_2\}$ be a basis for $\mathbb{R}^2$, and let L be a linear transformation of $\mathbb{R}^2$ so that

$$L(c_1\vec{v}_1 + c_2\vec{v}_2) = (c_1 + 3c_2)\vec{v}_1 + (2c_1 + 4c_2)\vec{v}_2$$

Find the matrix representing L with respect to the basis $\{\vec{v}_1, \vec{v}_2\}$.

b) Suppose that $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Find the matrix representing L with respect to the standard basis of $\mathbb{R}^2$.

Problem 7: Let $A = \begin{bmatrix} 7 & -2 \\ 15 & -4 \end{bmatrix}$. Define the linear map $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $L(\vec{x}) = A\vec{x}$

a) Find the matrix B for the linear map L with respect to the new basis $\vec{u}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $\vec{u}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

b) Suppose that $\vec{p}$ has coordinates $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ with respect to the basis $\{\vec{u}_1, \vec{u}_2\}$. Find $L(\vec{p})$ with respect to the basis $\{\vec{u}_1, \vec{u}_2\}$.

c) Suppose that $\vec{p} = \vec{u}_2$. Find $L^{101}(\vec{u}_2) = L(L(\ldots L(L(\vec{u}_2))\ldots))$.

Dot Product, Inner Products and Geometry

Problem 1: Find the cosine of the angle between the vectors $\vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

Problem 2: V is the vector space $C([-1,1])$ of continuous functions on the interval $[-1,1]$ with the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\, dx$$

a) Find the length of the vectors $f(x) = 1$ and $g(x) = x^2$.

b) Find the angle θ between the vectors $f(x) = 1$ and $g(x) = x^2$

Problem 3: V is the vector space $C([0,1])$ of continuous functions on the interval $[0,1]$ with the inner product

$$\langle f, g \rangle = \int_{0}^{1} f(x)g(x)\, dx$$

a) Find the length of the vectors $f(x) = x - 1$ and $g(x) = x + 1$.

b) Find the angle θ between the vectors $f(x) = x + 1$ and $g(x) = x - 1$

Vector Projection and Distance to Subspaces

Problem 1: For the point $\vec{p} = (1, 2, 3)^T$ and the plane **2x + y - 2z = 0**

a. Find a unit normal vector $\vec{n}$ to the plane.

b. Find the distance from the point $\vec{p}$ to the plane.

c. Find the projection onto the plane of the point $\vec{p}$. (Hint: use your answer to parts a) and b).)

Problem 2: Given the point $\vec{P} = (5, 5)$ and the line ℓ defined by $\mathbf{3x - 4y = 0}$

a) Find the point on the line ℓ closest to the point $\vec{P}$.

b) What is the distance from the point $\vec{P}$ to the line ℓ?

Problem 3: Find the distance from the point $\vec{p} = [3, 3, 3]$ to the plane $x - y + 3z = 0$.

Problem 4: Given a point $\vec{x} = (2, 4)$ and a line ℓ defined by $\mathbf{2y - x = 0}$

a) Find the point on the line ℓ closest to the point $\vec{x}$.

b) What is the distance from the point $\vec{x}$ to the line ℓ?

c) Find vectors $\vec{p}$ on the line ℓ and $\vec{z}$ perpendicular to the line ℓ so that $\vec{x} = \vec{p} + \vec{z}$.

Least Squares Solutions and Best Fit Curves

Problem 1: Find the least–squares solution to the system of equations

$$\begin{array}{rcrcr} -x & + & 2y & = & 1 \\ 2x & + & y & = & 1 \\ x & + & 2y & = & 1 \end{array}$$

Problem 2: Find the least-squares best-fit by a linear function $y = a + bx$ to the data

x	1	2	3	4
y	3	0	-1	-1

Problem 3: Find the least-squares best–fit solution to the system of equations

$$\begin{array}{rcrcr} -x & - & 2y & = & -2 \\ x & + & 2y & = & -1 \\ 2x & + & y & = & 0 \\ 2x & + & 2y & = & 2 \end{array}$$

Problem 4: Find the parabola $\mathbf{y = a + bx + cx^2}$ that passes through the three points $(\mathbf{-1, 3})$, $(\mathbf{0, 1})$, $(\mathbf{1, 4})$. (Hint: there are two ways to do the problem – one direct, and the other using "least squares".)

Problem 5: Find the least squares solution of $\begin{bmatrix} -1 & 0 \\ 2 & 3 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

b) Let V be the subspace of $\mathbb{R}^3$ spanned by the vectors $\vec{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix}$.

Find the point in V that is closest to the point $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Problem 6: a) Find the least-squares approximate-solution of the system of equations

$$\begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

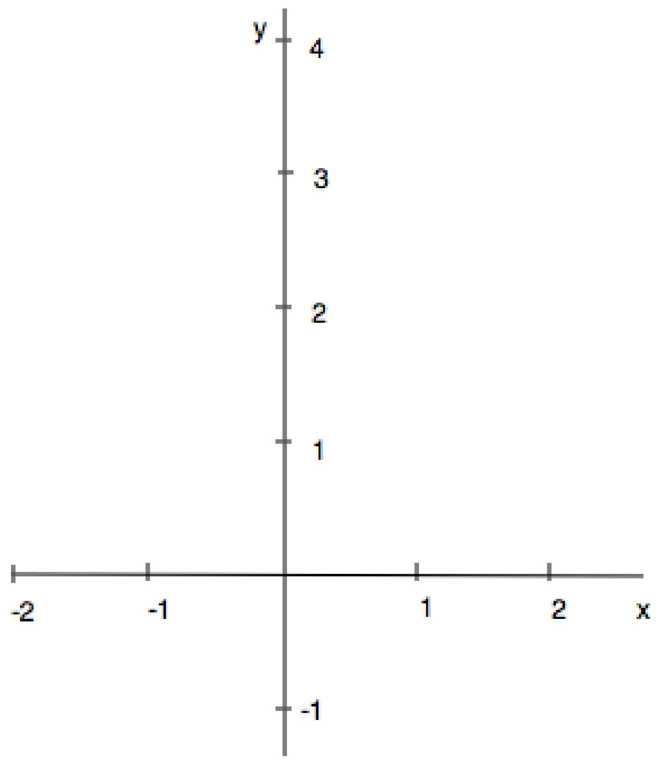
b) How large is the error?

Problem 7: Find the equation of the line $y = a + bx$ which gives a "least squares best fit" to the data $\{(-2, -3), (-1, -1), (1, 1), (1, 3)\}$

Problem 8: a) Find the best least squares fit by a linear function $y = a + bx$ to the data

x	-1	0	1	2
y	-1	1	3	4

b) Plot the linear function from part a) along with the data on the coordinate system below:



Orthonormal Sets and Gram-Schmidt

Problem 1: Given the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, use the Gram-Schmidt method to find orthonormal vectors $\{\vec{u}_1, \vec{u}_2\}$ so that $\vec{u}_1$ is colinear with $\vec{v}_1$ and $\text{Span}\{\vec{v}_1, \vec{v}_2\} = \text{Span}\{\vec{u}_1, \vec{u}_2\}$.

Problem 2: Use the Gram–Schmidt method to find orthonormal vectors $\vec{u}_1$ and $\vec{u}_2$ which span the same subspace as $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$.

Problem 3: Let P be the subspace of $\mathbb{R}^3$ consisting of vectors orthogonal to $\vec{x} = [1, 2, -1]^T$.

a) Find a basis for the subspace P.

a) Use your answer to part a) to find an orthonormal basis $\{\vec{u}_1, \vec{u}_2\}$ for P.

Problem 4: Let V be the subspace of $\mathbb{R}^3$ spanned by the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

a) Find a basis for V.

b) Use your answer for part a) and the Gram-Schmidt method to give an orthonormal basis for V.

Problem 5: Find an orthonormal basis for the subspace of $\mathbb{R}^3$ perpendicular to $\vec{v} = (4, 3, -3)$.

Problem 6: Let V be the subspace of $\mathbb{R}^3$ spanned by the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

a) Find a basis for the orthogonal space $V^{\perp}$.

b) Find a basis for V.

c) Use the Gram-Schmidt method to give an orthonormal basis for V.

d) Use your answer to a) to extend the orthonormal vectors of c) to an orthonormal basis of $\mathbb{R}^3$.

Problem 7: Find an *orthonormal* basis for the subspace V of $\mathbb{R}^3$ spanned by the vectors

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$$

Problem 8: Find a unit vector in $\mathbb{R}^4$ which is orthogonal to the span of the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ -4 \end{bmatrix}$$

Problem 9: Find an orthonormal basis for the subspace in $\mathbb{R}^4$ which is orthogonal to the span of the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ -4 \end{bmatrix}$$

Problem 10: Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

(a) Find an orthonormal basis for row space of A.

(b) Find an orthonormal basis for the orthogonal complement to the row space of A.

Problem 11: Let $C = \begin{bmatrix} 2 & 5 \\ 1 & 10 \end{bmatrix}$. Find an orthogonal matrix Q and a triangular matrix R so that $C = Q \cdot R$.

Problem 12: a) Find an orthonormal basis for the column space of $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$.

b) Use the answer to part a) to give the QR factorization of A.

(That is, $A^{3\times2} = Q^{3\times2} \cdot R^{2\times2}$ where Q has orthonormal columns and R is upper triangular).

Eigenvalues, Eigenvectors and Eigenspaces

Problem 1: Find the eigenvalues and corresponding eigenvectors for $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

Problem 2: Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = 3, \lambda_2 = 0, \lambda_3 = 0$.

a) Find the eigenvectors for these eigenvalues.

b) Note A is symmetric. Find an *orthogonal* matrix S with $S^{-1}AS = D$ diagonal.

Problem 3: Find the eigenvalues and eigenvectors for $A = \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$.

Systems of Differential Equations

Problem 1: Given the differential equations with initial conditions

$$\begin{array}{rcrcrcrcl} x' & = & 3x & + & 4y & ; & x(0) & = & 1 \\ y' & = & -2x & - & 3y & ; & y(0) & = & 2 \end{array}$$

Find the functions $x(t)$ and $y(t)$.

Problem 2: a) Give the general solution of the differential system

$$\begin{array}{rcrcr} y_1' & = & y_1 & + & y_2 \\ y_2' & = & -2y_1 & + & 4y_2 \end{array}$$

b) Give the particular solution when $y_1(0) = 3$ and $y_2(0) = 1$.

Finding Powers, Square Roots and Exponentials of Matrices

Problem 1: For the matrix $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$

a) Find 2×2 matrices S and D such that $A = S \cdot D \cdot S^{-1}$

b) Use your answer to part a) to calculate A^5.

Problem 2: For $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$ find the matrix e^A. (Your answer should be a 2×2 matrix.)

Problem 3: For $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$, find the 2×2 matrix e^{tA}.

Problem 4: For $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

a) Calculate A^2, A^3, A^4, A^5.

b) Find the eigenvalues and eigenvectors for A.

c) Use your answer to part b) to calculate A^{10}.

d) Use your answer to part b) to get a formula for A^n when n is a positive integer.

Problem 1: UIC plans to change their phone registration system so that students can change their schedule each day, throughout the semester. Math 310 is offered at two times, 10 AM and 2 PM. Each day, 1/10 of the students in the 10 AM class switch to the 2 PM class, while 1/5 of the students in the 2 PM class switch to 10 AM. The rest of the students stay in their respective classes (for that day at least).

a) Let $X_n = \begin{bmatrix} AM_n \\ PM_n \end{bmatrix}$ represent the number of students in the each of these classes on day n of the semester. Give a matrix equation for the relation between X_{n+1} and X_n.

b) 48 students are registered in Math 310 on the first day of class (this is the total for both classes.) How many students are in the 10 AM section at the end of the semester?

Problem 2: Assume there are 3 long-distance telephone companies, **ATT**, **MCI** and **Sprint**. Suppose also that each year

– 20% of ATT customers switch to MCI, 10% switch to Sprint, and the rest stay with ATT.

– 30% of MCI customers switch to ATT, 10% switch to Sprint, and the rest stay with MCI.

– 20% of Sprint customers switch to ATT, 20% switch to MCI, and the rest stay with Sprint.

Let $\mathbf{P_n} = \begin{bmatrix} A_n \\ M_n \\ S_n \end{bmatrix}$ be the number of customers for ATT, MCI and Sprint after n years.

a) Find the Markov matrix $\mathbf{T}$ expressing the vector $\mathbf{P_{n+1}}$ in terms of the vector $\mathbf{P_n}$.

b) Find an eigenvector for the steady-state distribution of customers (i.e. for the eigenvalue $\lambda = 1$.)

c) In year 0, assume there are 3000 ATT customers, 0 MCI customers, and 0 Sprint customers.

Find the number of customers for each company after a very long time (n = 100 for example.)

Problem 3: My garden has three kinds of plants, **New**, **Old** and **Dead**.

After a year, 1/2 of the **New** plants die, and the remaining **New** plants become **Old** plants.

1/4 of the **Old** plants die, while the remainder stay **Old**.

Of course, all the **Dead** plants stay **Dead** :-(

The number of each kind of plant in my garden after **k** years from this Spring is $P_k = \begin{bmatrix} N_k \\ O_k \\ D_k \end{bmatrix}$.

a) Find the matrix $\mathbf{T}$ so that $\mathbf{P_{k+1}} = \mathbf{T} \cdot \mathbf{P_k}$ gives the number of each kind of plant one year later.

b) Suppose that this Spring there are 10 **New** plants, 20 **Old** plants and 10 **Dead** plants in my garden. If I do not plant any more **New** plants ever again, use matrix algebra to calculate how many **Dead** plants there will be in my garden in 2 years.

c) How many **New** or **Old** plants will there be in my garden in 5 years?

Problem 4: There are 10 plants in my living room. Every Saturday, I move 1/2 of the plants in the east windows to the west windows, and 1/3 of the plants in the west windows to the east windows. Let $\mathbf{E_n}$ denote the number of plants in the east windows, and $\mathbf{W_n}$ the number of plants in the west windows on the n^{th} Saturday. In vector notation, the number of plants in the east and west windows at week $\mathbf{n}$ is

$$\mathbf{P_n} = \begin{bmatrix} \mathbf{E_n} \\ \mathbf{W_n} \end{bmatrix}$$

a) Find the Markov transition matrix $\mathbf{T}$ so that $\mathbf{P_{n+1}} = \mathbf{T} \cdot \mathbf{P_n}$.

b) Find a steady-state eigenvector for $\mathbf{T}$.

c) Assuming that I start with an equal number of plants in both the east and west windows, find steady-state number of plants in the west windows (i.e., the number of plants after many weeks.)

Problem 5: In the summer, my kitchen is full of ants. To be exact, there were 120 ants each day, all during last summer. Watching the ants, they like to stay either in my Trash, or on the Dishes in my sink. But each night, 10% of the ants in the Trash went over to the Dishes, while 30% of the ants on the Dishes went over to the Trash.

a) Find the Markov transition matrix $\mathbf{P}$ describing the behavior of the ants from one day to the next.

b) Find a steady-state solution of this Markov process. (ie, find the number of ants in the Trash and on the Dishes so that night after night, these numbers remain the same.)

Problem 5: There are 53 combined number of patrons at 3 all-night taverns in Cicero:

- H = He's Not Here
- S = Smitty's Lounge
- K = Kirk's Corner

Each hour, 1/3 of the patrons at H leave and go to S, and 1/4 leave and go to K.

Each hour, 1/3 of the patrons at S leave and go to H, and 1/3 leave and go to K.

Each hour, 1/2 of the patrons at K leave and go to S, and the rest remain at K.

a) Find the Markov transition matrix P describing the hourly behavior of the patrons.

b) Beginning on Thanksgiving Day, there are 20 patrons each at H and S, and 13 at K. They drink solidly, around-the-clock, until New Year's Day, following the above pattern. What is the distribution of patrons at these taverns on New Year's Day? (That is, what is the long term distribution of the patrons based on the Markov model?)

Problem 6: Given the Markov matrix $P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$.

a) Find the "steady state" eigenvector for M. (That is, the eigenvector for $\lambda = 1$.)

b) Diagonalize P and use this to get an expression for the power P^n.