5.2 Show how to obtain geometrically the two different presentations of the fundamental group of a Klein Bottle mentioned as an example in Section III.6.

5.3 Consider the presentation of the fundamental group of the Klein Bottle with two generators, a and b, and one relation, $baba^{-1}$. Prove that the subgroup generated by b is a normal subgroup, and that the quotient group is infinite cyclic. Prove also that the subgroup generated by a is infinite cyclic.

5.4 The fact that the connected sum of three projective planes is homeomorphic to the connected sum of a torus and a projective plane gives rise to two different presentations of the fundamental group (as in Problem 5.2). Prove algebraically that these presentations represent isomorphic groups.

5.5 For any integer $n > 2$, show how to construct a space whose fundamental group is cyclic of order n.

5.6 Prove that the fundamental group of a compact nonorientable surface of genus n has a presentation consisting of n generators, $\alpha_1, \ldots, \alpha_n$, and one relation, $\alpha_1\alpha_2 \ldots \alpha_n\alpha_1^{-1}\alpha_2^{-1} \ldots \alpha_{n-1}^{-1}\alpha_n$ (see Exercise I.8.10).

5.7 Prove that the fundamental group of a compact, orientable surface of genus n has a presentation consisting of $2n$ generators, $\alpha_1, \alpha_2, \ldots, \alpha_{2n}$, and one relation, $\alpha_1\alpha_2 \ldots \alpha_{2n}\alpha_1^{-1}\alpha_2^{-1} \ldots \alpha_{2n}^{-1}$ (see Exercise I.8.11).

6 Application to knot theory

A *knot* is, by definition, a simple closed curve in Euclidean 3-space. It is a mathematical abstraction of our intuitive idea of a knot tied in a piece of string; the two ends of the string are to be thought of as spliced together so that the knot can not become untied.

It is also necessary to define when two knots are to be thought of as equivalent or nonequivalent. Here it would be highly desirable to frame the definition so that it corresponds to the usual notion of two knots in two different pieces of string being the same. Of several alternative ways of doing this, the following definition is now universally accepted (as the result of many years of experience) as being the most suitable.

Definition Two knots K_1 and K_2 contained in $\mathbf{R}^3$ *are equivalent* if there exists an orientation-preserving homeomorphism $h : \mathbf{R}^3 \to \mathbf{R}^3$ such that $h(K_1) = K_2$.

Obviously, if K_1 and K_2 are equivalent according to this definition, then h maps $\mathbf{R}^3 - K_1$ homeomorphically onto $\mathbf{R}^3 - K_2$. Therefore, $\mathbf{R}^3 - K_1$ and $\mathbf{R}^3 - K_2$ have isomorphic fundamental groups. Thus, given two knots K_1 and K_2 in $\mathbf{R}^3$, if we can prove that the groups $\pi(\mathbf{R}^3 - K_1)$ and $\pi(\mathbf{R}^3 - K_2)$ are nonisomorphic, then we know the knots K_1 and K_2 are nonequivalent. This is the most common method of distinguishing between knots. The fundamental group $\pi(\mathbf{R}^3 - K)$ is calle[d] the *group of the knot* K.

We shall show how it is possible to use the Seifert-Van Kampe[n] theorem to determine a presentation of the group of certain knots, an[d] then discuss the problem of proving that these groups are nonisomorphic.

In certain cases, it will be convenient to think of the knots we shal[l] consider as being imbedded in the 3-sphere S^3,

$$S^3 = \{x \in \mathbf{R}^4 : |x| = 1\}$$

rather than being imbedded in $\mathbf{R}^3$. This makes little difference, becaus[e] S^3 is homeomorphic to the Alexandroff 1-point compactification of $\mathbf{R}^3$; this can be proved by stereographic projection (see Newman, M. H. A. *Elements of the Topology of Plane Sets of Points*. Cambridge: The University Press, 1951, pp. 64–65).

Exercise

6.1 If K is a knot in $\mathbf{R}^3$ and we regard S^3 as the 1-point compactification o[f] $\mathbf{R}^3$, prove that the fundamental groups $\pi(\mathbf{R}^3 - K)$ and $\pi(S^3 - K)$ are isomo[r]phic. (HINT: Use Theorem 4.1.)

We shall consider a class of knots called *torus knots* because they ar[e] contained in a torus imbedded in $\mathbf{R}^3$ in the standard way (i.e., the toru[s] is obtained by rotating a circle about a line in its plane). Recall that torus may be considered as the space obtained by identifying the opposit[e] edges of the unit square,

$$\{(x, y) \in \mathbf{R}^2 : 0 \leqq x \leqq 1, \qquad 0 \leqq y \leqq 1\},$$

or, alternatively, as the space obtained from the entire plane $\mathbf{R}^2$ b[y] identifying two points (x, y) and (x', y') if and only if $x - x'$ and $y - y'$ are both integers. Let $p : \mathbf{R}^2 \to T$ be the identification map. Let L be line through the origin in $\mathbf{R}^2$ with slope m/n, where $1 < m < n$, and m and n are relatively prime integers. It is readily seen that the image

$$K = p(L)$$

is a simple closed curve on the torus T; it spirals around the torus m times while going around it n times the other way. If we now assum[e] that T is imbedded in $\mathbf{R}^3$ in the standard way, then

$$K \subset T \subset \mathbf{R}^3,$$

and K is a knot in $\mathbf{R}^3$ called a *torus knot of type* (m, n). Such knots wi[ll] be our main object of study.

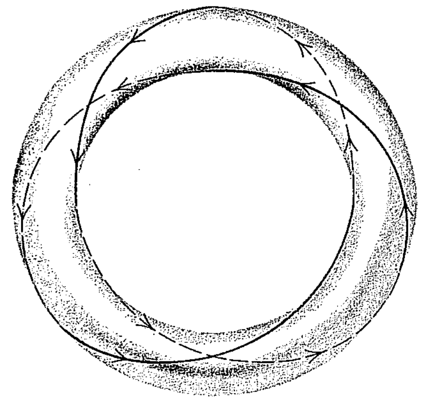

FIGURE 4.10 Torus knot of type (2, 3).

We shall also consider *unknotted circles* in $\mathbf{R}^3$, i.e., any knot equivalent to an ordinary Euclidean circle in a plane in $\mathbf{R}^3$.

To begin, we obtain a presentation of the group of a torus knot of type (m, n) and of the group of an unknotted circle. The first step is to obtain a certain decomposition of the 3-sphere S^3 into two pieces, which is necessary for the use of the Seifert-Van Kampen theorem. Let

$$A = \{(x_1, x_2, x_3, x_4) \in S^3 : x_1^2 + x_2^2 \leqq x_3^2 + x_4^2\},$$
$$B = \{(x_1, x_2, x_3, x_4) \in S^3 : x_1^2 + x_2^2 \geqq x_3^2 + x_4^2\}.$$

It is clear that A and B are closed subsets of S^3, that $A \cup B = S^3$, and that

$$A \cap B = \{(x_1, x_2, x_3, x_4) \in S^3 : x_1^2 + x_2^2 = \tfrac{1}{2} \text{ and } x_3^2 + x_4^2 = \tfrac{1}{2}\}$$

From this it is clear that $A \cap B$ is a torus; in fact, it is the Cartesian product of the circle $x_1^2 + x_2^2 = \frac{1}{2}$ [in the (x_1, x_2) plane] and the circle $x_3^2 + x_4^2 = \frac{1}{2}$ [in the (x_3, x_4) plane].

We now assert that A and B are each solid tori (i.e., homeomorphic to the product of a disc and a circle). We shall prove this by exhibiting a homeomorphism. Let

$$D = \{(x_1, x_2) \in \mathbf{R}^2 : x_1^2 + x_2^2 \leqq \tfrac{1}{2}\}$$
$$S^1 = \{(x_3, x_4) \in \mathbf{R}^2 : x_3^2 + x_4^2 = \tfrac{1}{2}\}$$

be a closed disc and a circle, each of radius $\frac{1}{2}\sqrt{2}$. Define a map

$$f : D \times S^1 \to A$$

by the formula

$$f(x_1, x_2, x_3, x_4) = (x_1, x_2, \sqrt{2}\, x_3[1 - (x_1^2 + x_2^2)]^{1/2}, \sqrt{2}\, x_4[1 - (x_1^2 + x_2^2)]^{1/2}).$$

This function is obviously continuous. We leave it to the reader to verify that it is one-to-one and onto, and hence a homeomorphism. A similar proof applies to the set B. It is also clear from this that the torus $A \cap B$ is the common boundary of the two solid tori A and B.

We leave it to the reader to verify that, under stereographic projection, the torus $A \cap B$ corresponds to a torus imbedded in $\mathbf{R}^3$ in the standard way.

First, we consider the group of an unknotted circle K in S^3. We can take as our unknotted circle the "center line" of the solid torus A:

$$K = \{(x_1, x_2, x_3, x_4) \in A : x_1 = x_2 = 0\}.$$

Then, K is the unit circle in the (x_3, x_4) plane. Clearly, the boundary of A is a deformation retract of $A - K$; therefore B is a deformation retract of $S^3 - K$. It is also clear that the center line of B,

$$\{(x_1, x_2, x_3, x_4) \in B : x_3 = x_4 = 0\},$$

is a deformation retract of B. Therefore, the center line of B is a deformation retract of $S^3 - K$. Hence, $S^3 - K$ has the homotopy type of a circle, and the group of K is infinite cyclic. Thus, we have proved

Proposition 6.1 *The group of an unknotted circle in* $\mathbf{R}^3$ *is infinite cyclic.*

Next, we consider a torus knot K of type (m, n) in S^3. We can consider K a subset of the torus $A \cap B \subset S^3$. It would be convenient to apply the Seifert-Van Kampen theorem to determine the fundamental group of $S^3 - K$ by using the fact that

$$S^3 - K = (A - K) \cup (B - K).$$

Then, $A - K$, $B - K$, and $(A - K) \cap (B - K)$ are all arcwise connected, but unfortunately $A - K$ and $B - K$ are not open subsets of $S^3 - K$. The way around this difficulty is clear: We enlarge A and B slightly to obtain open sets with the same homotopy type as A and B.

To be precise, choose a number $\varepsilon > 0$ small enough so that, if N denotes a tubular neighborhood of K of radius ε, then $S^3 - N$ is a deformation retract of $S^3 - K$. It is clear that this will be the case provided ε

is sufficiently small; the precise meaning of the phrase "sufficiently small" depends on the integers m and n. Then, let U and V be the $\frac{1}{2}\varepsilon$ neighborhoods of A and B, respectively. It is clear that U and V are each homeomorphic to the product of an open disc with a circle, and A and B are deformation retracts of U and V. Also, $U \cap V$ is a "thickened" torus, i.e., homeomorphic to the product of $A \cap B$ and the open interval $(-\frac{1}{2}\varepsilon, \frac{1}{2}\varepsilon)$. We can now use the fact that

$$S^3 - N = (U - N) \cup (V - N)$$

and apply the Seifert-Van Kampen theorem to arrive at a presentation of $\pi(S^3 - N) \approx \pi(S^3 - K)$.

First, $U - N$ and $V - N$ both have the homotopy type of a circle; in fact, the center lines of A and B are deformation retracts of these two spaces. Therefore, their fundamental groups are infinite cyclic.

Secondly, the spaces $(U - N) \cap (V - N) = (U \cap V) - N$ and $(A - K) \cap (B - K) = (A \cap B) - K$ both have the same homotopy type. In fact, the set $(A - N) \cap (B - N) = (A \cap B) - N$ is a deformation retract of each of these spaces. We can readily see that $(A \cap B) - K$ is a subset of the torus $A \cap B$ homeomorphic to the product of a circle with an open interval. It is a strip wound spirally around the torus, like a bandage. Its fundamental group is infinite cyclic.

Finally, we must determine the homomorphisms

$$\varphi_1 : \pi(U \cap V - N) \to \pi(U - N),$$

$$\varphi_2 : \pi(U \cap V - N) \to \pi(V - N).$$

Here we leave the details to the reader. The result is that one of these homomorphisms is of degree m, and the other is of degree n. (We say a homomorphism of one infinite cyclic group into another is of degree m if the image of a generator of the first group is the mth power of a generator of the second group.) If we combine this result with Exercise 4.1(c) we obtain the following result:

Proposition 6.2 *The group G of a torus knot of type (m, n) has a presentation consisting of two generators, $\{\alpha, \beta\}$, and one relation, $\alpha^m\beta^n$.*

There remains the task of proving that these groups are nonisomorphic for different values of the pair (m, n). This we now do by a method due to O. Schreier. Consider the element $\alpha^m = \beta^{-n}$ in this group. This element commutes with α and β, and hence with every element; thus it belongs to the center. Let N denote the subgroup generated by this element; it is obviously a normal subgroup. Consider the quotient group

G/N. Let α' and β' denote the coset of α and β in G/N. Obviously, G/I is generated by the elements α' and β', and it has the following presenta tion:

Generators: α', β'. Relations: α'^m, β'^n.

From this presentation, it follows that G/N is the free product of a cycl group of order m (generated by α') and a cyclic group of order n (ger erated by β'). The proof, which is not difficult, is left to the reader. W now apply Exercise III.4.1 to conclude that the center of G/N is {1 Because the image of the center of G is contained in the center of G/N it follows that N is the entire center of G. Thus, the quotient of G by i center is the free product of two cyclic groups (of order m and n). W can now apply the result of Exercise III.4.6 to conclude that the intege m, n are completely determined (up to their order) by G. Thus, we hav proved the following:

Proposition 6.3 *If torus knots of types (m, n) and (m', n') are equi alent, then $m = m'$ and $n = n'$, or else $m = n'$ and $n = m'$. No torus kn is equivalent to an unknotted circle (assuming $m, n > 1$).*

Thus, by means of torus knots we have constructed an infinite fami of nonequivalent knots.

Of course, most knots are not torus knots. The foregoing paragrapl should only be considered a brief introduction to the subject of kn theory. The reader who wishes to learn more about this subject ca consult the books of Crowell and Fox [3] or Neuwirth [5].

NOTES

Apparently a theorem along the lines of Theorem 2.1 was first proved by I Seifert in 1931 in a paper entitled "Konstruktion dreidimensionaler geschlossen Räume" (*Ber. Sächs. Akad. Wiss.*, *83*, 1931, pp. 26–66). A little later a simil theorem was discovered and proved independently by E. R. Van Kampen ("(the connection between the fundamental groups of some related spaces," *Am J. Math.*, *55*, 1933, pp. 261–267). In spite of this, it is usually referred to "Van Kampen's theorem" in American books and papers. Of course, the formul tion of the theorem as the solution of a universal mapping problem came lat Our exposition is based on a paper by R. H. Crowell [2], which was apparent inspired by lectures of R. H. Fox at Princeton; see their joint textbook [3].

The reader who is familiar with the theory of simplicial complexes ca easily derive Seifert's version of the Seifert-Van Kampen theorem (as stated Section 52 of Seifert and Threlfall [6]) from Theorem 2.1 and Lemma 3.2 of tl chapter. To do this, one makes use of the properties of a *regular neighborho* of a subcomplex of a simplicial complex as outlined in Chapter II, Section 9 of the text by S. Eilenberg and N. Steenrod (*Foundations of Algebraic Topology*. Princeton, N.J.: Princeton University Press, 1951).

Free products with amalgamated subgroups

Let $\{W\} \cup \{V_i : i \in I\}$ be a covering of X by arcwise-connected open sets such that $V_i \cap V_j = W$ if $i \neq j$ and $x_0 \in W$ (see Exercise 3.1). Assume that, for each index i, the homomorphism $\pi(W, x_0) \to \pi(V_i, x_0)$ is a *monomorphism*. Then, the fundamental group $\pi(X, x_0)$, as specified by Theorem 2.2, has a structure that has been well studied by group theorists; it is called a "free product with amalgamated subgroup." It is a quotient group of the free product of the groups $\pi(V_i)$ obtained by "amalgamating" or identifying the various subgroups which correspond to $\pi(W, x_0)$ under the given monomorphisms. Every element of such a free product with amalgamated subgroups has a unique expression as a "word in canonical form." Although such groups have played a role in group theory, so far they have only been of minor importance in topology. For further information on this subject, see the textbooks on group theory listed in the bibliography of Chapter III.

The Poincaré conjecture

It follows from the computations made in this chapter that any simply-connected, compact surface is homeomorphic to the 2-sphere S^2. H. Poincaré conjectured in the early 1900's that an analogous statement is true for 3-manifolds, namely, that a compact, simply-connected 3-manifold is homeomorphic to the 3-sphere S^3. In spite of the expenditure of much effort by many outstanding mathematicians over the years since Poincaré, it is still unknown whether or not this famous conjecture is true. It is easy to give examples of compact, simply-connected 4-manifolds which are not homeomorphic to S^4 (e.g., $S^2 \times S^2$). However, for all integers $n > 3$ there is an analog of the Poincaré conjecture, namely, that a compact n-manifold that has the homotopy type of an n-sphere is homeomorphic to S^n. This generalized Poincaré conjecture was proved for $n > 4$ by S. Smale in 1960 (see *Ann. Math.*, *74*, 1961, pp. 391–406). The case where $n = 4$ is also still open.

Until the classical Poincaré conjecture (the case where $n = 3$) is settled, we cannot hope to have a classification theorem for compact 3-manifolds.

Homotopy type vs. topological type for compact manifolds

From the computations of the fundamental groups of compact surfaces in this chapter, the following fact emerges: If two compact surfaces are not homeomorphic, then they do not have the same homotopy type. The analogous statement for compact 3-manifolds is known to be false; there are fairly simple examples of compact 3-dimensional manifolds which are of the same homotopy type, but not homeomorphic (the so-called "lens spaces"). The proof of this fact is the culmination of the work of mathematicians in several countries over a period of years. The details are rather elaborate.

Apparently no such examples are known for the case of higher dimensional manifolds.

Fundamental group of a noncompact surface

The fundamental group of any noncompact surface (with a countable basis) is a free group on a countable or finite set of generators. Any simply-connected, noncompact surface is homeomorphic to the plane $\mathbf{R}^2$. For a proof of these facts, see Ahlfors and Sario [1], Chapter I, or the exercises of Section VI.5.

Sketch of the proof that any finitely presented group can be the fundamental group of a compact 4-manifold

First, note that the fundamental group of $S^1 \times S^3$ is infinite cyclic. Hence, b forming the connected sum of n copies of $S^1 \times S^3$, we obtain an orientable compact 4-manifold whose fundamental group is a free group on n generator (see Exercise 3.7).

Next, suppose that M is a compact, orientable 4-manifold and C is a smooth simple closed curve in M; it may be shown that any sufficiently small, close tubular neighborhood N of C is homeomorphic to $S^1 \times E^3$ (this assertion woul not be true if M were nonorientable). Also, the boundary of N is homeomorph to $S^1 \times S^2$. Now $S^1 \times S^2$ is also the boundary of $E^2 \times S^2$, a 4-dimensiona manifold with boundary. Let M' denote the complement of the interior of N Form a quotient space of $M' \cup (E^2 \times S^2)$ by identifying corresponding point of the boundary of N and the boundary of $E^2 \times S^2$; denote the quotient space b M_1. Then, M_1 is readily seen to be a compact, orientable 4-manifold also; th process of obtaining M_1 from M is often called "surgery."

What is the fundamental group of M_1? We can answer this question by apply ing the Seifert-Van Kampen theorem twice. First, $M = M' \cup N$ and $M' \cap$ is homeomorphic to $S^1 \times S^2$. It is readily seen that the homomorphism $\pi(M'$ $N) \to \pi(N)$ (induced by the inclusion) is an isomorphism; therefore by Exercis 4.1(a) the homomorphism $\pi(M') \to \pi(M)$ is also an isomorphism. Next, $M_1 =$ $M' \cup (E^2 \times S^2)$ and $M' \cap (E^2 \times S^2) = M' \cap N$. Because $E^2 \times S^2$ is simpl connected, Theorem 4.1 is applicable, and we can conclude that $\pi(M') \to \pi(M$ is an epimorphism, and the kernel is the smallest normal subgroup containing th image of $\pi(M' \cap N) \to \pi(M')$; but it is readily seen that the images of $\pi(M' \cap N$ $\to \pi(M')$ and $\pi(C) \to \pi(M)$ are equivalent. (NOTE: Actually, each time we appl the Seifert-Van Kampen theorem, it is necessary to invoke Lemma 3.2, becaus M' and N are not open subsets of M.)

We can summarize the conclusion just obtained as follows: $\pi(M_1)$ is naturall isomorphic to the quotient of $\pi(M)$ by the smallest normal subgroup containin the image of $\pi(C) \to \pi(M)$. In other words, we have "killed off" the element of $\pi(M)$ represented by the closed path C. If the group $\pi(M)$ is presented b means of generators and relations, then $\pi(M_1)$ has a presentation consisting of th same set of generators, and having one additional relation, namely, α.

It is not difficult to show that any element $\alpha \in \pi(M)$ can be represented by smooth closed path C without any self-intersections, as required in the precedin argument. In fact, this is true for any orientable n-manifold M provided $n \geqq 3$. In a manifold of dimension $\geqq 3$ there is enough "room" to get rid of the self-intersections in any closed path by means of arbitrarily small deformations.

Now let G be a group which has a presentation consisting of n generators $x_1, \ldots, x_n$ and k relations, $r_1, r_2, \ldots, r_k$. Let M be the connected sum of n copies of $S^1 \times S^3$; then $\pi(M)$ is a free group on n generators, which we may denote by $x_1, \ldots, x_n$. We now perform surgery k times on M, killing off in succession the elements $r_1, \ldots, r_k$. The result will be a compact, orientable 4-manifold M_k such that $\pi(M_k) \approx G$, as required.[1]

This construction was utilized by A. A. Markov in his proof that there cannot exist any algorithm for deciding whether or not two given compact, orientable, triangulable 4-manifolds are homeomorphic. Markov's proof depends on the fact that there exists no general algorithm for deciding whether or not two given group presentations represent isomorphic groups (see *Proceedings of International Congress of Mathematicians*, 1958, pp. 300–306; also, a paper by W. Boone, W. Haken, and V. Poenaru in *Fundamenta Mathematica*).

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[1] This result is due to Seifert and Threlfall [6], p. 180.

CHAPTER FIVE

Covering Spaces

1 Introduction

Let X be a topological space; *a covering space of* X consists of a space . and a continuous map p of $\tilde{X}$ onto X which satisfies a certain very stro smoothness requirement. The precise definition is given below. Tl theory of covering spaces is important not only in topology, but also : related disciplines such as differential geometry, the theory of Lie Group and the theory of Riemann surfaces.

The theory of covering spaces is closely connected with the study the fundamental group. Many basic topological questions about coveri spaces can be reduced to purely algebraic questions about the fund mental groups of the various spaces involved. It would be practical impossible to give a complete exposition of either one of these two topi without also taking up the other.

2 Definition and some examples of covering spaces

In this chapter, we shall assume that *all spaces are arcwise connected a locally arcwise connected* (see Section II.2 for the definition) unless oth wise stated. To save words, we shall not keep repeating this assumptio On the other hand, it is not necessary to assume that the spaces we dealing with satisfy any separation axioms.

Definition Let X be a topological space. A *covering space* of X a pair consisting of a space $\tilde{X}$ and a continuous map $p : \tilde{X} \to X$ such tl the following condition holds: Each point $x \in X$ has an arcwise-co nected open neighborhood U such that each arc component of $p^{-1}($ is mapped topologically onto U by p [in particular, it is assumed tl $p^{-1}(U)$ is nonempty]. Any open neighborhood U that satisfies the co