On irreducible 3-manifolds which are sufficiently large*

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We are mainly concerned with the questions whether any homotopy equivalence between compact orientable PL 3-manifolds can be induced by a homeomorphism, and whether homotopic homeomorphisms are also isotopic. Fairly complete answers are obtained for the class of manifolds which is indicated in the title.

The restriction to irreducible manifolds has its main reason in the unproved Poincaré conjecture. It has the side effect that our manifolds neither have handles nor are connected sums; which is very convenient, but only partially necessary.

The second restriction is characteristic for the technique employed, which may be called an induction on dimension. Systematically, use is made of codimension 1 submanifolds which are "characteristic for the topology" *of the* "manifold." Following Haken, we adopt the name *incompressible surface* for such a submanifold (by this name, we will always refer to an orientable surface). We call a manifold *sufficienty large*, if it contains an incompressible surface. The important fact is that a sufficiently large irreducible manifold can be reduced to a ball, with the use of incompressible surfaces only, in the same way that a compact orientable 2-manifold, different from the 2-sphere, can be reduced to a disc by first splitting it at a non-contractible curve, if it is closed, and then splitting the resulting bounded 2-manifold at arcs (Haken). (Naturally, for 3-manifolds it takes a bit longer to get down.)

Another (and rather immediate) consequence of this fact is that the universal cover of such a manifold is a familiar space.

The result concerning the existence of homeomorphisms has been known for surface bundles over the circle (Stallings [16], Neuwirth [9]). Partial results on the existence of isotopies have been announced by Giffen [3].

I wish to thank Professor Schubert for discussions.

0. Notation

Up to § 7, we work in the piecewise linear category.

* This work was partially supported by a grant from the National Science Foundation., GP5804.

By a *manifold* we mean an orientable 3-manifold. It is compact, unless the contrary is admitted explicitly, or might emerge from a construction (e.g., a covering); the same applies to other spaces or subspaces. It will sometimes be convenient to abbreviate *system of submanifolds* by *manifold*; but, in general, a manifold is connected.

A *surface* is a connected 2-manifold. It is compact and/or orientable, unless the contrary is admitted explicitly.

A *surface F in the manifold M* is properly embedded, i.e., $F \cap \partial M = \partial F$, (where ∂ denotes *boundary*). A *surface in ∂M* is a submanifold of ∂M.

A *system of surfaces in M or ∂M* consists of finitely many, mutually disjoint components of the above two types.

For F, a system of surfaces in M or ∂M, the symbol $[\partial F]$ will denote the image under the boundary homomorphism of the 2-cycle represented by F and an orientation of F.

I, D, E, denote line, disc, and ball, respectively; I is occasionally identified with the unit interval $[0, 1]$.

Closure and interior over more than one symbol are denoted by $\overline{(\cdots)}$ and $\mathring{(\cdots)}$.

$U(\cdots)$ denotes a regular neighborhood. General practice: Choose a triangulation in which all subspaces, previously mentioned in the argument, are subcomplexes; construct its second derived, and take the closed star of the object in question. Special practice: In the presence of a product structure, we will sometimes require that $U(\cdots)$ is in some sense compatible with the product structure. This will be indicated in the text.

Let F be a surface in the manifold M. Then *the manifold M', obtained by splitting M at F*, has by definition the properties: $\partial M'$ contains surfaces F_1 and F_2 which are copies of F, and identification of F_1 and F_2 gives a natural projection $(M', F_1 \cup F_2) \to (M, F)$. M' is homeomorphic to $\overline{(M - U(F))}$, but we have to use both constructions.

An isotopy

(a) of a homeomorphism $h: X \to Y$ is a map $H: X \times I \to Y$ such that, for $h_\tau = H \mid X \times \tau$, we have $h_0 = h$, and $h_\tau: X \to Y$ is a homeomorphism onto Y,

(b) of subspaces Z_1, Z_2 in X is an isotopy of the identity map on X, such that $h_1(Z_1) = Z_2$,

(c) of embeddings is defined *via* (b).

Let F and G be surfaces in M or ∂M. F is *parallel* to G, if and only if there exists an embedding of $F \times I$ in M, such that $F = F \times 0$, and $G = \overline{(\partial(F \times I) - F \times 0)}$.

Note. If F is parallel to G, then G is parallel to F. Note also that, if the surfaces F and G in M are parallel, then they are isotopic by an isotopy which is constant on ∂M. (Here the phrase *surfaces in* M, rather than *surfaces in* M or ∂M is essential).

Frequently, a proof involves a sequence of constructions, each of which in turn involves alterations of some things. To avoid an orgy of notation in such cases, we often denote the altered things by the old symbols. The reader might adopt the point of view that such a proof proceeds by "induction on niceness." After having convinced ourselves that there is no obstruction to achieve some more niceness, we take up the same problem again, but with an improved induction hypothesis.

1. Definitions and preliminaries

(1.1) *Incompressible surfaces.*

Definition. Let M be a manifold. Let F be a system of surfaces in M or ∂M. F is *compressible in* M in either of the following two cases.

(a) There is a non-contractible simple closed curve k in $\mathring{F}$, and a disc D in M, $\mathring{D} \subset \mathring{M}$, such that $D \cap F = \partial D = k$.

(b) There is a ball E in M, such that $E \cap F = \partial E$.

F is *incompressible in* M, if and only if it is not compressible in M. Whenever there is no doubt about the manifold, we will abbreviate *incompressible in* M by *incompressible*.

Definition. The manifold M is *irreducible*, if and only if every 2-sphere in M is compressible. (Remember. If M is irreducible, and $\partial M \neq \emptyset$, then either M is a ball, or else genus $(\partial M) > 0$, and hence $H_1(M)$ is infinite.)

Definition. The manifold M is *boundary-irreducible*, if and only if ∂M is incompressible.

LEMMA 1.1.2. *Let M be a manifold. Let F be a system of surfaces in M or ∂M. F is incompressible in M, if and only if every component is.*

LEMMA 1.1.3. *Let M be a manifold. Let F be a surface in M or ∂M, F being not a 2-sphere. F is incompressible, if and only if*

$$\ker\big(\pi_1(F) \longrightarrow \pi_1(M)\big) = 0 .$$

Since F is either a submanifold of ∂M, or a two-sided proper surface in M, this follows from the loop theorem.

LEMMA 1.1.4. *Let M be a manifold. Let F be a system of incompressible surfaces in M or ∂M. Let $U(F)$ be a regular neighborhood of F, and*

$\tilde{M} = \overline{(M - U(F))}$. *Then*

(a) $\tilde{M}$ *is irreducible, if and only if* M *is.*

(b) $\ker(\pi_1(M') \to \pi_1(M)) = 0$, *where* M' *is a component of* $\tilde{M}$.

This is not difficult. A proof for (a) may be found in [17], and for (b) in [16]. The next one is a well known corollary of the sphere theorem.

LEMMA 1.1.5. *Suppose* M *is irreducible, and* $\pi_1(M)$ *is not finite. Then* M *is aspherical*, i.e., $\pi_j(M) = 0$, *for* $j \geqq 2$.

LEMMA 1.1.6. *Let* M *be an irreducible manifold.*

(a) *If* $\partial M \neq \emptyset$, *and* M *is not a ball, then there exists in* M *an incompressible surface* F *such that* $0 \neq [\partial F] \in H_1(\partial M)$.

(b) *If* $\partial M = \emptyset$, *then there exists in* M *an incompressible surface, if and only if either* $H_1(M)$ *is not finite or* $\pi_1(M)$ *is a non-trivial free product with amalgamation (or both).*

If F *is a separating incompressible surface in* M, $\partial M = \emptyset$, *then* $\pi_1(M)$ *is a non-trivial free product with amalgamation*, $\pi_1(M) \approx A *_C B$, *where* $C \approx \pi_1(F)$, *in a natural way.*

This seems to be widely known. A proof is given in [19].

Definition 1.1.7. Let M be an irreducible manifold, which is not a ball. M is *sufficiently large* if and only if there exists an incompressible surface in M.

Remark. There exist irreducible manifolds with infinite fundamental group, which are not sufficiently large [19].

(1.2) *Hierarchies.* Let M_1 be an irreducible manifold. A *hierarchy for* M_1 (*of length* n) is by definition a sequence of triples

$$M_j,\ \ F_j \subset M_j,\ \ U(F_j) \subset M_j\ ; \qquad M_{j+1} = \overline{(M_j - U(F_j))}\ ,$$

where j ranges from 1 to n ($\geqq 0$), such that

(a) F_j is an incompressible surface in M_j, $U(F_j)$ is a regular neighborhood of F_j in M_j.

(b) each component of M_{n+1} is a ball.

This concept (with technical differences, and with an additional condition on the surfaces which is inessential for our applications) has been introduced by Haken [7].

In our applications, it will be convenient (yet not essential) that as many as possible of the F_j have non-empty boundary and do not separate the respective M_j. The following existence theorem gives us a hierarchy which automatically has these properties.

THEOREM. *Let M_1 be an irreducible manifold with non-empty boundary. Then there exists a hierarchy for M_1,*

$$M_j,\ F_j \subset M_j,\ U(F_j) \subset M_j\ ;\quad M_{j+1} = \big(M_j - U(F_j)\big)\ ,$$

$j = 1, \cdots, n$, such that $0 \neq [\partial F_j] \in H_1(\partial M_j)$, $j = 1, \cdots, n$.

This is essentially a part of a result of Haken [7, 1f, p. 101]. Since details have not yet appeared, and since we rely heavily on this theorem, a proof will be given in the next section. Our method of proof is slightly simpler than the original one; but it cannot give Haken's result. In particular, it cannot give an upper bound for the length of the hierarchy.

Note. In the hierarchy, given by the theorem, every M_j is connected, and (by induction on (1.1.4)), every M_j is irreducible, and the inclusion homomorphisms $\pi_1(M_i) \to \pi_1(M_j)$, $i \geqq j$, are injections.

(1.3) *Maps.* Let F be a system of surfaces in M, and $U(F)$ a regular neighborhood of F. Then $U(F)$ may be given the structure of a line bundle $F \times I$, with $F = F \times 1/2$, and $F \times I \cap \partial M = \partial F \times I$.

A map $f: X \to M$ will be called *transverse with respect to F*, if there exists $U(F) = F \times I$ as above, such that f induces in $f^{-1}\big(U(F)\big)$ the structure of a line bundle, and f maps each fibre homeomorphically onto a fibre.

PROPOSITION. *Let M be an irreducible manifold, and F a system of incompressible surfaces in M. Let N be a manifold, and $f: N \to M$ a map. Then there exists a map g, homotopic to f, which is transverse with respect to F, and such that the system of surfaces in N, $G = g^{-1}(F)$, is incompressible in N. If $f \mid \partial N$ were transverse with respect to F, then the homotopy from f to g may be chosen constant on ∂N.*

This principle has been applied to 3-manifolds by Stallings [16]. The proof proceeds, roughly, by sliding f along the fibres to make it transverse, and then, if $f^{-1}(F)$ is not incompressible, to simplify $f^{-1}(F)$ by surgery. Details are provided by Lemmas 1.1; 2, 3, 4a, 5 above, and by [19, Lem. 1.1].

The same principle, applied in lower dimension, gives

LEMMA. *Let F be a surface, and k a system of simple arcs and non-contractible simple closed curves in F, $k \cap \partial F = \partial k$. Let G be a surface, and $f: G \to F$ a map. Then there is a map g, homotopic to f, which is transverse with respect to k, and such that $g^{-1}(k)$ does not contain a contractible closed curve. If $f \mid \partial G$ is transverse with respect to k, then the homotopy from f to g may be chosen constant on ∂G.*

(1.4) We state theorems of Baer and Nielsen (restricted to compact orientable 2-manifolds) in the form in which we use them. Proofs of (1.4.1) and (1.4.2) may be found in [2]. The simplest proof of (1.4.3) is analogous to our proof of (6.1), it uses Lemma (1.3). The closest reference to this type of proof seems to be [13].

LEMMA 1.4.1. *In the surface F, let k and l be either simple arcs or simple closed curves, such that $k \cup \partial F = \partial k = \partial l = l \cap \partial F$. Suppose k is homotopic to l by a homotopy which is constant on ∂k. Then there is an isotopic deformation of F, constant on ∂F, which carries k to l.*

LEMMA 1.4.2. *Let $h: F \to F$ be a homeomorphism, and $H: F \times I \to F$ a homotopy such that $H \mid F \times 0 = \mathrm{id}$, $H \mid F \times 1 = h$, $H(\partial F \times I) \subset \partial F$. Then h is isotopic to the identity. If the homotopy is constant on ∂F, then the isotopy may be chosen constant on ∂F.*

LEMMA 1.4.3. *Let $f: (G, \partial G) \to (F, \partial F)$ be a map such that*

$$\ker\left(f_*: \pi_1(G) \longrightarrow \pi_1(F)\right) = 0 .$$

Suppose $\pi_1(G) \neq 0$. Then there is a homotopy $f_\tau: (G, \partial G) \to (F, \partial F)$, $\tau \in I$, $f_0 = f$, such that either (a) *or* (b) *holds.*

(a) *G is an annulus, and $f_1(G) \subset \partial F$,*

(b) *$f_1: G \to F$ is a covering map.*

If $f \mid \partial G$ is locally homeomorphic, then the homotopy may be chosen so that $f_\tau \mid \partial G = f_0 \mid \partial G$, for all τ.

2. Existence of hierarchies
(Proof of Theorem 1.2)

We need the very simplest facts of Haken's theory [6]. We refer to Schubert's exposition [12]. Since our definitions slightly differ, we give a shorthand description of the concepts which we use. Instead of *normal decomposition*, we use the term *handle decomposition*.

(2.1) Let M be a manifold. A *handle decomposition* consists of collections of balls $N^0, N^{\mathrm{I}}, N^{\mathrm{II}}, N^{\mathrm{III}}$, with union M, such that the members of each family are mutually disjoint, and with the additional properties below. The members of $N^0, N^{\mathrm{I}}, N^{\mathrm{II}}$ will be called *Balls*, *Beams*, and *Plates*, respectively.

(1) $N^{\mathrm{III}} \cap (N^0 \cup N^{\mathrm{I}} \cup N^{\mathrm{II}}) = \partial N^{\mathrm{III}}$, *whence* $\partial M \subset N^0 \cup N^{\mathrm{I}} \cup N^{\mathrm{II}}$.

(2) *For each Beam, a fixed presentation as $I \times D$ is specified; for each Plate, a fixed presentation as $D \times I$ is specified.*

(3) *For any Beam, we have*

(a) $I \times D \cap N^0 = \partial I \times D$

(b) *$I \times D \cap N^{\mathrm{II}} = I \times d$, where d is a collection of arcs in ∂D; (d may be empty).*

(4) *For any Plate, we have*

(a) $D \times I \cap N^0 = e_1 \times I$

(b) *$D \times I \cap N^{\mathrm{I}} = e_2 \times I$, where e_1 and e_2 are collections of arcs in ∂D (neither empty), such that $e_1 \cup e_2 = \partial D$.*

(5) *For each component of $N^{\mathrm{I}} \cap N^{\mathrm{II}}$, the induced product structures agree.*

(2.2) To a handle decomposition we associate a triple (χ, η, ζ) of non-negative integers, which, in lexicographical ordering, will measure the "complexity" of the decomposition.

(δ) For a Beam, let δ denote the number of components of d in (2.1.3); define $\delta'' = \max(\delta - 2, 0)$, $\delta' = \max(\delta - 1, 0)$.

Then $\chi = \sum \delta''$; $\eta = \sum \delta'$; where the sums are over all Beams.

(ε) For a Ball, E define ε as the number of components of $E \cap (N^{\mathrm{I}} \cup N^{\mathrm{II}})$; (we have $\varepsilon > 0$, unless $N^{\mathrm{I}} = \varnothing = N^{\mathrm{II}}$).

Define $\zeta = \sum(\varepsilon - 1)$; where the sum is over all Balls.

(2.3) *Normal surfaces.* Let F be an incompressible surface in M, such that $0 \neq [\partial F] \in H_1(\partial M)$. Consider the following operations

(ρ) Let D be a disc in $\mathring{M}$, such that $D \cap F = \partial D$. Replace a neighbhorhood of $D \cap F$ in F by two copies of D. Since F is incompressible, the result will consist of a 2-sphere and a surface F'', which again is incompressible, and has the same boundary as F. We regard F'' as the result of the operation.

(σ) Let D be a disc in M, such that $D \cap (F \cup \partial M) = \partial D$, and each of $D \cap F$ and $D \cap \partial M$ is one arc. Replace a neighborhood of $D \cap F$ in F by two copies of D; call the result F'. Clearly, F' is incompressible. Also, if we give F and F' compatible orientations, then $[\partial F'] = [\partial F]$. As the result of the operation, we will consider F'', which is F', if F' is connected, and otherwise is a component of F', such that $[\partial F''] \neq 0$.

PROPOSITION. *Given a handle decomposition of M, and an incompressible surface F in M, such that $0 \neq [\partial F] \in H_1(\partial M)$, there exists an incompressible surface G, $[\partial G] \neq 0$, which may be obtained from F by operations (ρ) and (σ) and by isotopic deformations, and which is a normal surface in the following sense.*

(1) *$G \cap N^{\mathrm{III}} = \varnothing$.*

(2) *If $D \times I$ is any Plate, then $D \times I \cap G = D \times r$, where r is a collection of points in $\mathring{I}$; in particular, $\partial G \cap N^{\mathrm{II}} = \varnothing$.*

(3) *If $I \times D$ is any Beam, then $I \times D \cap G = I \times k$, where k is a*

system of arcs in D, $k \cap \partial D = \partial k$.

(4) *If* k_1 *is any component of* k *in* (3), *then the end points of* k_1 *are not contained in the same component of* d (cf. (2.1.3)), *or in the same component of* $\partial D - d$.

(5) *If* k_1 *is any component of* k *in* (3), *then the end points of* k_1 *are not contained in adjacent components of* d (cf. (2.1.3)), *and* $\partial D - d$.

(6) *The intersection of* G *and any Ball consists of discs.*

For the proof, we refer to [12], (2.2). The main difference between *normal decomposition*, as described there, and *handle decomposition*, as described here, is that we do not require that every member of N^0, N^{I}, N^{II} have connected intersection with ∂M. But this does not affect the normalization. After this normalization has been carried out, we are left with a surface G, which has the above properties, except possibly for (5), which has no analogue in [12].

In the situation which is forbidden by (5), there exist a Beam B, a Plate P, and a disc D in B, such that $D \cap P$ and $D \cap \partial M$ are one arc each, and $l = \overline{(\partial D - ((D \cap P) \cup (D \cap \partial M)))}$ is an arc which lies in G in such a way that $G \cap U(l) = l$, where $U(l)$ is a neighborhood of l in D. Then G is further simplified in a similar way as in step 9 in [12], (2.2).

Note that, in another respect (concering $G \cap N^0$), our definition of normal surface is much weaker than that in [12], namely, step 9 need be carried out only to the extent that our condition (4) is satisfied.

PROPOSITION 2.4. *Let* M *be a manifold with a handle decomposition of complexity* (χ, η, ζ). *Let* G *be a surface in* M, *such that* $0 \neq [\partial G] \in H_1(\partial M)$, *and such that* G *is a normal surface in the sense of* (2.3). *Let* $U(G)$ *be a regular neighborhood of* G. *Then* $M' = \overline{(M - U(G))}$ *has a handle decomposition of complexity* $(\chi', \eta', \zeta') < (\chi, \eta, \zeta)$.

PROOF. (A) We choose $U(G)$ small with respect to the handle decomposition. If now N is any Ball, Beam, or Plate of M, then we define the components of $\overline{(N - U(G))}$ to be *Balls*, *Beams*, or *Plates* of M' respectively. To define the product structures, we construct the same decomposition in a slightly different way. Construct the manifold M'' by splitting M at G, i.e., $\partial M''$ contains two copies of G, and identifying these, we obtain M from M''. Similarly as for M', we define a decomposition for M''. Here we have natural product structures in the Beams and Plates, and it is easily checked that the axioms (2.1) hold. Finally, M'' is homeomorphic to M' by a homeomorphism which respects the decompositions.

(B) Let D be a disc, d a collection of arcs in ∂D, and k a system of arcs in D, such that $k \cap \partial D = \partial k$, and $k \cap \partial d = \emptyset$. Let $U(k)$ be a regular neighborhood of k in D, which is small with respect to d. Then $\overline{(D - U(k))}$ consists of discs $D_1, D_2, \cdots$. Define systems of arcs $d_j = d \cap D_j$.

Let δ (resp. δ_j) be the number of components of d (resp. d_j), and define $\delta'' = \max(\delta - 2, 0)$, $\delta' = \max(\delta - 1, 0)$, and similarly δ''_j and δ'_j. We wish to compare δ'' and δ' with $\sum\delta''_j$ and $\sum\delta'_j$. We do this first in the case where k consists of the single component k_1, by distinguishing cases.

The end points of k_1 are contained in

(1) different components of $\partial D - d$, then $\delta = \delta_1 + \delta_2$, and $\delta_1, \delta_2 \geqq 1$, thus $\delta' > \sum\delta'_j$; $\delta'' \geqq \sum\delta''_j$

(2) the same component of $\partial D - d$, then, say, $\delta_1 = \delta$, $\delta_2 = 0$, thus $\delta' = \sum\delta'_j$, $\delta'' = \sum\delta''_j$

(3) non-adjacent components of d and $\partial D - d$, then $\delta + 1 = \delta_1 + \delta_2$, and $\delta_1, \delta_2 \geqq 2$, thus $\delta'' > \sum\delta''_j$

(4) adjacent components of d and $\partial D - d$, then, say, $\delta_1 = \delta$, $\delta_2 = 1$, thus $\delta'' = \sum\delta''_j$

(5) different components of d, then $\delta + 2 = \delta_1 + \delta_2$, and $\delta_1, \delta_2 \geqq 2$, thus $\delta'' = \sum\delta''_j$

(6) the same component of d.

Next, we take a general system of arcs, but subject to the conditions (2.3.4) and (2.3.5); i.e., no component of k has both its end points in the same component of d or $\partial D - d$, respectively, or has its end points in adjacent components of d and $\partial D - d$. Instead of removing $U(k)$ all at once, we remove one component after the other. We have three cases.

(α) The only general thing which we can claim is that, because of our conditions on k, we never come across an arc of type (6) above. Thus all steps are of types (1)–(5), whence $\delta'' \geqq \sum\delta''_j$.

(β) At least one component of k meets both d and $\partial D - d$. Then at least one step is of type (3) above, whence $\delta'' > \sum\delta''_j$.

(γ) $\partial k \subset \partial D - d$. Then all steps are of type (1) or (2) above, and at least one step is of type (1). Thus $\delta'' \geqq \sum\delta''_j$, and $\delta' > \sum\delta'_j$.

(C) Proof that $(\chi', \eta', \zeta') < (\chi, \eta, \zeta)$. *Case* 1. $G \cap N^{\mathrm{II}} \neq \emptyset$.

In (B) we checked the amount of complexity, which is contributed to (χ', η', ζ') by those Beams of M' which come from a single Beam of M. We found that $\delta'' \geqq \sum\delta''_j$ without exception. Since $G \cap N^{\mathrm{II}} \neq \emptyset$ and $G \cap \partial M \neq \emptyset$, one Beam at least gives rise to the situation (β). Thus, in fact, $\chi' < \chi$.

Case 2. $G \cap N^{\mathrm{II}} = \emptyset$, but, $G \cap N^{\mathrm{I}} \neq \emptyset$.

Then $\chi' \leqq \chi$. Since only the situation (γ) can occur, and at least one Beam is involved, we have $\eta' < \eta$.

Case 3. $G \cap N^1 = \emptyset$.

Then $\chi' = \chi$ and $\eta' = \eta$. Assume $\zeta' = \zeta$. It follows that the disc G is parallel to a disc in ∂M, which contradicts $[\partial G] \neq 0$.

PROOF OF THEOREM 1.2. Let M_1 be an irreducible manifold with non-empty boundary. Construct a handle decomposition, (2.1), for M_1, e.g., from a triangulation of M_1. Let $(\chi_1, \eta_1, \zeta_1)$ be its complexity, (2.2). If ∂M_1 consists of 2-spheres only, then M_1 is a ball, since it is irreducible; so there is nothing to prove. Otherwise, there exists an incompressible surface F_1 in M_1 such that $0 \neq [\partial F_1] \in H_1(\partial M_1)$, (1.1.6). By (2.3) we may assume, F_1 is normal with respect to the given handle decomposition. Then, by (2.4), $M_2 = \overline{(M_1 - U(F_1))}$ has a handle decomposition of complexity $(\chi_2, \eta_2, \zeta_2) < (\chi_1, \eta_1, \zeta_1)$. Use this handle decomposition to continue with the construction, and proceed inductively.

Assume the induction step can be carried out arbitrarily often. It follows that we can construct an infinite sequence of triples, $(\chi_j, \eta_j, \zeta_j)$, $1 \leqq j < \infty$, such that

$$(\chi_j, \eta_j, \zeta_j) > (\chi_{j+1}, \eta_{j+1}, \zeta_{j+1}) \geqq (0, 0, 0) .$$

But such a sequence does not exist.

3. Product line bundles

In this section, $M = F \times I$ is the product of the orientable surface F which is not the 2-sphere, and the interval. $p: M \to F$ denotes the projection onto the factor F. A subspace X of M is called *vertical*, if $X = p^{-1}(p(X))$.

PROPOSITION 3.1. *Let G be a system of incompressible surfaces in M. Suppose ∂G is contained in $F \times 1$. Then G is isotopic, by a deformation which is constant on ∂M, to a system G' such that $p \mid G'$ is homeomorphic on each component of G'.*

COROLLARY 3.2. *Each component of G is parallel to a surface in $F \times 1$.*

Proof of (3.1) *in the case where F is a disc, annulus, or 2-sphere with 3 holes.* Let H be a system of vertical discs in M, such that splitting at the arcs $H \cap F \times 1$ would reduce $F \times 1$ to a disc, and such that the intersection $(H \cap F \times 1) \cap G$ consists of the smallest possible number of points. In particular, we have general position at that intersection.

Deform G by an isotopy which leaves ∂G fixed, so that H and G intersect in general position, and that, in addition, $H \cap G$ is as small as possible. Then we have

(a) *Every component of $H \cap G$ is an arc.*

(b) *Define* $\tilde{M} = \overline{(M - U(H))}$, where $U(H)$ is a vertical regular neighborhood of H (small with respect to G), *and define* $\tilde{G} = G \cap \tilde{M}$. *Then* $\tilde{G}$ *is incompressible in* $\tilde{M}$.

ad (a) Assume there are closed curves in $H \cap G$. Then there is a disc D in H, such that $D \cap G = \partial D$. Since G is incompressible, ∂D bounds a disc D' in G. $D \cup D'$ is a non-singular 2-sphere, so it bounds a ball E in M, since M is irreducible. E shows that there is an isotopy of G which discards (at least) $D' \cap D$ from $H \cap G$, contrary to our assumption that $H \cap G$ is minimal.

ad (b) Assume the contrary. Then there is a disc D in $\tilde{M}$, $D \cap \tilde{G} = \partial D$, ∂D not bounding a disc in $\tilde{G}$. However ∂D bounds a disc in G, whence $H \cap G$ contains a closed curve, in contradiction to (a).

Near H deform G so that $p \mid G \cap U(H)$ is homeomorphic on each component. This is possible by (a). $\tilde{M}$ is a ball. Therefore by (b) each component of $\tilde{G}$ is a disc. We claim

(c) $p \mid \partial\tilde{G}$ *is homeomorphic on each component.*

For assume there is a component $\tilde{D}$ of $\tilde{G}$ for which $p \mid \partial\tilde{D}$ is not a homeomorphism. Let $\tilde{D}$ be part of the component G_1 of G. Any component of ∂G_1 intersects any component of H in at most one point. We have two cases.

Case 1. There is a component k of $G_1 \cap H$ such that those components k_1 and k_2 of ∂G_1 which contain the end points of k, bound an annulus in $F \times 1$. It makes sense to assume that this annulus does not contain any other component of ∂G_1, which we do. Let U be a regular neighborhood of that annulus, and the disc in H which is bounded by k and an arc in that annulus. $(\partial U \cap \mathring{M}) \cap G_1$ is a curve which bounds a disc in $(\partial U \cap \mathring{M})$. Consequently, being incompressible, G_1 must be an annulus. Therefore G_1 intersects any component of H in at most one arc, and it follows that $\tilde{D}$ cannot have been a counter-example.

Case 2. For none of the arcs in $G_1 \cap H$ we have *Case* 1. We deform G_1 in such a way that each of its boundary curves goes to that boundary curve of $F \times 1$ to which it is isotopic, and then slightly into $\partial F \times I$. Because of our assumption that we are in *Case* 2, we can keep $p \mid \partial G_1 \cup (G_1 \cap H)$ locally homeomorphic during this isotopy, and again it follows that $\tilde{D}$ cannot have been a counter-example.

By (c) we may span $\partial\tilde{G}$ by a system of discs in $\tilde{M}$, each component of which is mapped homeomorphically by p. Since this new system is isotopic to $\tilde{G}$ by a deformation of $\tilde{M}$ which is constant on $\partial\tilde{M}$, we have proved that there is a deformation of G, constant on ∂M, which makes $p|G$ locally homeomorphic.

Assume then $p \mid G$ is locally homeomorphic, but not globally on each component of G. This means there is a path l in G with end points q_1 and q_2

such that $p(q_1) = p(q_2)$. As a point x travels along l from the upper end point q_1 to the lower end point q_2, the intersection $p^{-1}(p(x)) \cap G$ generates (among other things) a path l' which starts at q_2 and ends at some point q_3. q_3 cannot lie in ∂G, since $\partial G \cap F \times 0 = \varnothing$. Hence q_3 must lie below q_2. By induction on this argument, we see that $p^{-1}(p(q_1)) \cap G$ contains an infinite number of points, which is absurd.

Proof of 3.1 *in the general case, by induction on* (*genus* F, *number of components of* ∂F) *in lexicographical ordering.* Let H be a vertical annulus in M, such that $H \cap F \times 1 = H = \mathring{F} \times 1$ is a non-contractible curve which is not parallel to a component of $\partial F \times 1$, and which is disjoint to ∂G. H is incompressible. By an isotopy which is constant on ∂M, we deform G so that H and G intersect in general position, and that $H \cap G$ consists of as few curves as possible. Then by similar arguments as in (a) and (b) in the special case above, we prove (a) and (b) below.

(a) Each of the curves $H \cap G$ is in H parallel to $H \cap F \times 1$.

(b) $\tilde{G} = G \cap \tilde{M}$ is incompressible in $\tilde{M} = \overline{(M - U(H))}$, where $U(H)$ is a vertical regular neighborhood of H (which is small with respect to G).

By (a), we may assume that G has been deformed near H in such a way that $p \mid G \cap U(H)$ is homeomorphic on each component.

To make use of the induction hypothesis, we argue with $\tilde{M}$, as follows. We push upward and slightly into $\tilde{M} \cap F \times 1$ those boundary curves of $\tilde{G}$ which lie in $U(H) \cap \tilde{M}$. This can be done by an isotopy which always keeps $p \mid \partial\tilde{G}$ homeomorphic on each component. We know then by (b) and the induction hypothesis that, after an isotopy of $\tilde{G}$ constant on $\partial\tilde{M}$, we will have $p \mid \tilde{G}$ homeomorphic on each component. Finally, we wish to push back $\partial\tilde{G}$ to its original position. Let G_1 be a component of $\tilde{G}$, and assume it is G_1's turn to have its boundary curve k pushed into the component H_1 of $\tilde{M} \cap U(H)$. We have two cases.

Case 1. So far, there is nothing of ∂G_1 in H_1. Then, in pushing k into H_1, we can keep $p \mid G_1$ homeomorphic.

Case 2. The boundary curve l of G_1 is already in H_1. Then there is an annulus G_1' in $\partial\tilde{M}$, which is bounded by l and k and which contains $H_1 \cap F \times 1$. We know from the corollary to the induction hypothesis, that G_1 is parallel to a surface in $\partial\tilde{M} - F \times 0$; there is no other choice for this surface but G_1'. Tracing back G_1 and G_1' to the time just before we started pushing, we find two curves in $H \cap G$ bounding an annulus in G which is parallel to an annulus in H. But this contradicts our minimality assumption on $H \cap G$.

Thus we see that we may assume $p \mid G$ locally homeomorphic. That this forces $p \mid G$ to be homeomorphic on each component of G, follows exactly as in the special case above.

Definition 3.3. A homeomorphism $h: M \to M$ is *level-preserving*, if and only if it can be written as $h(x, y) = (f_y(x), y)$, for $x \in F$, $y \in I$. An isotopy is level-preserving if and only if it goes through level-preserving homeomorphisms.

LEMMA 3.4. *In M, let G be a system, such that each component of G is either a disc which intersects $\partial F \times I$ in two vertical arcs, or an incompressible annulus which has one boundary curve in $\mathring{F} \times 0$, and the other one in $\mathring{F} \times 1$. Then there is an isotopy, constant on $F \times 0 \cup \partial F \times I$, which makes G vertical. This isotopy may be composed of isotopies which are either constant on ∂M, or level-preserving and constant on $F \times 0 \cup \partial F \times I$.*

PROOF. Let G_1 be the first component of G. Define $k_1 = G_1 \cap F \times 0$; $k_2 = G_1 \cap F \times 1$. Using the projection $p: M \to F$, we define k_1' as lying in $F \times 1$ over k_1. We have $k_1' \cap \partial(F \times 1) = k_2 \cap \partial(F \times 1) = \partial k_1' = \partial k_2$. The projection of G_1 to $F \times 1$ defines a homotopy from k_2 to k_1' which is constant on ∂k_2. Thus, by Baer's theorem there is an isotopy, constant on $\partial(F \times 1)$ which carries k_2 to k_1'. We extend this isotopy to a level-preserving isotopy of M, which is constant on $F \times 0 \cup \partial F \times I$. Denote by G_1' the vertical object determined by k_1. We have then $\partial G_1' = \partial G_1$.

Case 1. *G_1 is a disc.* After a deformation of G, constant on ∂M, we may assume that $G_1 \cap G_1'$ consists of their common boundary and a number of simple closed curves in the interior, at which the intersection is transversal. Assume the number of these curves is minimal. Then there are no such curves, by the usual argument. Thus $G_1 \cup G_1'$ bounds a ball, whence there is a deformation, constant on ∂M, which takes G_1 to G_1'. Splitting then M at G_1, we have an induction.

Case 2. *G_1 is an annulus.* $k_1 = G_1 \cap F \times 0$ is a non-contractible curve in $F \times 0$. Therefore there exists $k \subset F \times 0$, $k \cap \partial(F \times 0) = \partial k$, which is a simple closed curve or arc according to whether F is closed or not, such that $k \cap k_1$ consists of one or two points, and cannot be made smaller by an isotopy of k. Let H be the vertical object over k. After G has been adjusted by a small deformation, constant on ∂M, $G_1 \cap H$ will consist of simple closed curves and arcs. Assume that among the arcs in $G_1 \cap H$, the arc l has both its end points in $F \times 0$; then by our choice of H, $\partial l = \partial H \cap k_1$, and projection of the disc which is split off G_1 by l, will show that $\partial H \cap k_1$ can be made empty, contrary to the definition of H. Thus (since $\partial H \cap k_1$ and $\partial H \cap k_2$ have the same number of points), any arc in $G_1 \cap H$ must intersect both $F \times 0$ and

$F \times 1$. In particular, any closed curve in $G_1 \cap H$ is contractible in both G_1 and H, and so these can be removed in the usual way. We conclude that there is a deformation of G, composed of one which is constant on ∂M, and one which is level-preserving and constant on $F \times 0 \cup \partial F \times I$, which makes $G_1 \cap H$ consist of one or two vertical arcs.

Splitting now at H (and forgetting for the moment about $G - G_1$), we obtain a manifold $\tilde{M}$ and a system $\tilde{G}$ in $\tilde{M}$, which comes from G_1. To $\tilde{G}$ in $\tilde{M}$ *Case* 1 applies. Thus there is in fact a deformation of M, of our special sort, which makes the component G_1 of G vertical. Splitting then M at G_1, we have an induction.

LEMMA 3.5. *Let $h: M \to M$ be a homeomorphism such that*

$$h \mid (F \times 0 \cup \partial F \times I)$$

is the identity. Then there is an isotopy, constant on ∂M, which makes h a level-preserving homeomorphism.

PROOF. We first show h can be deformed into the identity by isotopies which are either constant on ∂M, or level-preserving and constant on $F \times 0 \cup \partial F \times I$.

Case 1. $\partial F \neq \emptyset$; Let G be a system of vertical discs such that splitting at $p(G)$ will reduce F to a disc. By (3.4), we may assume $h(G) = G$, (and each component of G is mapped to itself). Further deformations of our special type will give us consecutively $h \mid G = id \mid G$, and $h \mid \partial\tilde{M} = id \mid \partial\tilde{M}$, where $\tilde{M}$ is obtained from M by splitting at G. An application of Alexander's theorem to the ball $\tilde{M}$ will complete the proof.

Case 2. $\partial F = \emptyset$. Since F is not a 2-sphere, there exists in M an incompressible vertical annulus G. By (3.4), we may assume $h(G) = G$. Further deformations which are constant on ∂M or level-preserving and constant on $F \times 0$, will give us $h \mid G = \text{id} \mid G$. Splitting then M at G, we reduce *Case* 2 to *Case* 1.

Let h_τ, $\tau \in I$, be the isotopy obtained in the end. Clearly, we may write

$$h_\tau = f_{n\tau} g_{n\tau} \cdots f_{1\tau} g_{1\tau}, \ \tau \in I ,$$

where the isotopies $f_{j\tau}$, $\tau \in I$, are constant on ∂M, and the $g_{j\tau}$ are level-preserving and constant on $F \times 0 \cup \partial F \times I$; rewriting gives

$$h_\tau = g_{n\tau} \cdots g_{1\tau}(g_{1\tau}^{-1} \cdots g_{n\tau}^{-1} f_{n\tau} g_{n\tau} \cdots g_{1\tau}) \cdots (g_{1\tau}^{-1} f_{1\tau} g_{1\tau}) .$$

Taking the composition of the bracketed factors only, we obtain the required isotopy.

4. Twisted line bundles

PROPOSITION 4.1. *Let $M = F \times I$, where F is a closed surface, different from the 2-sphere. Let N be a manifold with connected boundary. Suppose M is a 2-sheeted cover of N. Then N is homeomorphic to a line bundle over a non-orientable closed surface.*

PROOF. Denote by $f\colon M \to N$ the covering map, and by $g\colon M \to M$ the covering translation. To prove the proposition, it suffices to construct a fibering of M which is invariant under g. This will be done in several steps.

(4.2) *There exists an incompressible annulus G_1 in M, such that* $G_1 \cap F \times 0 \neq \emptyset$, $G_1 \cap F \times 1 \neq \emptyset$, and $G_1 \cap g(G_1) = \emptyset$.

PROOF. Let G be a vertical incompressible annulus in M. The map $f \mid G\colon G \to N$ has no local singularities. Therefore, looking at $f(G)$, we find small isotopic deformations of G, after which the singularities of $f \mid G$ will be simple closed double curves and simple double arcs (with transversal intersection) only. There are four types to be considered.

(1) *There is a disc D in G, such that $D \cap g(G) = \partial D$.* Since G is incompressible, ∂D bounds a disc D' in $g(G)$. In $g(G)$, replace D' by a disc near D, "at the other side"; and do the corresponding (i.e., *via* g) change at G. Since the intersection at ∂D has been transversal, at least one intersection curve has vanished. So we assume such a D does not exist.

(2) *There is a disc D in G, such that $D \cap (\partial G \cup g(G)) = \partial D$, and $D \cap g(G)$ is one arc k.* Then there is a disc D' in $g(G)$ such that $\partial D' \subset k \cup g(\partial G)$. In $g(G)$, replace D' by a disc near D, "at the other side"; and do the corresponding change at G. Since the intersection at k had been transversal, at least k has vanished. So we assume such a D does not exist.

(3) *$G \cap g(G)$ consists of closed curves only, each of which is parallel in G to the boundary curves of G.* Take a regular neighborhood $U(f(G))$ in N, and define $V = f^{-1}(U(f(G)))$. The system $(\overline{\partial V - \partial M})$ contains four annuli. These are incompressible, and[1] at least two of them intersect both $F \times 0$ and $F \times 1$. Let G_1 be one of the latter. Then either $g(G_1) \cap G_1 = \emptyset$, and we are through; or, $g(G_1) = G_1$. In the latter case, $f(G_1)$ is a two-sided Moebius strip in N. So this case cannot occur.

(4) *$G \cap g(G)$ consists of arcs only, each of which intersects both $F \times 0$ and $F \times 1$.* In this case, we are forced to argue with $f(G)$, and to admit singular things at intermediate steps. Let l be a double arc of $f(G)$. There are two possibilities to do a cut (*Umschaltung*) at l. One choice for the cut

[1] *Added in proof.* This may be false if F is a torus. Replace the argument in (3) by one involving cuts, similar to (4) below.

will decompose our singular annulus into two things which are either singular annuli or singular Moebius strips. The other cut will have as its result one such object. The important thing is that at any step we are free to make our choice. We find that we can always obtain a singular annulus or Moebius strip with non-contractible boundary. Let H be the end-result. H is a non-singular annulus or Moebius strip. $f^{-1}(H)$ consists accordingly of two or one annuli which are incompressible and intersect both boundary components of M. In the former case, we take as G_1 a component of $f^{-1}(H)$. In the latter case, g interchanges the sides of $f^{-1}(H)$, so we need only push $f^{-1}(H)$ slightly to one of its sides.

(4.3) Let G_1 be the annulus which was constructed in (4.2). Define $G_2 = g(G_1)$. By (3.4), there is an isotopy of M, which makes $G_1 \cup G_2$ vertical. We are careful, however, not to move G_1 and G_2. Instead, we use the inverse isotopy to deform the fibering of M. We have then induced fiberings on G_1 and G_2. We proceed to make $g \mid G_1$ fibre-preserving. We do this by deforming M near G_2 in such a way that the induced deformation on G_2 carries that fibering of G_2, which is induced by the inclusion in M, to that one which is defined by $G_2 = g(G_1)$. Finally, we achieve that there are vertical neighborhoods $U(G_1)$ and $U(G_2)$, such that $g\big(U(G_1)\big) = U(G_2)$, and $g \mid U(G_1)$ is fibre-preserving. Roughly, we achieve this by removing G_1 and G_2 from M, and inserting instead $U(G_1)$ and $U(G_2)$.

(4.4) Let M' be a component of $\overline{\big(M - \big(U(G_1) \cup U(G_2)\big)\big)}$. Assume $g(M') = M'$. Let D be a vertical disc in M', such that ∂D is not contractible in $\partial M'$, and $g(l_i) \cap l_j = \emptyset$, $i, j = 1, 2$, where l_1 and l_2 are the arcs $D \cap \big(U(G_1) \cup U(G_2)\big)$. By similar arguments as in (4.2), we find a disc D' in M', such that $\partial D'$ is not contractible in $\partial M'$, $\big(D' \cup g(D')\big) \cap \big(U(G_1) \cup U(G_2)\big) = \big(D \cup g(D)\big) \cap \big(U(G_1) \cup U(G_2)\big)$, and $g(D') \cap D' = \emptyset$. Then, by (3.4), there is a deformation of the fibering of M', constant on $\overline{(\partial M' \cap \mathring{M})}$, after which $D' \cup g(D')$ will be vertical. And finally, as in (4.3), we find further deformations of the fibering of M' (constant on $\overline{(\partial M' \cap \mathring{M})}$) and regular neighborhoods which are vertical, such that $g\big(U(D')\big) = U\big(g(D')\big)$, and $g \mid U(D')$ is fibre-preserving.

(4.5) We repeat step (4.4) as often as possible: i.e., we construct a submanifold M^* in M, (the union of all those neighborhoods), such that M^* is vertical, $g(M^*) = M^*$, and $g \mid M^*$ is fibre-preserving, and that finally we have: *If M' is any component of $\overline{(M - M^*)}$, then either $g(M') \neq M'$, or there is no such disc D in M' as was used in* (4.4). In the latter case, M' must be a ball, and so again $g(M') \neq M'$, since g has no fix-point. Thus, whenever $g \mid M'$ is not fibre-preserving, we may define a new fibering of $g(M')$, precisely by requiring $g \mid M'$ to be fibre-preserving.

5. Isotopic surfaces

LEMMA 5.1. *Let M be an irreducible manifold which need not be compact. Let F be an incompressible (compact, closed) boundary component of M. In $\partial M - F$, let F' be an incompressible surface which need neither be closed nor compact. Suppose: if k is any closed curve in F, then some non-null multiple of k is homotopic to a curve in F'. Then, M is homeomorphic to $F \times I$.*

PROOF. Let r be the genus of F. Choose simple closed curves $k_1, \cdots, k_{2r}$ in F such that $k_i \cap k_j = \emptyset$, if $i \neq j \pm 1$, and k_j and k_{j+1} intersect (transversely) in exactly one point, for $j = 1, \cdots, 2r - 1$. Such a system of curve is easily constructed from a usual "meridian-longitude-system"; the complement of $\bigcup k_j$ is an open disc.

Let l be the circle. By assumption there exists a map $f_j: l \times I \to M$, such that $f_j(l \times 1) \subset F'$, $f_j(l \times \mathring{I}) \subset \mathring{M}$, and that $f_j(l \times 0)$ is a non-null multiple of k_j. The generalized loop theorem [18] gives us a non-singular annulus G_j, $G_j \cap \partial M = \partial G_j$, which has one boundary curve in F, near k_j, and the other boundary curve in F', such that not both its boundary curves are contractible in M. Since both F and F' are incompressible, G_j has in fact both its boundary curves non-contractible in M; whence that boundary curve in F must be isotopic to k_j. So we assume it is k_j.

Consider a fixed pair G_i, G_j. After a small deformation, if necessary, $G_i \cap G_j$ will consist of mutually disjoint simple closed curves and arcs. If $i = j \pm 1$, and only then, there is a distinguished one among the intersection arcs which has one end point in F and one end point in F'. Any other intersection arc has both its end points in F'. Any closed intersection curve is either contractible in both G_i and G_j, or non-contractible in both G_i and G_j, because of the incompressibility of F. We proceed to reduce the number of intersections by performing a cut (*Umschaltung*) either at a closed intersection curve or at a non-distinguished intersection arc. By what we said above, there is at each step a correct one among the two possibilities. The annuli which show up at intermediate steps may have singularities. But in the end we are left with a pair of non-singular annuli, again denoted by G_i, G_j, such that $G_i \cap G_j$ is a distinguished intersection arc if $i = j \pm 1$, and empty otherwise.

Next, we take up some other pair, and do with it the same things we did with G_i, G_j, and so on. In the course of this construction, we may be forced to take up several times the "same" pair. But finiteness may be seen thus: In the beginning, we might have normalized the G_j in such a way, that the intersections of $\bigcup G_j$ consisted of double curves and arcs, and triple points, only.

The pair (t, d), where t denotes the number of triple points, and d the number of double curves and arcs, will then be decreased, in the sense of lexicographical ordering, every time a cut is performed.

We conclude that in the end, $\bigcup G_j$ is homeomorphic to $(\bigcup k_j) \times I$. Taking regular neighborhoods, we see that $(\overline{\partial U(F \cup \bigcup G_j)} \cap \mathring{M})$ is a disc. Since F' is incompressible, $(F' - (\mathring{U}(F \cup \bigcup G_j) \cap F'))$ is a disc, too (whence in particular, F' is closed). So, $(\partial U(F \cup \bigcup G_j \cup F') \cap \mathring{M})$ is a 2-sphere. Since M is irreducible, this 2-sphere bounds a ball in M, and the lemma follows.

LEMMA 5.2. *Let M be an irreducible manifold which need not be compact. Let G be a boundary surface of M which need not be compact. Let F be a (compact) surface in G, such that $\partial F \neq \emptyset$. Suppose both F and $G - F$ are incompressible. And, any arc k in F, $k \cap \partial F = \partial k$, is homotopic to an arc k' in G, $k' \cap F = \partial k'$, by a homotopy which is constant on ∂k. Then, F is parallel to $G - \mathring{F}$.*

PROOF. Let $k_1, \cdots, k_r$, $r \geqq 0$, be a system of (disjoint) arcs in F, $F_j \cap \partial F = \partial k_j$, such that splitting at the k_j would reduce F to a disc. By assumption, there exists a singular disc $f_j\colon D \to M$, such that $f_j(\partial D) \subset G$, and $f_j \mid f_j^{-1}(f_j(\partial D) \cap F)$ is a homeomorphism onto k_j.

The curve $f_j(\partial D)$ is essential in G modulo that normal subgroup of $\pi_1(G)$ which is generated by $\pi_1(G - F)$. Therefore the loop theorem gives us a non-singular disc D_j near $f_j(D)$, $D_j \cap \partial M = \partial D_j$, such that ∂D_j is essential in G modulo that same normal subgroup. We recall that in the proof of the loop theorem the disc D_j is actually constructed in a very special way. This enables us to conclude in our present case that $D_j \cap F$ is either empty or is an arc which is isotopic in F to k_j. Thus, we may assume $D_j \cap F = k_j$.

The rest is similar to the proof of (5.1). We construct new discs which are pairwise disjoint, and then an argument involving regular neighborhoods will complete the proof.

LEMMA 5.3. *Let M be an irreducible manifold. Let G be an incompressible surface in ∂M. Let F be an incompressible surface in M, such that $\partial F \subset G$. Suppose there is a surface H and a map $f\colon H \times I \to M$, such that $f \mid H \times 0$ is a covering map onto F, and $f(\partial(H \times I) - H \times 0) \subset G$. Then F is parallel to a surface F' in G.*

PROOF. F is not a 2-sphere. If F is a disc, then the assertion follows immediately from the fact that G is incompressible and M irreducible. So we assume, F is not a disc. Then no boundary curve of F is contractible (in either G or M).

Let us look first at the special case $f^{-1}(F) \cap (\mathring{H \times I}) = \emptyset$. We construct the manifold M' by splitting M at F. By our assumption on f, there exists a lifting $f'\colon H \times I \to M'$ of f. Since F was incompressible, M' is irreducible, and since no boundary curve of F was contractible, the system G' (which is G, split at F) is incompressible. Therefore we can apply either (5.1) or (5.2), and the lemma follows.

We return to the general case. Our aim is to reduce it to the special case, by constructing a "homotopy" of the special sort.

First, we may assume that $\partial F \cap \partial G = \varnothing$. For otherwise, we enlarge G slightly to G'' which also is incompressible; the surface F' which we are going to detect in G'', will nevertheless be contained in G, since it cannot contain boundary points of G'' in its interior.

Thus, we may add to the hypotheses about f

(1) There exists a regular neighborhood $U(H \times 0 \cup \partial H \times I)$, such that $f^{-1}(F) \cap U = H \times 0$.

Next, we apply our normalization procedure (1.3) to f:

(2) By a deformation of f, constant on U, we induce a deformation of $f \mid H \times 1\colon H \times 1 \to G$, which makes this map transverse with respect to ∂F; we choose the deformation so that $f^{-1}(\partial F) \cap \overline{(H \times 1 - U)}$ will not contain a contractible curve, and that (in addition) the number of these curves will be as small as possible.

After this deformation, $f \mid \partial\overline{(H \times I - U)}$ is transverse with respect to F. So, another application of (1.3) gives a deformation which is constant on $H \times 1 \cup U$, and which makes $f^{-1}(F) \cap \overline{(H \times I - U)}$ a system of incompressible surfaces in $\overline{(H \times I - U)}$, and hence also in $H \times I$.

If this system is empty, our reduction is complete. So we assume the component H' exists. By (3.2), H' is parallel to a surface $H'' \subset H \times 1$. Let N be the submanifold which is bounded by $H' \cup H''$. By another application of (3.2), it makes sense to assume that H' is "next" to $H \times 1$, i.e. that $N \cap f^{-1}(F) = H'$; which we do. If now $f \mid H'\colon (H', \partial H') \to (F, \partial F)$ is homotopic (by a homotopy of pairs) to a covering map, then, looking at $f \mid N$, we see that again our reduction is complete. So we assume it is not. A commutative diagram shows, $\ker (f \mid H')_* = 0$. Therefore, by Nielsen's theorem, (1.4.3), there are only two cases left:

Case 1. *H' is a disc.* Then H'' is a disc, too, which contradicts (2) above.

Case 2. *H' is an annulus, and $f \mid H'\colon (H', \partial H') \to (F, \partial F)$ contracts*; in particular, $f(\partial H')$ is contained in one component k, of ∂F. We have again two cases, according to whether $f \mid H''\colon (H'', \partial H'') \to (G, k)$ does or does not contract into (k, k). In the second case it follows from (1.4.3) that G is a torus, that $G \subset f(H'')$ and hence (since $(f \mid H'')^{-1}(\partial F) = (f \mid H'')^{-1}(k)$) that $F \cap G = k \neq \partial F$, which contradicts $\partial F \subset G$. The first case contradicts (2) above.

PROPOSITION 5.4. *Let M be an irreducible manifold. In M let F and G be incompressible surfaces, such that $\partial F \subset \partial F \cap \partial G$, and $F \cap G$ consists of mutually disjoint simple closed curves, with transversal intersection at any curve which is not in ∂F. Suppose there is a surface H and a map $f: H \times I \to M$, such that $f \mid H \times 0$ is a covering map onto F, and*

$$f(\partial(H \times I) - H \times 0) \subset G .$$

Then there is a surface $\tilde{H}$ and an embedding $\tilde{H} \times I \to M$, such that

$$\tilde{H} \times 0 = \tilde{F} \subset F, \quad \overline{(\partial(\tilde{H} \times I) - \tilde{H} \times 0)} = \tilde{G} \subset G$$

(i.e., a small piece of F is parallel to a small piece of G), *and that moreover $\tilde{F} \cap G = \partial\tilde{F}$, and either $\tilde{G} \cap F = \partial\tilde{G}$, or $\tilde{F}$ and $\tilde{G}$ are discs.*

PROOF. *Case* 1. *The intersection curve $k \subset F \cap G$ is contractible in F or G.* Then there is a disc D in G, say, which is bounded by k. D contains an innermost disc D', i.e., $D' \cap F = \partial D'$. Since F is incompressible, there is a disc D'' in F, such that $\partial D'' = \partial D'$. Let $\tilde{F}$ be an innermost disc in D''; so $\tilde{F} \cap G = \partial\tilde{F}$. Since G is incompressible, there is a disc $\tilde{G}$ in G, such that $\partial\tilde{G} = \partial\tilde{F}$. Because of our choice of $\tilde{F}$, the 2-sphere $\tilde{F} \cup \tilde{G}$ is non-singular; since M is irreducible, this 2-sphere bounds a ball, and the proposition follows in *Case* 1.

Case 2. *None of the intersection curves is contractible in F or G.* Our aim is to reach a situation where we can apply (5.3); the construction of this situation is similar to the proof of (5.3).

Using small deformations of f, we add to our hypotheses the following.

(1) There is an open neighborhood of $H \times 1 \cup \partial H \times \mathring{I}$, the interior of which is disjoint to $f^{-1}(G)$.

(2) For any boundary curve k of H, there exists a regular neighborhood $U(k \times 0)$, such that $U(k \times 0) \cap f^{-1}(G) = (U(k \times 0) \cap \partial H \times I) \cup X$, where X is either empty, or is an annulus such that $X \cap \partial U(k \times 0) = \partial X$, and $X \cap \partial(H \times I) = k \times 0$.

By (1) and (2), provided (2) is done carefully, there exists a regular neighborhood U of $H \times 1 \cup \partial H \times I$, such that $f \mid \partial\overline{(H \times I - U)}$ is transverse with respect to G. So, by (1.3), there is a deformation of f, constant on $U \cup H \times 0$, which makes $f^{-1}(G) \cap \overline{(H \times I - U)}$ a system of incompressible surfaces in $\overline{(H \times I - U)}$. We have then $f^{-1}(G) = H \times 1 \cup \partial H \times I \cup \bigcup H_j$, where $\bigcup H_j$ is a system of incompressible surfaces in $H \times I$; (in fact, for any

$U(k \times 0)$ (cf. (2)), $\bigcup H_j \cap U(k \times 0)$ is at most one annulus. Therefore each of the H_j is non-singular, and any two are disjoint; incompressibility is clear).

We assume the component H' of $\bigcup H_j$ exists (the other case is quite similar, and is simpler). By (3.2), H' is parallel to $H'' \subset H \times 0$; let N be the submanifold which is bounded by $H' \cup H''$. We assume H' is "next" to $H \times 0$, i.e., $f^{-1}(G) \cap N = H'$.

Construct the manifold M' by splitting M at G. Then a map $f'\colon N \to M'$ exists which is a lifting of $f \mid N\colon N \to M$. In $\partial M'$ there are two copies of G; denote by G' that one which contains $f'(H')$; G' is incompressible in M' (trivially).

Because of our general position assumptions on $F \cap G$, the subspace F' of M' which by $M' \to M$ is projected onto F, is a system of surfaces in M'. F' is incompressible in M'. For otherwise, there exists a disc D in $\mathring{M}'$ such that $D \cap F' = \partial D$, ∂D not bounding a disc in F'. Since the image of ∂D in M bounds a disc in F, we find a curve in $F \cap G$, which is contractible in F, contrary to our assumption that we are not in *Case* 1.

Let F'' be that component of F' which contains $f'(H'')$. We conclude that with M', G', F'', $f'\colon N \to M'$, we are exactly in the hypotheses of (5.3). Therefore there is a surface $G'' \subset G'$ which is parallel to F''. If now $G'' \cap F' = G'' \cap F''$, then the proposition is proved. Otherwise, denote by M'' that submanifold of M' which is bounded by $F'' \cup G''$. Applying (3.2) to the system $F' \cap M''$ in M'', we find a component $\widetilde{F}$ of $F' \cap M''$, which is next to G'', and which is parallel to $\widetilde{G} \subset G''$.

COROLLARY 5.5. *Let M be an irreducible manifold. Let F and G be incompressible surfaces in M. Suppose there is a homotopy from F to G, which is constant on ∂F. Then, F is isotopic to G by a deformation which is constant on ∂M.*

PROOF. By an isotopy which is constant on ∂M, move F so that the intersection $F \cup G$ consists of mutually disjoint simple closed curves, the number of which is as small as possible. Consider those tiny pieces $\widetilde{F}$ and $\widetilde{G}$ which were found in (5.4). Suppose $\widetilde{F}$ and $\widetilde{G}$ are discs, and $F \cap \widetilde{G} \neq \partial\widetilde{G}$. Then the ball which is bounded by $\widetilde{F} \cup \widetilde{G}$, contains part of F its interior. Pushing out these things across $\widetilde{G}$, we reduce the intersection $F \cap G$. But this contradicts our minimality condition, and so this case cannot occur. Therefore there is an isotopic deformation of F, constant on $\overline{(F - \widetilde{F})}$, which takes $\widetilde{F}$ to $\widetilde{G}$. Suppose $F \neq \widetilde{F}$. Then we can push off $\widetilde{F}$ slightly to the other side of $\widetilde{G}$, while keeping $\widetilde{F} \cap \partial F$ fixed, and again we achieve the impossible. Therefore $F = \widetilde{F}$. We have also $G = \widetilde{G}$. For, we have just seen that $\partial\widetilde{G} \subset \partial M$, and since $G \cap \partial M = \partial G$, there is no other choice for G.

6. Existence of homeomorphisms

THEOREM 6.1. *Let M and N be manifolds which are irreducible and boundary-irreducible. Suppose that M is sufficiently large, and that $\pi_1(N) \neq 0$. Let $f\colon (N, \partial N) \to (M, \partial M)$ be a map which induces an injection $f_*\colon \pi_1(N) \to \pi_1(M)$. Then there exists a homotopy $f_\tau\colon (N, \partial N) \to (M, \partial M)$, $\tau \in I$, $f_0 = f$, such that either* (a) *or* (b) *holds.*

(a) *N is the product line bundle over a closed orientable surface, and $f_1(N) \subset \partial M$,*

(b) *$f_1\colon N \to M$ is a covering map.*

If $f \mid \partial N$ is locally homeomorphic, then the homotopy may be chosen so that $f_\tau \mid \partial N = f_0 \mid \partial N$, for all τ.

Proof of (6.1) *in the case $\partial M \neq \emptyset$.* Let R be a boundary component of N; R is not a 2-sphere. Let S be that boundary component of M which contains $f(R)$. Since N is boundary-irreducible, it follows from $\ker f_* = 0$, that $\ker (f \mid R)_* = 0$. Therefore, by Nielsen's theorem, $f \mid R$ is homotopic to a covering map. We perform a homotopy of f which induces such homotopies at all boundary components of N. We compose it with a general position homotopy, to make sure that $f^{-1}(\partial M) = \partial N$. If $f \mid \partial N$ was locally homeomorphic in the beginning, there has been no necessity so far to alter it, since in what follows there will be no necessity either, the last assertion in (6.1) will be established.

Choose a hierarchy for $M_1 = M$, (cf. (1.2), theorem):

$$M_j,\ F_j \subset M_j,\ U(F_j) \subset M_j\ ,\quad M_{j+1} = \overline{(M_j - U(F_j))}, \qquad j = 1, \cdots, n\ .$$

So far, we proved that, for $r = 1$, the following holds.

Induction hypothesis: $f \mid f^{-1}\bigl(\partial M \cup \bigcup_{j<r} U(F_j)\bigr)$ *is locally homeomorphic.*

Suppose, we have proved it for $r = n + 1$. M_{n+1} is a ball. Let N^* be a component of $f^{-1}(M_{n+1})$. We assumed $\pi_1(N) \neq 0$, whence $N^* \neq N$. Since there are no covering maps onto a 2-sphere other than homeomorphisms, it follows from the fact that N is irreducible, that there is a homotopy of $f \mid N^*$, constant on ∂N^*, which will make $f \mid N^*$ a homeomorphism; and *Case* (b) of the theorem will follow. Thus we attempt to show that the induction step can be made. When we fail, it will turn out that we can prove *Case* (a) of the theorem.

Let N' be a component of $f^{-1}(M_r)$. Denote by f' the restriction $f \mid N'$. We have $f'\colon (N', \partial N') \to (M_r, \partial M_r)$. By the induction hypothesis, $f' \mid \partial N'$ is locally homeomorphic. By (1.3), there is a homotopy of f', constant on $\partial N'$, such that afterwards f' is transverse with respect to F_r, and that $f'^{-1}(F_r)$ is a system of incompressible surfaces in N'. We prove easily, $\ker f'_* = 0$. From this follows $\ker(f' \mid G)_* = 0$, where G is any component of $f'^{-1}(F_r)$. We would like to conclude that $f' \mid G \colon G \to F_r$ is homotopic to a covering map by a homotopy which is constant on ∂G. If this conclusion holds for any G, and for any choice of N', then the induction step follows immediately.

Assume then that the conclusion is false for G. Remembering that G cannot be a 2-sphere, we find ourselves left with the following two possibilities.

(1) F_r is a disc; G too; and the covering map $f' \mid \partial G$ is not a homeomorphism.

(2) F_r is not a disc. By Nielsen's theorem (1.4.3), G is an annulus, and $f' \mid G \colon (G, \partial G) \to (F_r, \partial F_r)$ contracts to $(\partial F_r, \partial F_r)$; in particular, $f'(\partial G)$ is contained in one boundary curve of F_r.

In both cases, there exists a simple arc l in G, $l \cap \partial G = \partial l$, with the properties: $f'(\partial l)$ is one point; $f' \mid l \colon (l, \partial l) \to (F_r, f'(\partial l))$ contracts. Composing l, if necessary, with two suitable arcs (obtained e.g. by lifting an arc which joins $f'(\partial l)$ inside $\bigcup_{j<r} U(F_j)$ to ∂M), *we find a simple arc k in N, such that ∂k consists of two different points, p_1 and p_2, in ∂N, and such that $f \mid k \colon (k, \partial k) \to (M, f(\partial k))$ contracts*, (in particular $f(p_1) = f(p_2)$).

Denote by S that boundary surface of M which contains $f(p_1)$. Using $f(p_1)$ twice as base point, we have an obvious inclusion homomorphism $i_* \colon \pi_1(S) \to \pi_1(M)$. Let R_1 and R_2 be those boundary components of N which contain p_1 and p_2. Using p_1 twice as base point, we define $i_{1*} \colon \pi_1(R_1) \to \pi_1(N)$. Finally, we define $i_{2*} \colon \pi_1(R_2) \to \pi_1(N)$ using the path k. All these inclusion homomorphisms are injective. We have $f_* i_{1*} = i_*(f \mid R_1)_*$, (by naturality), and $f_* i_{2*} = i_*(f \mid R_2)_*$, (since $f \mid k \colon (k, \partial k) \to (M, f(\partial k))$ contracts).

Since all three, R_1, R_2, S, are closed, and since $f \mid R_1$ and $f \mid R_2$ are coverings, $(f \mid R_1)_*(\pi_1(R_1))$ and $(f \mid R_2)_*(\pi_1(R_2))$ have finite index in $\pi_1(S)$. Thus, by the above, *$i_{1*}(\pi_1(R_1))$ and $i_{2*}(\pi_1(R_2))$ intersect in a subgroup which has finite index in both.*

We now distinguish three cases.

(a) $R_1 \neq R_2$. By (5.1), N is homeomorphic to $R_1 \times I$. Consider the covering $\widetilde{M}$ of M which is associated to $i_*(\pi_1(S))$; denote by $\widetilde{S}$ a copy over S, for which $\pi_1(\widetilde{S}) \to \pi_1(S)$ is an isomorphism. Let $\tilde{f} \colon N \to \widetilde{M}$ be a lifting of f, such that $\tilde{f}(\partial N) \cap \widetilde{S} \neq \emptyset$. Then, in fact, $f(\partial N) \subset \widetilde{S}$, because $f \mid k \colon (k, \partial k) \to (M, f(\partial k))$ contracted. Observing that $\widetilde{M}$ deformation-retracts to $\widetilde{S}$, we find that we have proved *Case* (a) of the theorem.

(b) $R_1 = R_2$. *$(k, \partial k) \to (N, R_1)$ does not contract into (R_1, R_1).* Consider the covering $\tilde{N}$ of N which is associated to $i_{1*}(\pi_1(R_1))$. Denote by R' a copy over R_1 for which $\pi_1(R') \to \pi_1(R_1)$ is an isomorphism. Let k' be a copy over k, which originates at R'. Denote by R'' that copy over R_1 which contains the other end point of k'; R'' is different from R', (and may be non-compact). That identification of subgroups in $\pi_1(N)$ along k, lifts to an identification of subgroups in $\pi_1(\tilde{N})$ along k', one of the subgroups concerned being a subgroup of finite index in $\pi_1(R')$. Thus, by (5.1), $\tilde{N}$ is homeomorphic to $R' \times I$. Consequently, $\tilde{N} \to N$ is a 2-sheeted covering, whence, by (4.1), N is homeomorphic to a line bundle over a non-orientable closed surface. Since $f \mid k: (k, \partial k) \to (M, S)$ contracts, $\pi_1(N)$ is isomorphic to a subgroup of $\pi_1(S)$. Since S is orientable, this is absurd.

(c) $R_1 = R_2$. *There is a homotopy of k, fixed on ∂k, which sends k to an arc in R_1.* Call this arc $\tilde{k}$. $f(\tilde{k})$ defines a based loop in S, which is not contained in the subgroup $(f \mid R_1)_*(\pi_1(R_1))$. On the other hand, $f(\tilde{k})$ is homotopic in M to the based loop $f(k)$, which is contractible. Since S is incompressible, it follows that $f(\tilde{k})$ is contained in any subgroup of $\pi_1(S)$.

Proof of 6.1 *in the case $\partial M = \emptyset$.* By our conditions on M, there is an incompressible surface in F in M. Since M is closed, F has to be closed, too. Homotope f to make it transverse with respect to F, and to make $f^{-1}(F)$ a system of incompressible surfaces in N, (1.3). Choose the homotopy in such a way that in addition, the number of components of $f^{-1}(F)$ is as small as possible. Let G be a component of $f^{-1}(F)$ (at the present stage, we are not claiming that $f^{-1}(F)$ is non-empty). Since N is closed, G is closed, too. A commutative diagram shows that $\ker (f \mid G)_* = 0$. Since G is not a 2-sphere, Nielsen's theorem tells us that $f \mid G$ is homotopic to a covering map. Thus we may assume that there is a regular neighborhood $U(F)$, such that $f \mid f^{-1}(U(F))$ is a covering map on each component.

Consider then $\tilde{M}$, which is component of $M - \mathring{U}(F)$; $\tilde{N}$, which is a component of $f^{-1}(\tilde{M})$; and $\tilde{f} = f \mid \tilde{N}: \tilde{N} \to \tilde{M}$. Another diagram (cf. (1.1.4)) shows that $\ker \tilde{f}_* = 0$. Since $f \mid \partial \tilde{N}$ is locally homeomorphic, the formerly proved part of the theorem shows that there is a homotopy of $\tilde{f}$, constant on $\partial \tilde{N}$, with two possibilities for its end result. The first possibility would result in a contraction into $\partial \tilde{M}$. This is ruled out by our minimality condition on f. So there is in fact a homotopy of $\tilde{f}$, constant on $\partial \tilde{N}$, which makes $\tilde{f}$ a covering map.

Definition 6.2. Let M and N be manifolds. Let $\psi: \pi_1(N) \to \pi_1(M)$ be a homomorphism. *ψ is a homomorphism of group systems* or *respects the* *peripheral structure*, if and only if the following holds. For each boundary surface F of N, there exists a boundary surface G of M, such that $\psi(i_*(\pi_1(F))) \subset A$, and A is conjugate in $\pi_1(M)$ to $i_*(\pi_1(G))$. (Here i_* denotes inclusion homomorphisms. The definition does not depend on the choice of the i_*.)

LEMMA 6.3. *Let M and N be manifolds, such that M is irreducible and boundary-irreducible, and has infinite fundamental group. Let $\psi: \pi_1(N) \to \pi_1(M)$ be a homomorphism. Then there exists a map $f: (N, \partial N) \to (M, \partial M)$ which induces ψ, if and only if ψ respects the peripheral structure.*

PROOF. One direction is obvious. We come to the other. By the usual argument, M is aspherical. Therefore a map $f': N \to M$ can be constructed which induces ψ. To prove the lemma, it will suffice to prove *let F be a boundary component of N, and $g_0 = f' \mid F$. Then there exists a homotopy $g_\tau: F \to M$, $\tau \in I$, such that $g_1(F) \subset \partial M$.* We construct this homotopy piecewise. Inspection of (6.2) reveals that g_τ can be defined on the 1-skeleton of F. Next, we define g_1, compatible with g_τ on the 1-skeleton. The obstruction to do this lies in $\ker(\pi_1(G) \to \pi_1(M))$, which is 0, where G is the boundary component involved. The obstruction to fill in the rest, lies in $\pi_2(M)$, which is 0, too.

COROLLARY 6.4. *Let M and N be manifolds which are irreducible and boundary-irreducible. Suppose M is sufficiently large; N is not homeomorphic to a product line bundle over a closed surface, and $\pi_1(N) \neq 0$. Let $\psi: \pi_1(N) \to \pi_1(M)$ be an injection which respects the peripheral structure. Then there exists a covering map $f: N \to M$, which induces ψ.*

PROOF. We apply (6.3) to obtain a map $g: (N, \partial N) \to (M, \partial M)$, with $g_* = \psi$. From g, we obtain f by (6.1). (If in the construction of f we moved the base point, we move it back in the end.)

COROLLARY 6.5. *Let M and N be manifolds which are irreducible and boundary-irreducible. Suppose M is sufficiently large. Let $\psi: \pi_1(N) \to \pi_1(M)$ be an isomorphism which respects the peripheral structure. Then there exists a homeomorphism $f: N \to M$, which induces ψ.*

PROOF. If N is not excluded in the hypotheses of (6.4), we apply (6.4) to obtain a 1-sheeted covering map. If N is a product line bundle, then ψ^{-1} also respects the peripheral structure. Since *sufficiently large* depends only on the homotopy type, we try to apply (6.4) to ψ^{-1}. If this should fail, too, Nielsen's theorem will save the corollary.

7. Existence of isotopies

THEOREM 7.1. *Let M be a manifold which is irreducible and sufficiently* *large. Let $h: M \to M$ be a homeomorphism which is homotopic to the identity map by the homotopy $H: M \times I \to M$. And suppose that either* (a) *or* (b) *holds.*

(a) $H(\partial M \times I) \subset \partial M$.

(b) *M is boundary-irreducible. If M is homeomorphic to a line bundle, then h is orientation-preserving.*

Then h is isotopic to the identity.

If in case (a), *$H \mid \partial M \times I$ is projection onto the first factor, then the isotopy from h to the identity may be chosen constant on ∂M.*

In the proof we shall consider four cases.

Case 1. *$\partial M \neq \emptyset$; the homotopy is constant on ∂M.* Choose a hierarchy for $M_1 = M$, (cf. (1.2), theorem)

$$M_j,\ F_j \subset M_j,\ U(F_j) \subset M_j\ ,\quad M_{j+1} = \overline{(M_j - U(F_j))}\ ,\qquad j = 1, \cdots, n\ .$$

By assumption, the following holds for $r = 1$.

Induction hypothesis. $H \mid (\partial M \cup \bigcup_{j<r} U(F_j)) \times I$ is projection onto the first factor.

As a consequence of the induction hypothesis, we have $h \mid M_r$ is a homeomorphism onto M_r, and $h \mid \partial M_r$ is the identity map.

Let F be a surface which is homeomorphic to F_r, and define the map $f: F \times I \to M$ as the restriction $H \mid F_r \times I$.

LEMMA 7.2. *There is a homotopy of f, constant on $\partial(F \times I)$, after which $f(F \times I) \subset M_r$.*

PROOF. Assume as induction hypothesis, that $f(F \times I) \subset M_s$, for $s < r$. $f \mid \partial(F \times I)$ is trivially transverse with respect to F_s; thus, by (1.3), there is a homotopy of $f: F \times I \to M_s$, constant on $\partial(F \times I)$, which makes $f^{-1}(F_s)$ a system of incompressible surfaces. Since these surfaces have to be closed, and since $\partial F \neq \emptyset$, (3.2) shows that $f^{-1}(F_s)$ is empty. We finally push $f(F \times I)$ out of $\mathring{U}(F_s)$, i.e., into M_{s+1}.

By (7.2) we may apply (5.5) to the surfaces F_r and $h(F_r)$ in M_r; i.e., we can find an isotopic deformation of $h \mid M_r$, constant on ∂M_r, such that afterwards $h(F_r) = F_r$. Thus we assume, this holds true.

LEMMA 7.3. *There is a homotopy of f, constant on $\partial(F \times I)$, after which $f(F \times I) \subset F_r$.*

PROOF. At the present stage of our normalization, we have $f(F \times 0) \subset F_r$, $f(F \times 1) \subset F_r$, and $f \mid \partial F \times I$ is "projection onto the first factor" anyway. Denote by f_τ, $\tau \in I$, the homotopy which we are going to construct. Define $f_0 = f$, and $f_\tau \mid \partial(F \times I) = f_0 \mid \partial(F \times I)$, for all τ. Since $\partial F \neq \emptyset$, the interior of $F \times I$ admits a decomposition into open 2- and 3-cells only.

Therefore, the only obstructions to extending $f_1 \mid \partial(F \times I)$ to $f_1\colon F \times I \to F_r$, lie in $\ker\big(\pi_1(F_r) \to \pi_1(M_r)\big)$ and $\pi_2(F_r)$, which are 0. Similarly, the obstructions to defining the rest of the homotopy, lie in $\pi_2(M_r)$ and $\pi_3(M_r)$, which are 0, too.

By (7.3) we may assume that $f\colon F \times I \to M_r$ is in fact a map $f\colon F \times I \to F_r$. So, by Baer's theorem, $h \mid F_r$ is isotopic to $\mathrm{id} \mid F_r$ by an isotopy which is constant on ∂F_r. So we assume, $h \mid M_r$ has been deformed (by an isotopy which is constant on ∂M_r) so that $h \mid F_r = \mathrm{id} \mid F_r$.

Looking again at $f\colon F \times I \to F_r$, we find a homotopy, constant on $\partial(F \times I)$, from f to the *projection onto the first factor*. In fact, there is no obstruction to construct this homotopy, because the two maps agree on $\partial(F \times I)$; F_r is aspherical; $\mathring{(F \times I)}$ admits a decomposition into open 2- and 3-cells only.

We recall now that f was initially defined as the restriction to F_r of $H\colon M \times I \to M$. And we observe that all the deformations of f may be extended to deformations of $H \mid M_r \times I$ which are constant on $\partial M_r \times I$. Finally, extending our normalizations to a neighborhood, we make

$$H \mid \big(\partial M \cup \textstyle\bigcup_{j<r+1} U(F_j)\big) \times I$$

projection onto the first factor.

After n induction steps have been performed, $h \mid \overline{(M - M_{n+1})}$ will be the identity map. Since M_{n+1} is a ball, Alexander's theorem will complete the proof in *Case* 1.

Case 2. $\partial M \neq \emptyset$; $H(\partial M \times I) \subset \partial M$. Let F be a boundary component of M. Consider $f\colon F \times I \to F \times I$, defined by $f(x, y) = \big(H(x, y), y\big)$, for $x \in F$, $y \in I$. $f \mid F \times 0$ and $f \mid F \times 1$ are homeomorphisms. Therefore there is a homotopy of f, constant on $F \times \partial I$, which makes f a level-preserving homeomorphism, by (6.1) and (3.5). We change H near $F \times I$ according to this homotpy of f. After this change, $H \mid F \times I$ describes the ideal isotopy of $h \mid F$; namely, we perform this isotopy (actually, induce it by an isotopy of h near F), while making $H \mid F \times I$ the constant homotopy. *Case* 2 is thus reduced to *Case* 1.

Case 3. $\partial M \neq \emptyset$; *$M$ is boundary-irreducible; if M is homeomorphic to a line bundle, then h is orientation-preserving.* Assume, *Case* 3 cannot be reduced to *Case* 2. Then for some component F of ∂M, and for the map $f\colon F \times I \to M$, defined as $H \mid F \times I$, there is no deformation (of pairs) of $f\colon (F \times I, F \times \partial I) \to (M, \partial M)$ into $(\partial M, \partial M)$. It follows from (6.1) that there is a homotopy of f, constant on $F \times \partial I$, which makes f a covering map. Since $f \mid F \times 0$ is a homeomorphism, this covering is 1- or 2-sheeted.

In the first case, h interchanges the boundary components of M; since h

is homotopic to the identity map, it must be orientation-reversing.

In the second case, we argue as follows. We know from (4.1) that M is homeomorphic to a line bundle over a closed non-orientable surface. We compose the homeomorphism h with a homeomorphism which is reflection on each line. Denote the composition by h'. There is a natural homotopy H' from h' to the identity map. What we assumed on H, implies that

$$f': (F\times I, F\times\partial I)\to(M, \partial M) ,$$

defined as $H' \mid F \times I$, does contract to $(\partial M, \partial M)$. Thus, since F is all of ∂M, we deduce from *Case* 2 that h' is isotopic to the identity map. Whence h was orientation-reversing.

To handle *Case* 4, we need the following lemma.

LEMMA 7.4. *In the closed irreducible manifold N, let G be an incompressible surface. Let $h\colon N\to N$ be a homeomorphism, such that $h(G)=G$. If h is homotopic to the identity map, then h does not interchange the sides of G.*

PROOF. If G is non-separating, look at a closed curve which intersects G in one point. Since h induces the identity on $H_1(N)$, the assertion follows.

If G is separating, then $\pi_1(N) \approx A *_C B$ in a non-trivial (and natural) way, where C, A, B stand for $\pi_1(G), \pi_1(N_1), \pi_1(N_2)$ respectively, N_1, N_2 being the closures of $N - G$, (cf. (1.1.6)). If our assertion were wrong, there would exist an inner automorphism of $A *_C B$ which interchanges A and B. Let a be an element which effects such an inner automorphism. Present a as $a = a_1\cdot \cdots \cdot a_m$, where a_j is an element of A or B, and not both a_j and a_{j+1} belong to either A or B. If $a_m \in C$, then $m = 1$, $a = a_m$, and conjugation by a cannot interchange A and B; so assume $a_m \notin C$, and $a_m \in A$, say. Select $b \in B$, $b \notin C$. Then

$$aba^{-1} = a_1\cdot \cdots \cdot a_m b a_m^{-1}\cdot \cdots \cdot a_1^{-1} .$$

But this is not an element of A [8, Satz 2, p. 340; Kor. p. 341].

Case 4. *M is closed*. Let F be an incompressible surface in M. By (5.5), we may assume $h(F) = F$. Let G be a surface which is homeomorphic to F. Define $f\colon G \times I \to M$ as the restriction $H \mid F \times I$. A small homotopy of f, constant on $G \times \partial I$, will give us $f^{-1}(F) \cap U(G \times \partial I) = G \times \partial I$, where $U(G \times \partial I)$ is a regular neighborhood of $G \times \partial I$. Applying then (1.3), we find a homotopy of f, constant on $U(G \times \partial I)$, which makes $f \mid \overline{(G \times I - U(G \times \partial I))}$ transverse with respect to F, and

$$f^{-1}(F) \cap \overline{(G \times I - U(G \times \partial I))} = G_1 \cup \cdots \cup G_m ,$$

a system of incompressible surfaces in $\overline{(G \times I - U(G \times \partial I))}$, and hence also in $G \times I$. The G_j are closed, so by (3.2), each G_j is parallel to $G \times 0$, and any two are parallel. By a commutative diagram, $\ker (f \mid G_j)_* = 0$. Therefore, using Nielsen's theorem, we may assume $f \mid G_j$ is a covering map for any j. Finally we assume, f has been deformed (by a homotopy which is constant on $G \times \partial I$) so that it has the above properties and that, in addition, the number m is as small as possible.

Any two components of $G \times \partial I \cup G_1 \cup \cdots \cup G_m$ bound a domain $G \times I'$. If these components are adjacent, there is a lifting of $f \mid G \times I'$ to $\tilde{f}\colon G \times I' \to \tilde{M}$, where $\tilde{M}$ is obtained fron M by splitting at F. Applying (6.1) to all these $\tilde{f}$, and remembering our minimality condition on m, we find a deformation of f, constant on $G \times \partial I \cup G_1 \cup \cdots \cup G_m$, with one of the following four cases as its end result.

(a) $f(G \times I) \subset F$

(b, c, d) $f\colon (G \times I, G \times \partial I \cup G_1 \cup \cdots \cup G_m) \to (M, F)$ *is locally homeomorphic.*

(b) $\tilde{M}$ *is homeomorphic to* $F \times I$.

(c) $m = 0$, *and* (*at least*) *one component of* $\tilde{M}$ *is the twisted line bundle with* F *as its boundary.*

(d) $m > 0$, *and both components of* $\tilde{M}$ *are the twisted line bundle with* F *is its boundary.*

(In the conclusions (b, c, d), we used that $f \mid G \times 0$ is a homeomorphism, and (4.1).)

ad (a). In the same way as in *Case* 2 above, we make the homotopy constant on F. By (7.4), h does not interchange the sides of F. So we can take a regular neighborhood $U(F)$, make $h \mid U(F)$ the identity map, and make the homotopy constant on $U(F)$. Next, we construct a hierarchy for $\overline{(M - U(F))}$, if $\overline{(M - U(F))}$ is connected, respectively, hierarchies for the components of $\overline{(M - U(F))}$ in the other case. The proof proceeds then as *Case* 1 from the beginning, with the difference only that the induction in (7.2) starts with $M_0 = M$, and $F_0 = F$.

ad (b). There is an obvious isotopy which slides around F. After this has been performed, the homotopy can be made constant on F. Thus we are in *Case* (a).

ad (c, d). We show these cannot happen.

ad (c). Let M' and M'' be the closures of $M - F$; let M' be that submanifold onto which $f\colon G \times I$ is a covering map. There is a 2-sheeted covering $p\colon N \to M$, such that $p^{-1}(M')$ is homeomorphic to $F \times I$, and $p^{-1}(M'')$ has two components, each of which is mapped homeomorphically by p.

Denote by h_τ the homotopy of the identity map on M, defined by H.

There exists a homotopy h'_τ of the identity map on N, such that $p \circ h'_\tau = h_\tau \circ p$, Define $h' = h'_1$. The map $h': N \to N$ is a lifting of the homeomorphism h; it has no choice but to be a homeomorphism itself. Consider now a lifting f' of $f: G \times I \to M$. $f': G \times I$ is a homeomorphism onto $p^{-1}(M')$. Thus, if we denote $\partial(p^{-1}(M'))$ by $F' \cup F''$, it follows that h' interchanges F' and F''. Hence h' interchanges the components of $p^{-1}(M'')$. Since F'' is parallel to F', we can deforme h' into a homeomorphism $h'': N \to N$, which maps F' to itself, and interchanges its sides. Since h'' is homotopic to the identity map, this contradicts (7.4).

ad (d). Make the same construction as in (c). And consider a lifting $f': G \times I \to N$. If $f'^{-1}(p^{-1}(M')) = f^{-1}(M')$ has an odd number of components, it follows again that h' interchanges F' and F'', so the same contradiction comes out.

Otherwise, $f'(G \times \partial I) \subset F'$, say. Denote by N' and N'' the closures of $N - F'$. Denote by α and β the number of components of $f'^{-1}(N')$ and $f'^{-1}(N'')$, respectively. $\alpha + \beta$ is equal to the number of components of $f^{-1}(M'')$. Neither α nor β can be 0 unless the other is 1. Thus, repeating the construction of (c), we finally get our contradiction.

As an immediate consequence of (6.5) and (7.1), we have the following.

COROLLARY 7.5. *Let M be an irreducible and boundary-irreducible manifold which is sufficiently large, and which is not homeomorphic to a line bundle. Let $\mathcal{H}_0(M)$ be the quotient group of the group of auto-homeomorphisms of M by the subgroup of those which are isotopic to the identity map. The set of those automorphisms of $\pi_1(M)$ which respect the peripheral structure is a group, and its quotient group by the subgroup of inner automorphisms, is naturally isomorphic to $\mathcal{H}_0(M)$.*

Remark. There is a long way from the above isomorphism to the actual calculation of $\mathcal{H}_0(M)$ for a given manifold. For a few manifolds, there is another, more geometric, approach to $\mathcal{H}_0(M)$, which will be indicated now.

Let N be a compact orientable Seifert fibre space which is "big enough" in the (slightly more restricted) sense that there exists an incompressible surface in N, which is not boundary-parallel, and which receives an induced fibering from N. Denote by $\mathcal{G}(N)$ the group of fibre-preserving homeomorphisms of N, and by $\mathcal{G}'(N)$ the subgroup of those which are isotopic to the identity map by fibre-preserving isotopies. And consider the natural homomorphism $\mathcal{G}(N)/\mathcal{G}'(N) \to \mathcal{H}_0(N)$. This homomorphism is surjective if N is not one of a finite number of exceptions [17], (10.1). It is also injective. (This is not too difficult, and goes roughly as follows. There is a hierarchy for N, in which the first surfaces (and neighborhoods) receive an induced fibering from N, and the remaining surfaces are discs (being essentially meridian surfaces in fibre-neighborhoods of the exceptional fibres.) Given a fibre-preserving homeomorphism of N and an isotopy from it to the identity map, we treat this isotopy as a homotopy, and start playing the game by which we proved (7.1), using the above hierarchy. The essential step is to make the homotopy constant on those first surfaces of the hierarchy. We achieve this by referring explicitly to (5.4), (instead of (5.5) in the proof of (7.1)). The small region of parallelity which we find this way does not contain an exceptional fibre, by [17, Lemmas (7.4) and (7.6)], and so the situation can be improved by a fibre-preserving isotopy.

The calculation of $\mathcal{G}(N)/\mathcal{G}'(N)$ may be considered as a $(2+\varepsilon)$-dimensional problem. It should be practicable quite generally. (Clearly, there is an exact sequence $A \longrightarrow \mathcal{G}/\mathcal{G}' \longrightarrow B \longrightarrow 0$, where each element of A is represented by a homeomorphism which sends each fibre to itself, and B is a kind of braid group.)

8. Universal covers

Let M be a compact connected orientable PL 3-manifold, which is irreducible and sufficiently large (in the sense of (1.1.7)). Denote by $\tilde{M}$ the universal cover of M. Let E be the unit ball in euclidean 3-space.

THEOREM 8.1. *There is an embedding $f\colon \tilde{M} \longrightarrow E$, such that $f(\tilde{M}) \supset \mathring{E}$.*

Let F be an incompressible (PL) surface in M, $U(F)$ a regular neighborhood of F, and $N = \overline{(M - U(F))}$. Because of (1.2) it will suffice to prove

If (8.1) *holds for N (respectively, for the two components of N), then it holds for M, too.*

The subspace of $\tilde{M}$ which projects onto F by the covering map, consists of a number (countable at most) of components, each of which is homeomorphic to the universal cover of F. We denote them by $G_1, G_2, \cdots$. The subspace of $\tilde{M}$ which projects onto $U(F)$ may be written as $\bigcup (G_j \times I)$ in a natural way, with G_j identified with $G_j \times 1/2$. Each component of $\overline{(\tilde{M} - \bigcup (G_j \times I))}$ is homeomorphic to the universal cover of N (respectively of one of the components of N). We denote them by $N_1, N_2, \cdots$. We arrange the numbering of the N_j and G_j, and define the $N^{(j)}$, in such a way that the following holds.

$$N^{(1)} = N_1\,; \qquad N^{(j)} \cap (G_j \times I) = G_j \times 0\,; \qquad (G_j \times I) \cap N_{j+1} = G_j \times 1\,;$$
$$N^{(j+1)} = N^{(j)} \cup (G_j \times I) \cup N_{j+1}\,.$$

Suppose, an embedding $N^{(j)} \longrightarrow E$ has been costructed, such that $N^{(j)} \supset \mathring{E}$. Then in particular, $G_j \times 0$ is embedded in ∂E.

On the other hand, $G_j \times 0$ is homeomorphic to a submanifold G'_j of the disc D, with $G'_j \supset \mathring{D}$. We identify D with the unit disc in the plane $z = 0$ in (another) euclidean 3-space. Let p and q be points on the z-axis with z-coordinates $z_p = -1$, and

$$(8.2) \qquad -1/j < z_q < 0 ,$$

and let P and Q be the cones from p and q to G'_j. Finally, let $G'_j \times I/2$ be the cylinder, determined by G'_j and by $0 \leqq z \leqq 1/2$. We define an embedding $Q \cup (G'_j \times I/2) \to Q$ as follows. For any straight line which contains q, we map the closure of its intersection with $Q \cup (G'_j \times I/2)$ linearly onto ist intersection with Q.

Using this embedding and the natural homeomorphism from P to the cone over $G_j \times 0$ (with cone-point the center of E), we define an embedding $N^{(j)} \cup (G_j \times [0, 1/2]) \to E$. Then, again, $(N^{(j)} \cup (G_j \times [0, 1/2])) \supset \mathring{E}$, and moreover, the closure of $G_j = G_j \times 1/2$ in ∂E is a disc. In the same way we construct from the embedding of N_{j+1} in the ball E' an embedding of $(G_j \times [1/2, 1]) \cup N_{j+1}$ in E', such that the closure of G_j in $\partial E'$ is a disc. If we use both times the same homeomorphism to $G'_j \subset D$ (i.e., *via* the correspondence $G_j \times 0 \leftrightarrow G_j \times 1$), we find that the identification map $\partial E \supset G_j \leftrightarrow G_j \subset \partial E'$ extends to a homeomorphism of the closures of G_j. Thus, matching ∂E and $\partial E'$ along these closures, we define an embedding of $N^{(j+1)}$ in the ball $E \cup E'$, with $N^{(j+1)} \supset (E \cup E')^{\circ}$.

We finally map $E \cup E'$ onto E by a homeomorphism $h: E \cup E' \to E$ with the properties

(8.3) For any $x \in E$, the distance of $h(x)$ from ∂E is not less than that of x from ∂E. If $x \in E$ has distance at least $1/j$ from ∂E, or if x lies in the cone from the center of E to $\overline{(\partial E - (E \cap E'))}$, then $h(x) = x$.

Repeating the induction step, we construct embeddings $N^{(j)} \to E$ for arbitrary large j. Because of (8.2) and (8.3), a limit map is defined. It is the required embedding.

Remark. Of those irreducible manifolds, known to me, which have infinite fundamental group and are not sufficiently large [19], some (and possibly all) have a finite cover which is sufficiently large. Moreover, due to their fibre structure, it is easily seen directly that their universal cover is indeed euclidean 3-space. Thus, (8.1) is by no means best possible.

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(Received April 11, 1967)